ROBUST H_∞ CONTROL OF MARKOVIAN JUMP LINEAR SYSTEMS WITH UNCERTAIN SWITCHING PROBABILITIES [1]

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Abstract: This paper is concerned with the robust H_∞ control of Markovian jump linear systems with uncertain switching probabilities. The uncertain system under consideration involves norm-bounded uncertainties in system matrices and element-wise bounded uncertainties in the mode transition rate matrix. A new criterion is established to test the robust H_∞ performance in terms of linear matrix inequalities and a sufficient condition is addressed for constructing a robust state-feedback controller such that the closed-loop system is robustly mean square stable and has the desired H_∞ performance level. A globally convergent algorithm involving convex optimization is also proposed to find such controllers effectively. Finally, a numerical example illustrates the usefulness of the developed theory.

Keywords: Linear matrix inequalities, Markovian parameters, Robust H_∞ control, Uncertainties

1. INTRODUCTION

A great deal of attention has been devoted to the Markovian jump systems over the past decade. This family of systems may represent a large class of dynamic systems subject to random abrupt variations in their structures. It is because that very often dynamic systems are inherently vulnerable to random failures of the components, sudden disturbances of the environment, changes of the subsystems interconnections, abrupt modifications of the operating points of a linearized model of a nonlinear system and so on. Applications of such systems may be found in target tracking problems, manufactory processes and fault-tolerant systems (Mariton, 1990). A Markovian jump system normally consists of a finite collection of deterministic systems. Each deterministic system represents one operation mode of the Markovian jump system. The change of operation modes from one to another is governed by a Markov process. The problems related to controllability (Mariton, 1990; Ji and Chizeck, 1990), observability (Ji and Chizeck, 1990), stability (Ji and Chizeck, 1990; Feng *et al.*, 1992; Ghaoui and Rami, 1996; Boukas *et al.*, 1999), H_2 control (Costa *et al.*, 1999; do Val *et al.*, 2002), H_∞ control (de Farias *et al.*, 2000) have been extensively studied. As for Markovian jump linear systems with parametric uncertainties in system matrices, the robust H_∞ control problem has also been well investigated (Cao and Lam, 2000; Mahmoud and Shi, 2003).

However, only a little work has been addressed to deal with the uncertainties in the mode transition rate matrix. In a more realistic situation, it is reasonable to assume that the values of mode transition rates are not known precisely a priori and only the estimated values with error bounds

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are available when controlling a Markovian jump system. The errors between the true values and the estimated values are referred to as switching probability uncertainties. In the literature, two types of descriptions about the uncertainties in the mode transition rate matrix have been considered in the context of robust stabilization. The first one is the polytopic description where the mode transition rate matrix is assumed to belong to a convex hull with known vertices (Ghaoui and Rami, 1996). The other is described in an element-wise way where every mode transition rate is measured and an associated error bound is given at the same time (Boukas *et al.*, 1999). Although the element-wise description can be reformulated into an equivalent polytopic description, the total number of vertex matrices increases exponentially with respect to the total number of the mode transition rates which have uncertainties (Xiong *et al.*, 2005). Consequently, the technique developed in (Ghaoui and Rami, 1996) becomes impractical when the Markovian jump system has many uncertain switching rates.

On the other hand, the robust H_2 control problem for Markovian jump systems where the transition rate matrix belongs to a fixed polytope has been studied in (Costa *et al.*, 1999), and the technique developed in (Boukas *et al.*, 1999) has also been used to deal with the robust H_∞ control problem for Markovian jump linear systems (Mahmoud and Shi, 2003). In this paper, we consider the element-wise uncertainties in the mode transition rate matrix and propose a new technique to tackle the robust H_∞ control problem.

This paper is concerned with the robust H_∞ control problem for a class of Markovian jump linear systems with uncertain switching probabilities. The objective is to design a state-feedback control law such that the closed-loop system is robustly mean square stable and guarantees a prescribed H_∞ performance level over all admissible uncertainties both in the system matrices and in the mode transition rate matrix. We show that the analysis problem can be tackled in terms of the feasibility of a set of coupled linear matrix inequalities, and the associated synthesis problem can be solved effectively using a convergent algorithm involving convex optimization. Finally, a numerical example is used to illustrate the usefulness of the developed theory.

