INVARIANT SETS FOR SWITCHED DISCRETE TIME SYSTEMS SUBJECT TO BOUNDED DISTURBANCES ⋆

P. Grieder * S.V. Raković ** M. Morari * D.Q. Mayne **

* *Automatic Control Laboratory, Swiss Federal Institute of Technology, Physikstrasse 3, ETL K13.2, CH-8092 Zurich*
** *Imperial College London, Exhibition Road, London SW7 2BT, United Kingdom*

Abstract: This paper addresses invariant set computation for discrete-time switched systems subject to bounded disturbances. Specifically, we show how to compute the maximal robust switched invariant set $\tilde{\mathcal{C}}^{\mathrm{S}}_{\infty}$, which we define to be the set of states which can be made robust invariant by an appropriate switching law. Furthermore it is demonstrated how the maximal robust reachable set $\tilde{\mathcal{K}}^{\mathrm{S}}_{\infty}(\Omega)$ can be computed, i.e., all states which can be robustly steered into the target set Ω by an appropriate switching law. We also show how these sets may be used to obtain a minimum time controller for switched systems, i.e., a controller which robustly steers the state into a given target set in minimal time. *Copyright© 2005 IFAC.*

Keywords: switched systems, robust invariant sets, minimum time control

1. INTRODUCTION

Switched systems are of interest since they represent a wide range of practical problems, e.g., gear selection for a car, controller switching for linear systems. Furthermore, in hybrid control design, the goal is often to achieve the desired closed-loop behavior by switching between various control schemes which each optimize different control objectives. These schemes are able to tackle complex design tasks such as multi-objective problems. A survey of hybrid control design methods is given in (Antsaklis and Koutsoukos, 2003).

The results presented here are directly applicable to switched systems although we extend our observations to general *Autonomous Piecewise Affine* (APWA) systems. The standard APWA system consists of affine systems not subject to external inputs ($x^+ = Ax + b$) which are defined over a polyhedral partition, i.e., a unique affine dynamic is associated to every state. Here, we consider APWA systems which are defined over polyhedra that may overlap. Therefore, switched systems which correspond to a set of linear or affine dynamics valid for the entire state-space are a subset of the type of APWA systems considered here.

A wide range of literature exists on PWA systems, since they represent a powerful tool for approximating non-linear systems (Sontag, 1981) and because of their equivalence to certain classes of hybrid systems (Heemels *et al.*, 2001). Specifically, we consider APWA systems subject to bounded and persistent disturbances. This class of disturbances is useful for dealing with measurement noise and system uncertainties. Although the APWA systems considered here may be seen as a subset of PWA systems, the specific structure in the APWA dynamics can be exploited, allowing us to use more efficient computational tools in the subsequent sections. Specifically, when reachabil-

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ity computations are considered, the structure of the APWA dynamics allows us to refrain from resorting to projection methods as is necessary for standard PWA systems (e.g. (Kerrigan, 2000; Raković *et al.*, 2004)).

This paper presents methods for computing the maximal robust switched invariant set $\tilde{\mathcal{C}}_{\infty}^{\mathrm{S}}$ for APWA systems defined over possibly overlapping polyhedra and demonstrates how these sets may serve to obtain robust switching sequences for APWA systems subject to bounded and persistent disturbances. We base our contribution on invariant set results for linear and general PWA systems (Blanchini, 1999; Kolmanovsky and Gilbert, 1998; Kerrigan, 2000; Raković *et al.*, 2004; Saint-Pierre, 1994) and on recent results for linear switched systems (Julius and van der Schaft, 2002; De Santis *et al.*, 2004). Our results agree with the general and abstract viability theory framework elaborated in (Aubin, 1991). However, we are concerned with efficient computational procedures for the APWA class of systems and we show how computational geometry and polyhedral algebra can be efficiently exploited to perform the required computations. We also show how the robust invariant set may be used to obtain minimum time controllers which guarantee robust convergence to a predefined target set.

The paper is structured as follows. Section 2 introduces the problem and various definitions and preliminaries. Section 3 presents a general algorithm for computing robust invariant sets for switched systems which is subsequently used in Section 4 to obtain a robust switch control law. Numerical examples are given in Section 5 before concluding in Section 6.