Notation: The notations in this paper are standard. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote, respectively, the n-dimensional Euclidean space and the set of all $m \times n$ real matrices. $\mathbb{R}^+$ denotes the set of strictly positive real numbers. $\mathbb{S}^{n\times n}$ is the set of all $n \times n$ real symmetric positive definite matrices and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are real symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimensions. The superscript "T" denotes the transpose for vectors or matrices, and trace$(\cdot)$ stands for the trace of a square matrix. $\|\cdot\|_2$ refers to the Euclidean norm for vectors and induced 2-norm for matrices. Moreover, let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and P is the probability measure on $\mathcal{F}$. $\mathrm{E}(\cdot)$ stands for the mathematical expectation operator. $\mathbb{L}_2^p$ denotes the Hilbert space formed by the stochastic processes $w \triangleq \{w(t) : w(t) \in \mathbb{R}^p, t \geq 0\}$ such that

$$\|w\|_2^2 \triangleq \int_0^\infty \mathrm{E}(\|w(t)\|_2^2)dt < \infty$$

2. PROBLEM FORMULATION

Consider the following class of Markovian jump linear systems with uncertain switching probabilities on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$:

$$\dot{x}(t) = \hat{A}(\hat{r}(t))x(t) + \hat{B}_1(\hat{r}(t))u(t) + \hat{B}_2(\hat{r}(t))w(t) \quad (1a)$$

$$z(t) = \hat{C}(\hat{r}(t))x(t) + \hat{D}_1(\hat{r}(t))u(t) + \hat{D}_2(\hat{r}(t))w(t) \quad (1b)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^p$ is an exogenous non-zero disturbance input, $z(t) \in \mathbb{R}^q$ is the regulated output, $t \geq 0$ is the time. The mode jumping process $\{\hat{r}(t) : t \geq 0\}$ is a continuous-time, discrete-state Markov process on the probability space, takes values in a finite mode space $\mathcal{S} \triangleq \{1, 2, \ldots, s\}$, and has the mode transition probabilities

$$\Pr(\hat{r}(t+\delta t) = j \mid \hat{r}(t) = i) = \begin{cases} \hat{\pi}_{ij}\delta t + o(\delta t), \text{if } j \neq i \\ 1 + \hat{\pi}_{ii}\delta t + o(\delta t), \text{if } j = i \end{cases}$$

where $\delta t > 0$, $\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0$, and $\hat{\pi}_{ij} \geq 0$ $(i, j \in \mathcal{S}, j \neq i)$ denotes the switching rate from mode i at time t to mode j at time $t + \delta t$ and $\hat{\pi}_{ii} \triangleq -\sum_{j=1, j\neq i}^{s} \hat{\pi}_{ij}$ for all $i \in \mathcal{S}$.

Whenever $i \in \mathcal{S}$, the matrices $\hat{A}_i \triangleq \hat{A}(\hat{r}(t) = i)$, $\hat{B}_{1i} \triangleq \hat{B}_1(\hat{r}(t) = i)$, $\hat{B}_{2i} \triangleq \hat{B}_2(\hat{r}(t) = i)$, $\hat{C}_i \triangleq \hat{C}(\hat{r}(t) = i)$, $\hat{D}_{1i} \triangleq \hat{D}_1(\hat{r}(t) = i)$ and $\hat{D}_{2i} \triangleq \hat{D}_2(\hat{r}(t) = i)$ are assumed to be not known precisely, but have the following form of norm-bounded uncertainties:

$$\begin{bmatrix} \hat{A}_i & \hat{B}_{1i} & \hat{B}_{2i} \\ \hat{C}_i & \hat{D}_{1i} & \hat{D}_{2i} \end{bmatrix} = \begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_i & D_{1i} & D_{2i} \end{bmatrix} + \begin{bmatrix} E_{0i} \\ E_{1i} \end{bmatrix} F_i \begin{bmatrix} H_{0i} & H_{1i} & H_{2i} \end{bmatrix} \quad (2)$$

where matrices A_i, B_{1i}, B_{2i}, C_i, D_{1i}, D_{2i}, E_{0i}, E_{1i}, H_{0i}, H_{1i} and H_{2i} are known constant real matrices of appropriate dimensions, while F_i is unknown, denotes the uncertainties in the system matrices, and satisfies $F_i^T F_i \leq I$ for all $i \in \mathcal{S}$. Additionally, the mode transition rate matrix $\hat{\Pi} \triangleq$