2. PROBLEM DESCRIPTION

We consider the autonomous piecewise affine switched system

$$x^+ = f(x, i, w), \tag{1}$$

where x^+ denotes the successor state, x and w denote the current state and disturbance, respectively, and i denotes the active dynamic.

Definition 1. A *polyhedron* is the intersection of a finite number of closed half-spaces, a *polytope* is a closed and bounded *polyhedron* and a *P-collection* is the (possibly non-convex) union of a finite number of polyhedra.

For switched systems, the dynamic i may be chosen by the controller. The function $f(\cdot)$ is affine in each of a finite number of polyhedra $\{Q_i\},\ i \in \mathcal{I} \triangleq \{1, 2, \dots, J\}$, with possibly overlapping interiors that cover the region of state space of interest. Note that this system model covers classic switched systems where we usually have $Q_i = \mathbb{R}^n, \quad \forall i \in \mathcal{I}$. The system satisfies:

$$f(x, i, w) \triangleq A_i x + c_i + w,\ x \in Q_i. \tag{2}$$

and is subject to the constraints

$$(x, w) \in \mathbb{X} \times \mathbb{W} \subset \mathbb{R}^n \times \mathbb{R}^n \tag{3}$$

where *both* $\mathbb{X}$ *and* $\mathbb{W}$ *are polytopic sets containing the origin in their interior.*

Definition 2. The set $\Omega \subseteq \mathbb{X}$ is said to be a robust switched invariant set for the PWA system in (2) subject to the constraints in (3) if for every $x \in \Omega$ there exists an $i \in \mathcal{I}$ such that $f(x, i, w) \in \Omega$ for all $w \in \mathbb{W}$.

The problem we consider is the efficient computation of the maximal robust switched invariant set $\tilde{\mathcal{C}}_{\infty}^{\mathrm{S}}$ and the maximal robust switched attractive set $\tilde{\mathcal{K}}_{\infty}^{\mathrm{S}}$ for switched PWA systems which are defined as:

Definition 3. The maximal robust switched invariant set, $\tilde{\mathcal{C}}_{\infty}^{\mathrm{S}}$, for the PWA system in (2) subject to the constraints in (3) is defined by

$$\begin{aligned}\tilde{\mathcal{C}}_{\infty}^{\mathrm{S}} = \{x(0) \in \mathbb{R}^n |\ &\exists i(k) \in \mathcal{I}, \text{s.t. } x(k) \in \mathbb{X} \cap Q_{i(k)},\\ &x(k+1) = f(x(k), i(k), w(k)),\\ &\forall w(k) \in \mathbb{W},\ \forall k \geq 0\}.\end{aligned}$$

Definition 4. The N-step robust switched attractive set, $\tilde{\mathcal{K}}_{N}^{\mathrm{S}}(\Omega)$, where Ω is a non-empty set in $\mathbb{R}^n$, for the switched system (2) subject to the constraints in (3) is defined by:

$$\begin{aligned}\tilde{\mathcal{K}}_{N}^{\mathrm{S}}(\Omega) = \{x(0) \in \mathbb{R}^n |\ &\exists i(k) \in \mathcal{I},\ \text{s.t.}\\ &x(k) \in \mathbb{X} \cap Q_{i(k)},\ x(N) \in \Omega,\\ &x(k+1) = f(x(k), i(k), w(k)),\\ &\forall w(k) \in \mathbb{W},\ k \in \{0, \dots, N\}\}.\end{aligned}$$

The set $\tilde{\mathcal{K}}_{N}^{\mathrm{S}}(\Omega)$ contains all states which can be robust steered into the set Ω in N-steps. The maximal robust switched attractive set $\tilde{\mathcal{K}}_{\infty}^{\mathrm{S}}(\Omega)$ is defined as the union of all N-step attractive sets with $N \in \mathbb{N}$. Here, $\mathbb{N} \triangleq \{1, 2, \dots\}$ denotes the set of positive integers.

Lemma 1. For system (2) subject to constraints (3) the following holds: (i) $\tilde{\mathcal{K}}_{\infty}^{\mathrm{S}}(\Omega) \subseteq \tilde{\mathcal{C}}_{\infty}^{\mathrm{S}}$ if $\Omega \subset \mathbb{X}$ and (ii) $\tilde{\mathcal{K}}_{\infty}^{\mathrm{S}}(\Omega) = \tilde{\mathcal{C}}_{\infty}^{\mathrm{S}}$ if $\Omega = \mathbb{X}$.

If Ω is invariant and no limit cycles (i.e. closed orbits) are contained inside Ω, the set $\tilde{\mathcal{K}}_{\infty}^{\mathrm{S}}(\Omega)$ does not contain limit cycles. The set $\tilde{\mathcal{C}}_{\infty}^{\mathrm{S}}$, on the other hand, may always contain limit cycles.

3. THE MAXIMAL ROBUST SWITCHED INVARIANT SET

In this section we present an algorithm to compute the maximal robust switched invariant set $\tilde{\mathcal{C}}^{\mathrm{S}}_{\infty}$ and the maximal robust switched attractive set $\tilde{\mathcal{K}}^{\mathrm{S}}_{\infty}(T_{\mathrm{set}})$ for a given target set T_{set}, as given in Definition 3 and 4, respectively.

Given the non-empty set $\Omega \subset \mathbb{R}^n$ we define the set of states $Pre(\Omega, i)$ that robustly evolve in one step to Ω when dynamic i is active as

$$\begin{aligned} Pre(\Omega, i) \triangleq& \{x \in \mathbb{X} \mid f(x, i, \omega) \in \Omega,\ \forall \omega \in \mathbb{W}\} \\ =& \{x \in \mathbb{X} \mid f(x, i, 0) \in \Omega \ominus \mathbb{W}\}. \end{aligned}$$

where $\ominus$ denotes the Minkowski set subtraction (Pontryagin difference) defined by:

$$\Omega \ominus \mathbb{W} \triangleq \{x \in \mathbb{R}^n \mid x + w \in \Omega,\ \forall w \in \mathbb{W}\}.$$

Remark 1. If the set Ω is a P-collection then the set $Pre(\Omega, i)$ is a P-collection, since the Pontryagin difference of a P-collection and a polytope is a P-collection and the dynamics $f(x, i, \omega)$ are affine for all $i \in \mathcal{I}$. Clearly, the set of states that can robustly evolve in one step to Ω is given by

$$\bigcup_{i \in \mathcal{I}} Pre(\Omega, i).$$

It is well known that computation of the Pontryagin difference is easily implemented if Ω and $\mathbb{W}$ are polytopes by solving a number of linear programs (Kolmanovsky and Gilbert, 1998; Kerrigan, 2000). However, if the set Ω is a P-collection, this operation becomes very complex. Details on Pontryagin difference computation for P-collections and polytopes are given in (Kerrigan, 2000; Raković *et al.*, 2004) and software tools to perform this operation are available (Kvasnica *et al.*, 2003; Veres, 2003).

The following algorithm can be used for computation of the maximal robust switched invariant set $\tilde{\mathcal{C}}^{\mathrm{S}}_{\infty}$ or the maximal robust switched attractive set $\tilde{\mathcal{K}}^{\mathrm{S}}_{\infty}(\Omega_0)$, for a given target set Ω_0 (Aubin, 1991; Saint-Pierre, 1994):

Algorithm 3.1.

(1) Specify initial set $\mathcal{H}_0 = \Omega_0$ and set $k = 0$.
(2) $\Omega^i_{k+1} = Pre(\mathcal{H}_k, i), \quad \forall i \in \mathcal{I}$.
(3) $\mathcal{H}_{k+1} = \bigcup_{i \in \mathcal{I}} \Omega^i_{k+1}$.
(4) If $\mathcal{H}_{k+1} = \mathcal{H}_k$, return; Else, set $k = k+1$ and goto 2.

An algorithm for checking whether two P-collections are equal can be found in (Raković *et al.*, 2003; Baotić and Torrisi, 2003). This functionality is also contained in various software tools (e.g., (Kvasnica *et al.*, 2003)).

Lemma 2. Let the set $\mathcal{H}_{k^*}$ be a fixed point (i.e., $\mathcal{H}_{k^*} = \mathcal{H}_{k^*+1}$) of Algorithm 3.1, then $\mathcal{H}_{k^*}$ is a robust switched invariant set.

The proof for Lemma 2 follows from results reported in (Julius and van der Schaft, 2002; Raković *et al.*, 2004) and is well known in the general viability theory (Aubin, 1991; Saint-Pierre, 1994).