$(\hat{\pi}_{ij})$ is also assumed to be not exactly known and has the element-wise uncertainties

$$\hat{\Pi} = \Pi + \Delta\Pi \tag{3}$$

with $\Pi \triangleq (\pi_{ij})$ satisfying $\pi_{ij} \geq 0$ $(i, j \in \mathcal{S}, j \neq i)$ and $\pi_{ii} \triangleq -\sum_{j=1,j\neq i}^{s} \pi_{ij}$ for all $i \in \mathcal{S}$, where π_{ij} denotes the estimated value of $\hat{\pi}_{ij}$ in practice, and $\Delta\Pi \triangleq (\Delta\pi_{ij}) = (\hat{\pi}_{ij} - \pi_{ij})$ where $|\Delta\pi_{ij}| \leq 2\varepsilon_{ij}$, $\varepsilon_{ij} \geq 0$. $\Delta\pi_{ij}$ denotes the error between $\hat{\pi}_{ij}$ and π_{ij} for all $i, j \in \mathcal{S}, j \neq i$, and $\Delta\pi_{ii} \triangleq -\sum_{j=1,j\neq i}^{s} \Delta\pi_{ij}$ for all $i \in \mathcal{S}$.

Consider the state-feedback control law

$$u(t) = K(\hat{r}(t))x(t) \tag{4}$$

where $K_i \triangleq K(\hat{r}(t) = i) \in \mathbb{R}^{m\times n}, i \in \mathcal{S}$, is the controller to be designed. Substituting (4) into (1) yields the closed-loop system

$$\dot{x}(t) = \hat{A}_{cl}(\hat{r}(t))x(t) + \hat{B}_2(\hat{r}(t))w(t) \tag{5a}$$

$$z(t) = \hat{C}_{cl}(\hat{r}(t))x(t) + \hat{D}_2(\hat{r}(t))w(t) \tag{5b}$$

where $\hat{A}_{cli} = (A_i + B_{1i}K_i) + E_{0i}F_i(H_{0i} + H_{1i}K_i)$ and $\hat{C}_{cli} = (C_i + D_{1i}K_i) + E_{1i}F_i(H_{0i} + H_{1i}K_i)$ for all $i \in \mathcal{S}$.

The objective of the paper is to tackle the robust H_∞ performance analysis and synthesis problems for the uncertain Markovian jump system (1) with the admissible uncertainty domains (2) and (3). Throughout this paper, we have the following definitions.

Definition 1. (de Farias *et al.*, 2000) The nominal Markovian jump system of uncertain system (1) with $u(t) \equiv 0$ is said to be mean square stable if

$$\lim_{t\to\infty} \mathrm{E}(\|x(t, x_0, r_0)\|_2^2) = 0$$

for any initial conditions $x_0 \triangleq x(0) \in \mathbb{R}^n$ and $\hat{r}_0 \triangleq \hat{r}(0) \in \mathcal{S}$.

Definition 2. (Zhang *et al.*, 2003) Consider the nominal Markovian jump system of uncertain system (1) with $u(t) \equiv 0$, let G_{zw} denote the operator from $w(t)$ to $z(t)$, the H_∞-norm of the operator G_{zw} is defined as $\|G_{zw}\|_{H_\infty} \triangleq \inf \gamma$ such that

$$\|z\|_2 < \gamma \|w\|_2$$

for all non-zero stochastic processes $w \in \mathbb{L}_2^p$.