Theorem 1. Suppose that $\Omega_0 = \mathbb{X}$ in Step 1 of Algorithm 3.1 and that there exists a $k^* \in \mathbb{N}$ such that $\mathcal{H}_{k^*} = \mathcal{H}_{k^*+1}$. Then, Algorithm 3.1 terminates and $\tilde{\mathcal{C}}^{\mathrm{S}}_{\infty} = \mathcal{H}_{k^*}$.

PROOF. $\Omega_0 = \mathbb{X}$ is the largest feasible set and $\mathcal{H}_k \supseteq \mathcal{H}_{k+1}$. If $\mathcal{H}_k = \mathcal{H}_{k+1}$ then $\mathcal{H}_k$ is a fixed point of the Algorithm 3.1 and it is the maximal robust control invariant set contained in $\mathbb{X}$, i.e., $\tilde{\mathcal{C}}^{\mathrm{S}}_{\infty} = \mathcal{H}_k$. □

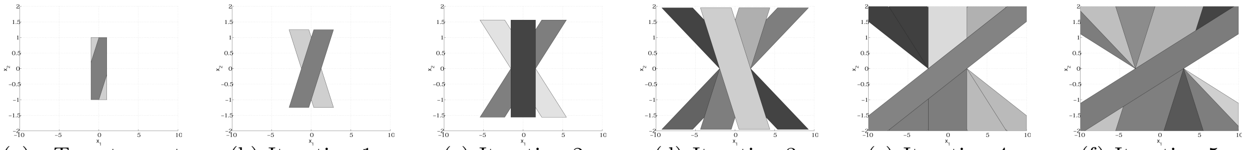
Theorem 2. Suppose that $\Omega_0 = T_{\mathrm{set}}$, where $T_{\mathrm{set}} \subset \mathbb{R}^n$ is a non-empty robust switched invariant set. If in Step 1 of Algorithm 3.1 there exists a $k^* \in \mathbb{N}$ such that $\mathcal{H}_{k^*} = \mathcal{H}_{k^*+1}$, then Algorithm 3.1 terminates and $\tilde{\mathcal{K}}^{\mathrm{S}}_{\infty}(T_{\mathrm{set}}) = \mathcal{H}_{k^*}$.

PROOF. It holds that $\mathcal{H}_k \subseteq \mathcal{H}_{k+1}$. If $\mathcal{H}_k = \mathcal{H}_{k+1}$, it follows that there does not exist a state $x \notin \mathcal{H}_k$ such that $f(x, i, \omega) \in \mathcal{H}_k$ for any $i \in \mathcal{I}$ and $\omega \in \mathbb{W}$. Therefore $\tilde{\mathcal{K}}^{\mathrm{S}}_{\infty}(T_{\mathrm{set}}) = \mathcal{H}_k$ if $\mathcal{H}_k = \mathcal{H}_{k+1}$. □

In general, the sets Ω^i_k and $\mathcal{H}_k$ are P-collections and the sets Ω^i_k may be overlapping due to the definition of $f(\cdot)$ in (1). Although most of the computations in Algorithm 3.1 are performed on P-collections it should be noted that the necessary computation for APWA systems is relatively straightforward, since no projections need to be performed as is the case for PWA systems (e.g. (Kerrigan, 2000; Raković *et al.*, 2004)).

4. MINIMUM TIME CONTROL

This section demonstrates how the set computation schemes of the previous sections can serve to obtain switching laws which are robust towards additive but bounded disturbances. The proposed switching scheme guarantees convergence to a target set, whereby the target set T_{set} is chosen here to be a small set containing the origin in its interior. However, the size of T_{set} is lower-bounded by the magnitude of the persistent disturbance $w \in \mathbb{W}$.



(a) Target set T_{set}. (b) Iteration 1. (c) Iteration 2. (d) Iteration 3. (e) Iteration 4. (f) Iteration 5.

Fig. 1. Reach-set computation according to Algorithm 4.1, with $\mathbb{W} = \{0\}$ for Example 1. Here, the first 5 iterations are depicted. The different polytopes defining the P-collections are depicted in different colors.

4.1 Controller Computation

We propose the following algorithm:

Algorithm 4.1.