Proposition 3. (Cao and Lam, 2000) Given a prescribed scalar $\gamma_{H_\infty} \in \mathbb{R}^+$, the nominal Markovian jump system of uncertain system (1) with $u(t) \equiv 0$ is mean square stable and has H_∞ performance $\|G_{zw}\|_{H_\infty} < \gamma_{H_\infty}$ if there exist matrices $P_i \in \mathbb{S}^{n\times n}, i \in \mathcal{S}$, such that

$$\begin{bmatrix} A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j + C_i^T C_i & P_i B_{2i} + C_i^T D_{2i} \\ B_{2i}^T P_i + D_{2i}^T C_i & -\gamma_{H_\infty}^2 I + D_{2i}^T D_{2i} \end{bmatrix} < 0$$

for all $i \in \mathcal{S}$.

3. ROBUST H_∞ CONTROL

In the section, the robust H_∞ performance analysis is proposed first in terms of coupled linear matrix inequalities, then the associated synthesis is dealt with in terms of the solvability of a set of coupled linear matrix inequalities with equality constraints, which can be further tackled using some well-developed algorithms (Ghaoui *et al.*, 1997; Leibfritz, 2001) effectively.

The following theorem provides a new criterion for testing the robust H_∞ performance level of the uncertain Markovian jump system (1) over the admissible uncertainty domains (2) and (3) in terms of coupled linear matrix inequalities.

Theorem 4. For a prescribed scalar $\gamma_{H_\infty} \in \mathbb{R}^+$, uncertain Markovian jump system (1) with $u(t) \equiv 0$ is robustly mean square stable and satisfies $\|G_{zw}\|_\infty < \gamma_{H_\infty}$ over the uncertainty domains (2) and (3) if there exist matrices $P_i \in \mathbb{S}^{n\times n}$, $T_{ij} \in \mathbb{S}^{n\times n}$, and $\lambda_i \in \mathbb{R}^+$, $i, j \in \mathcal{S}, j \neq i$, such that the coupled linear matrix inequality

$$\begin{bmatrix} Q_{1i} & * & * & * \\ M_{1i}^T & Q_{2i} & * & * \\ E_{0i}^T P_i + E_{1i}^T C_i & E_{1i}^T D_{2i} & -\lambda_i I + E_{1i}^T E_{1i} & * \\ M_{2i}^T & 0 & 0 & -\Lambda_i \end{bmatrix} < 0 \tag{6}$$

holds for all $i \in \mathcal{S}$, where

$$\begin{aligned} Q_{1i} &= A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j + \sum_{j=1,j\neq i}^{s} \varepsilon_{ij}^2 T_{ij} \\ &\quad + C_i^T C_i + \lambda_i H_{0i}^T H_{0i} \\ Q_{2i} &= -\lambda_{H_\infty}^2 I + D_{2i}^T D_{2i} + \lambda_i H_{2i}^T H_{2i} \\ M_{1i} &= P_i B_{2i} + C_i^T D_{2i} + \lambda_i H_{0i}^T H_{2i} \\ M_{2i} &= \begin{bmatrix} P_i - P_1 & \cdots & P_i - P_{i-1} & P_i - P_{i+1} & \cdots & P_i - P_s \end{bmatrix} \\ \Lambda_i &= \mathrm{diag}(T_{i1}, \ldots, T_{i(i-1)}, T_{i(i+1)}, \ldots, T_{is}) \end{aligned}$$

PROOF. According to Proposition 3, for a prescribed scalar $\gamma_{H_\infty} \in \mathbb{R}^+$, the uncertain Markovian jump system (1) with $u(t) \equiv 0$ is robustly mean square stable and has H_∞ performance $\|G_{zw}\|_{H_\infty} < \gamma_{H_\infty}$ if there exist matrices $P_i \in \mathbb{S}^{n\times n}, i \in \mathcal{S}$, such that

$$\Xi_i \triangleq \begin{bmatrix} \hat{A}_i^T P_i + P_i \hat{A}_i + \sum_{j=1}^{s} \hat{\pi}_{ij} P_j & P_i \hat{B}_{2i} & \hat{C}_i^T \\ \hat{B}_{2i}^T P_i & -\gamma_{H_\infty}^2 I & \hat{D}_{2i}^T \\ \hat{C}_i & \hat{D}_{2i} & -I \end{bmatrix} < 0$$