(1) Define a target set $\mathcal{H}_0 = T_{\text{set}}$ and set $k = 0$.
(2) Compute $\mathcal{S}_{k+1}^i = Pre(\mathcal{H}_k, i) \cap T_{\text{set}}, \quad \forall i \in \mathcal{I}$.
(3) Set $\mathcal{H}_{k+1} = \bigcup_{i \in \mathcal{I}} \mathcal{S}_{k+1}^i$.
(4) If $\mathcal{H}_{k+1} \neq \mathcal{H}_k$, set $k = k+1$ and goto step 2; Else, set $k^* = k$.
(5) Compute $\mathcal{S}_{k+1}^i = Pre(\mathcal{H}_k, i), \quad \forall i \in \mathcal{I}$.
(6) Set $\mathcal{H}_{k+1} = \bigcup_{i \in \mathcal{I}} \mathcal{S}_{k+1}^i$.
(7) If $\mathcal{H}_{k+1} = \mathcal{H}_k$, return; Else, set $k = k+1$, and goto step 5.

Algorithm 4.1 first computes an invariant target set in steps 1 to 4. In fact, the set $\mathcal{H}_{k^*}$ is the maximal robust switched set contained in T_{set}. The target set $\mathcal{H}_{k^*}$ is subsequently used to compute the set of states which can be driven into it in $k - k^*$ steps (see Figure 1). This allows for a switching scheme which drives the state to the target set in minimum time. As a consequence of the initialization for $\mathcal{H}_0$, the target set T_{set} is reached from $\mathcal{H}_k$ in at most $k+1$ steps. In general, Algorithm 4.1 may not terminate in finite time. However, a sensible criterion for its termination can be specified. For instance, it is to possible to abort the Algorithm at step 7 after a predefined number of iterations or after the state space of interest is covered. In this case, the minimum time controller covers merely a subset of $\tilde{\mathcal{K}}_\infty^{\text{S}}(T_{\text{set}})$.

4.2 Target Set Computation

In particular cases, where there exists a set of dynamics defined in the region of state space that contain the origin in their interior, Algorithm 4.1 can be simplified as we illustrate next.

Let $\mathcal{I}_0 \subseteq \mathcal{I}$, be defined by:

$$\mathcal{I}_0 \triangleq \{i \in \mathcal{I} \mid 0 \in \text{int}(Q_i),\ |\lambda_{max}(A_i)| < 1,\ c_i = 0\}$$

where 0 is the origin of the state-space and $\lambda_{max}(A_i)$ denotes the largest eigenvalue of A_i. Thus, $\mathcal{I}_0$ is a subset of $\mathcal{I}$ and contains the stable linear dynamics defined over the region of state space containing the origin as an interior point. It is shown in (Kouramas, 2002; Raković *et al.*, 2005) that a robust positively invariant approximation of the minimal robust positively invariant set (Kolmanovsky and Gilbert, 1998) for strictly stable linear discrete time system $x^+ = Ax + w,\ w \in \mathbb{W}$ can be computed in a finite number of iterations. Given the disturbance set $\mathbb{W}$, *containing the origin in its interior*, a scalar $0 \leq \alpha < 1$ and a strictly stable matrix A there exists an integer s such that $A^s\mathbb{W} \subseteq \alpha\mathbb{W}$ and the set $F(\alpha, s)$ (Kouramas, 2002; Raković *et al.*, 2005) defined by

$$F(\alpha, s) \triangleq (1-\alpha)^{-1} \bigoplus_{j=0}^{s-1} A^i\mathbb{W}$$

is a robust positively invariant set for the system $x^+ = Ax + w$ ($\bigoplus$ denotes the Minkowski set addition). For any $i \in \mathcal{I}_0$ let

$$F_i(\alpha_i, s_i) \triangleq (1-\alpha_i)^{-1} \bigoplus_{j=0}^{s_i-1} A^i\mathbb{W} \tag{4}$$

Note that for any $i \in \mathcal{I}_0$, the set $F_i(\alpha_i, s_i)$ is robust positively invariant for the dynamics A_i, if $F_i(\alpha_i, s_i) \subseteq (\mathbb{X} \cap Q_i)$. Let

$$\mathcal{I}_0^* = \{i \in \mathcal{I}_0 \mid F_i(\alpha_i, s_i) \subseteq (\mathbb{X} \cap Q_i)\}$$

If the set $\mathcal{I}_0^* \neq \varnothing$, it is clear that $\bigcup_{i \in \mathcal{I}_0^*} F_i(\alpha_i, s_i)$ is a robust switched invariant set. In fact any of the sets $F_i(\alpha_i, s_i),\ i \in \mathcal{I}_0^*$ is a robust switched invariant set. This gives us some extra degrees of freedom in choice of an appropriate terminal set for minimum time control scheme, since then the first four steps in the Algorithm 4.1 are not needed. The algorithm can be started with step 5, by setting $k^* = 0$ and choosing $\mathcal{H}_0$ to be any of the sets $F_i(\alpha_i, s_i),\ i \in \mathcal{I}_0^*$ or any union of the sets $F_i(\alpha_i, s_i),\ i \in \mathcal{I}_0^*$.

4.3 On-Line Switching

For each $(i, k) \in \mathcal{I} \times \mathbb{N}^+$ let $I(x, k)$ denote the set of admissible switches for a given state x, such that $f(x, i, w) \in \mathcal{H}_{k-1}$, $\forall w \in \mathbb{W}$ and $i \in \mathcal{I}$. Thus:

$$I(x,k) \triangleq \{i \in \mathcal{I} \mid x \in Q_i \cap \mathcal{H}_k,\ f(x,i,w) \in \mathcal{H}_{k-1},\ \forall w \in \mathbb{W}\}$$

The on-line implementation of the proposed controller can be efficiently applied recursively for any given state $x \in \tilde{\mathcal{K}}^{\mathrm{S}}_{\infty}(T_{\mathrm{set}})$ by the following simple algorithm:

Algorithm 4.2.

(1) $c = \min_k\{k \in \mathbb{N} \mid I(x,k) \neq \varnothing\}$.
(2) Choose any switch $i \in I(x,c)$.

Remark 2. It follows from the construction of Algorithms 4.1 and 4.2 that any state $x \in \mathcal{H}_k,\ k \geq k^*$ robustly evolves into $\mathcal{H}_{k-1}$ for all $w \in \mathbb{W}$ and for any choice of $i \in I(x,k)$ which in turn implies that a trajectory starting at any state $x \in \mathcal{H}_k,\ k \geq k^*$ robustly evolves into $\mathcal{H}_{k^*}$ in no more than $k - k^*$ time steps. Once the trajectory reaches the set $\mathcal{H}_{k^*}$ it remains inside $\mathcal{H}_{k^*}$ forever, since $\mathcal{H}_{k^*}$ is a switched robust positively invariant set by construction.

This scheme allows the user to specify preferences concerning the active dynamics without influencing robust constraint satisfaction of the system. It is conceivable to choose $i \in I(x,k)$ such that a predefined cost-function is minimized.

Note that the proposed procedure is different from control schemes for general piecewise affine systems (e.g. (Borrelli, 2003; Bemporad *et al.*, 2002)) since it is not based on multi-parametric programming. Multi-parametric programming (Bemporad *et al.*, 2002) may be seen as a form of projection which is computationally expensive and excessive for the type of problem considered here.

If Algorithm 4.1 is implemented directly, the number of polytopic sets Ω_j representing the P-collection $\mathcal{H}_k = \bigcup_{j\in J}\Omega_j$ increases significantly as k grows due to the combinatorial nature of the problem. This is a direct consequence of the dynamical behavior of APWA systems. In order to avoid this problem to some extent, it is important to reduce the number of stored sets J (i.e., $\mathcal{H}_k = \bigcup_{j\in J}\Omega_j$) before incrementing the iteration counter. Specifically, one can remove all Ω_{j^*} for which the following holds: $\Omega_{j^*} \subseteq \bigcup_{j\in J\setminus\{j^*\}}\Omega_j$. This corresponds to checking whether a polytope is covered by a P-collection, which is addressed in (Baotić and Torrisi, 2003). Although this reduction procedure reduces complexity significantly, the number of regions may still grow quickly with increasing prediction horizon; moreover this complexity reduction scheme subsequently makes available only a subset of the admissible switches in the controller implementation.

5. NUMERICAL EXAMPLES

This section illustrates the application of Algorithm 4.1 on the following numerical example.