for all $i \in \mathcal{S}$. Notice that the structures of the uncertainty domains (2) and (3), we have

$$\Xi_i = \begin{bmatrix} A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j & P_i B_{2i} & C_i^T \\ B_{2i}^T P_i & -\gamma_{H_\infty}^2 I & D_{2i}^T \\ C_i & D_{2i} & -I \end{bmatrix} + \begin{bmatrix} \sum_{j=1,j\neq i}^{s} [\Delta\pi_{ij}(P_j - P_i)] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} P_iE_{0i} \\ 0 \\ E_{1i} \end{bmatrix} F_i \begin{bmatrix} H_{0i}^T \\ H_{2i}^T \\ 0 \end{bmatrix}^T + \begin{bmatrix} H_{0i}^T \\ H_{2i}^T \\ 0 \end{bmatrix} F_i^T \begin{bmatrix} P_iE_{0i} \\ 0 \\ E_{1i} \end{bmatrix}^T$$

$$\leq \begin{bmatrix} A_i^TP_i + P_iA_i + \sum_{j=1}^{s} \pi_{ij}P_j & P_iB_{2i} & C_i^T \\ B_{2i}^TP_i & -\gamma_{H_\infty}^2 I & D_{2i}^T \\ C_i & D_{2i} & -I \end{bmatrix}$$

$$+ \begin{bmatrix} \sum_{j=1,j\neq i}^{s} [\varepsilon_{ij}^2 T_{ij} + (P_i - P_j)T_{ij}^{-1}(P_i - P_j)] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+\frac{1}{\lambda_i} \begin{bmatrix} P_iE_{0i} \\ 0 \\ E_{1i} \end{bmatrix} \begin{bmatrix} P_iE_{0i} \\ 0 \\ E_{1i} \end{bmatrix}^T + \lambda_i \begin{bmatrix} H_{0i}^T \\ H_{2i}^T \\ 0 \end{bmatrix} \begin{bmatrix} H_{0i}^T \\ H_{2i}^T \\ 0 \end{bmatrix}^T$$

holds for any $\lambda_i \in \mathbb{R}^+$, $T_{ij} \in \mathbb{S}^{n\times n}, (i, j \in \mathcal{S}, j \neq i)$. In view of Schur complement equivalence, $\Xi_i < 0$ holds if the inequality

$$\begin{bmatrix} Q'_{1i} + \lambda_i H_{0i}^T H_{0i} & P_iB_{2i} + \lambda_i H_{0i}^T H_{2i} & C_i^T & P_iE_{0i} \\ B_{2i}^TP_i + \lambda_i H_{2i}^T H_{0i} & -\lambda_{H_\infty}^2 I + \lambda_i H_{2i}^T H_{2i} & D_{2i}^T & 0 \\ C_i & D_{2i} & -I & E_{1i} \\ E_{0i}^TP_i & 0 & E_{1i}^T & -\lambda_i I \end{bmatrix} < 0$$

holds, where

$$Q'_{1i} = A_i^TP_i + P_iA_i + \sum_{j=1}^{s} \pi_{ij}P_j + \sum_{j=1,j\neq i}^{s} [\varepsilon_{ij}^2 T_{ij} + (P_i - P_j)T_{ij}^{-1}(P_i - P_j)]$$

Pre- and post-multiply both sides of the above inequality by

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}$$

we have

$$\begin{bmatrix} Q'_{1i} + \lambda_i H_{0i}^T H_{0i} & P_iB_{2i} + \lambda_i H_{0i}^T H_{2i} & P_iE_{0i} & C_i^T \\ B_{2i}^TP_i + \lambda_i H_{2i}^T H_{0i} & -\lambda_{H_\infty}^2 I + \lambda_i H_{2i}^T H_{2i} & 0 & D_{2i}^T \\ E_{0i}^TP_i & 0 & -\lambda_i I & E_{1i}^T \\ C_i & D_{2i} & E_{1i} & -I \end{bmatrix} < 0$$

The above inequality is equivalent to (6) based on Schur complement equivalence again. This completes the proof. □

In the following, we provide a solution to the *robust H_∞ control problem* (RH_∞P) for the uncertain system (1) with uncertain switching probabilities in terms of coupled linear matrix inequalities with equality constraints.

Theorem 5. Consider uncertain Markovian jump system (1), for a prescribed scalar $\gamma_{H_\infty} \in \mathbb{R}^+$, there exists a state-feedback controller (4) such that the closed-loop system (5) is robustly mean square stable and has H_∞ performance $\|G_{zw}\|_\infty < \gamma_{H_\infty}$ over the uncertainty domains (2) and (3) if there exist matrices $P_i \in \mathbb{S}^{n\times n}$, $X_i \in \mathbb{S}^{n\times n}$, $V_i \in \mathbb{S}^{n\times n}$, $Z_i \in \mathbb{S}^{n\times n}$, $Y_i \in \mathbb{R}^{m\times n}$, $T_{ij} \in \mathbb{S}^{n\times n}$, and $\alpha_i \in \mathbb{R}^+$, $i, j \in \mathcal{S}$, $j \neq i$, such that the coupled linear matrix inequalities

$$\begin{bmatrix} Q_{3i} & * & * & * \\ M_{3i}^T & Q_{4i} & * & * \\ M_{4i}^T & H_{2i}D_{2i}^T & -\alpha_i I + H_{2i}H_{2i}^T & 0 \\ X_i & 0 & 0 & -Z_i \end{bmatrix} < 0 \quad (7)$$

$$\begin{bmatrix} Q_{5i} & M_{2i} \\ M_{2i}^T & -\Lambda_i \end{bmatrix} \leq 0 \quad (8)$$

with equality constraints

$$P_iX_i = I, \qquad V_iZ_i = I \quad (9)$$

hold for all $i \in \mathcal{S}$, where

$$\begin{aligned} Q_{3i} &= (A_iX_i + B_{1i}Y_i)^T + (A_iX_i + B_{1i}Y_i) \\ &\quad + B_{2i}B_{2i}^T + \alpha_i E_{0i}E_{0i}^T \\ Q_{4i} &= -\lambda_{H_\infty}^2 I + D_{2i}D_{2i}^T + \alpha_i E_{1i}E_{1i}^T \\ Q_{5i} &= -V_i + \sum_{j=1}^{s} \pi_{ij}P_j + \sum_{j=1,j\neq i}^{s} \varepsilon_{ij}^2 T_{ij} \\ M_{2i} &= \begin{bmatrix} P_i - P_1 & \cdots & P_i - P_{i-1} & P_i - P_{i+1} & \cdots & P_i - P_s \end{bmatrix} \\ M_{3i} &= (C_iX_i + D_{1i}Y_i)^T + B_{2i}D_{2i}^T + \alpha_i E_{0i}E_{1i}^T \\ M_{4i} &= (H_{0i}X_i + H_{1i}Y_i)^T + B_{2i}H_{2i}^T \\ \Lambda_i &= \mathrm{diag}(T_{i1}, \ldots, T_{i(i-1)}, T_{i(i+1)}, \ldots, T_{is}) \end{aligned}$$

In this case, a controller (4) is given by $K_i = Y_iP_i$, $i \in \mathcal{S}$.

PROOF. Consider the closed-loop system (5), let $\bar{A}_i = A_i + B_{1i}K_i$, $\bar{C}_i = C_i + D_{1i}K_i$ and $\bar{H}_{0i} = H_{0i} + H_{1i}K_i$. Similarly to the argument of Theorem 4, we have that the closed-loop system (5) is robustly mean square stable and has H_∞ performance $\|G_{zw}\|_{H_\infty} < \gamma_{H_\infty}$ if there exist matrices $P_i \in \mathbb{S}^{n\times n}$, $T_{ij} \in \mathbb{S}^{n\times n}$, and $\alpha_i \in \mathbb{R}^+$, $i, j \in \mathcal{S}, j \neq i$, such that