Example 1. Assume a switched system (2) with $Q_1 = Q_2 = \mathbb{R}^n$ and the following dynamics

$$A_1 = \begin{bmatrix} 0.8 & 1 \\ 0 & 0.8 \end{bmatrix}, \qquad c_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.8 & -1 \\ 0 & 0.8 \end{bmatrix}, \qquad c_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The system is subject to state constraints $\mathbb{X} = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 10\}$, i.e., each state is constrained between ± 10. The objective of the controller should be to drive the state into the box $T_{\mathrm{set}} = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}$ as quickly as possible.

We initially consider no additive disturbance, i.e., $\mathbb{W} = \{0\}$. The maximal positively invariant set for each of the dynamics is depicted in Figure 2(a). Figure 2(b) depicts the partition of the switching controller obtained with Algorithm 4.1. It is clear from the figures that the proposed switching scheme enlarges the set of controllable states. The controller consists of 346 reach-sets which were obtained in 23 iterations and a runtime of 34 seconds [1] .

Let us now assume the system is subject to additive disturbance bounded by $\mathbb{W} = \{w \in \mathbb{R}^n \mid \|w\|_\infty \leq 0.1\}$. For this case, there is no robust invariant subset contained inside the target box, if we assume no switching occurs. However, if we allow for switches, the maximal robust switched invariant set is depicted in Figure 2(c).

Although the proposed computation scheme may be expensive, it is performed off-line. The on-line effort reduces to evaluating a look-up table which can be efficiently implemented.

6. CONCLUSION

We have shown how to compute the maximal robust switched invariant set and the maximal robust attractive set for autonomous piecewise affine (APWA) systems. By considering APWA systems instead of general piecewise affine systems, it is possible to compute these sets efficiently, since projection operations are not needed. We have furthermore shown, how the obtained sets may serve as appropriate target sets when computing robust minimum-time controllers. The resulting controllers guarantee robust convergence to a user defined target set in minimum time.

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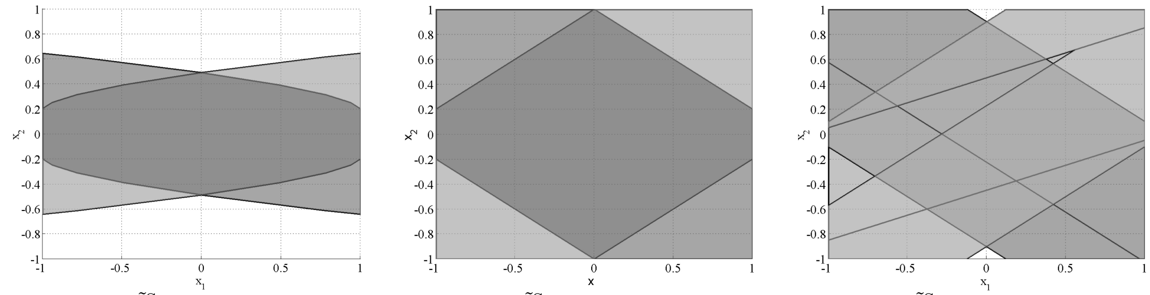
[1] Pentium IV, 2.4GHz using the MPT toolbox (Kvasnica *et al.*, 2003).



(a) The set $\tilde{\mathcal{C}}_{\infty}^{\mathrm{S}}$ for dynamic 1 and 2, respectively ($\mathbb{W} = \{0\}$, no switching).

(b) The set $\tilde{\mathcal{C}}_{\infty}^{\mathrm{S}}$ obtained with Algorithm 4.1 ($\mathbb{W} = \{0\}$).

(c) The set $\tilde{\mathcal{C}}_{\infty}^{\mathrm{S}}$ obtained with Algorithm 4.1 ($\mathbb{W} = \{w \in \mathbb{R}^n | \; ||w||_{\infty} \leq 0.1\}$).

Fig. 2. Controllable state space with and without switching, contained in $T_{\mathrm{set}} = \{x \in \mathbb{R}^n | \; \|x\|_{\infty} \leq 1\}$. The system subject to disturbances $\mathbb{W} = \{w \in \mathbb{R}^n | \; ||w||_{\infty} \leq 0.1\}$ is not controllable without switching.

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