$$\begin{bmatrix} \bar{A}_i^TP_i + P_i\bar{A}_i + \sum_{j=1}^{s} \pi_{ij}P_j & P_iB_{2i} & \bar{C}_i^T \\ B_{2i}^TP_i & -I & D_{2i}^T \\ \bar{C}_i & D_{2i} & -\gamma_{H_\infty}^2 I \end{bmatrix}$$

$$+ \begin{bmatrix} \sum_{j=1,j\neq i}^{s} [\varepsilon_{ij}^2 T_{ij} + (P_i - P_j)T_{ij}^{-1}(P_i - P_j)] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+\alpha_i \begin{bmatrix} P_iE_{0i} \\ 0 \\ E_{1i} \end{bmatrix} \begin{bmatrix} P_iE_{0i} \\ 0 \\ E_{1i} \end{bmatrix}^T + \frac{1}{\alpha_i} \begin{bmatrix} \bar{H}_{0i}^T \\ H_{2i}^T \\ 0 \end{bmatrix} \begin{bmatrix} \bar{H}_{0i}^T \\ H_{2i}^T \\ 0 \end{bmatrix}^T < 0 \quad (10)$$

holds for all $i \in \mathcal{S}$. Now let $V_i \in \mathbb{S}^{n\times n}$ such that

$$\sum_{j=1}^{s} \pi_{ij}P_j + \sum_{j=1,j\neq i}^{s} [\varepsilon_{ij}^2 T_{ij} + (P_i - P_j)T_{ij}^{-1}(P_i - P_j)] \leq V_i$$

which is equivalent to (8) in view of Schur complement equivalence. Now, pre- and post-multiply both sides of (10) by $\mathrm{diag}(P_i^{-1}, I, I)$ and apply the changes of variables $X_i \triangleq P_i^{-1}$, $Z_i \triangleq V_i^{-1}$ and $Y_i \triangleq K_i X_i$, one obtains

$$\begin{bmatrix} Q'_{3i} & B_{2i} & (C_iX_i + D_{1i}Y_i)^T \\ B_{2i}^T & -I & D_{2i}^T \\ C_iX_i + D_{1i}Y_i & D_{2i} & -\lambda_{H_\infty}^2 I \end{bmatrix} + \alpha_i \begin{bmatrix} E_{0i} \\ 0 \\ E_{1i} \end{bmatrix} \begin{bmatrix} E_{0i} \\ 0 \\ E_{1i} \end{bmatrix}^T$$
$$+\frac{1}{\alpha_i} \begin{bmatrix} (H_{0i}X_i + H_{1i}Y_i)^T \\ H_{2i}^T \\ 0 \end{bmatrix} \begin{bmatrix} (H_{0i}X_i + H_{1i}Y_i)^T \\ H_{2i}^T \\ 0 \end{bmatrix}^T < 0$$

for all $i \in \mathcal{S}$, where

$Q'_{3i} = (A_iX_i+B_{1i}Y_i)^T+(A_iX_i+B_{1i}Y_i)+X_iZ_i^{-1}X_i$

The above inequality is equivalent to

$$\begin{bmatrix} Q'_{3i} + \alpha_i E_{0i}E_{0i}^T & * & * & * \\ B_{2i}^T & -I & * & * \\ C_iX_i + D_{1i}Y_i + \alpha_i E_{1i}E_{0i}^T & D_{2i} & -\lambda_{H_\infty}^2 I + \alpha_i E_{1i}E_{1i}^T & * \\ H_{0i}X_i + H_{1i}Y_i & H_{2i} & 0 & -\alpha_i I \end{bmatrix} < 0$$

which is also equivalent to

$$\begin{bmatrix} Q'_{3i} + \alpha_i E_{0i}E_{0i}^T & * & * & * \\ C_iX_i + D_{1i}Y_i + \alpha_i E_{1i}E_{0i}^T & -\lambda_{H_\infty}^2 I + \alpha_i E_{1i}E_{1i}^T & * & * \\ H_{0i}X_i + H_{1i}Y_i & 0 & -\alpha_i I & * \\ B_{2i}^T & D_{2i}^T & H_{2i}^T & -I \end{bmatrix} < 0$$

The above inequality is equivalent to (7) in view of Schur complement equivalence. This completes the proof. □

To solve the set of the coupled linear matrix inequalities (7), (8) with equality constraints (9) more effectively, we modify (7) with a sufficiently small number $\beta \in \mathbb{R}^+$ to

$$\begin{bmatrix} Q_{3i} + \beta I & * & * & * \\ M_{3i}^T & Q_{4i} & * & * \\ M_{4i}^T & H_{2i}D_{2i}^T & -\alpha_i I + H_{2i}H_{2i}^T & 0 \\ X_i & 0 & 0 & -Z_i \end{bmatrix} \leq 0 \qquad (11)$$

and weaken equality constraints (9) to the semi-definite programming conditions

$$\begin{bmatrix} P_i & I \\ I & X_i \end{bmatrix} \geq 0, \qquad \begin{bmatrix} V_i & I \\ I & Z_i \end{bmatrix} \geq 0 \qquad (12)$$

Now, we are ready to employ the sequential linear programming method (Leibfritz, 2001) to solve the robust H_∞ control problem. The solution of RH_∞P is summarized below.

Algorithm 1. (RH_∞P). For a given precision $\delta \in \mathbb{R}^+$, let N be the maximum number of iterations and a sufficiently small number $\beta \in \mathbb{R}^+$ be given.

(1) Determine P_i^0, X_i^0, V_i^0, Z_i^0, Y_i^0, T_{ij}^0 and α_i^0, $i, j \in \mathcal{S}, j \neq i$ satisfying (8), (11) and (12), let $k := 0$.

(2) Solve the convex optimization problem for the variables P_i, X_i, V_i, Z_i, Y_i, T_{ij}, α_i, $i, j \in \mathcal{S}, j \neq i$.
$$\min \sum_{i=1}^{s} \text{trace}(P_iX_i^k + P_i^kX_i + V_iZ_i^k + V_i^kZ_i)$$
subject to (8), (11) and (12) for all $i \in \mathcal{S}$.

(3) Let $T_i^k := P_i$, $L_i^k := X_i$, $U_i^k := V_i$, $R_i^k := Z_i$ for all $i \in \mathcal{S}$.

(4) If
$$\left| \sum_{i=1}^{s} \text{trace}(T_i^kX_i^k + P_i^kL_i^k + U_i^kZ_i^k + V_i^kR_i^k) - 2\sum_{i=1}^{s} \text{trace}(P_i^kX_i^k + V_i^kZ_i^k) \right| < \delta$$
then go to step (7), else go to step (5).

(5) Compute $\theta^* \in [0,1]$ by solving
$$\min_{\theta \in [0,1]} g(\theta)$$
where
$$g(\theta) \triangleq \text{trace}([P_i^k + \theta(T_i^k - P_i^k)][X_i^k + \theta(L_i^k - X_i^k)] + [V_i^k + \theta(U_i^k - V_i^k)][Z_i^k + \theta(R_i^k - Z_i^k)])$$

(6) Let
$$P_i^{k+1} := P_i^k + \theta^*(T_i^k - P_i^k)$$
$$X_i^{k+1} := X_i^k + \theta^*(L_i^k - X_i^k)$$
$$V_i^{k+1} := V_i^k + \theta^*(U_i^k - V_i^k)$$
$$Z_i^{k+1} := Z_i^k + \theta^*(R_i^k - Z_i^k)$$
for all $i \in \mathcal{S}$, and $k := k+1$. If $k < N$, then go to step (2), else go to step (7).

(7) Stop. If $\sum_{i=1}^{s} \text{trace}(P_i^kX_i^k + V_i^kZ_i^k) = 2sn$, then a solution is found successfully, else a solution cannot be found.

Remark 6. As explained in (Leibfritz, 2001), Algorithm RH_∞P always generates a strictly decreasing sequence of values of the objective function $f(k) \triangleq \sum_{i=1}^{s} \text{trace}(P_i^kX_i^k + V_i^kZ_i^k)$. That is,
$$\sum_{i=1}^{s} \text{trace}(P_i^{k+1}X_i^{k+1} + V_i^{k+1}Z_i^{k+1}) < \sum_{i=1}^{s} \text{trace}(P_i^kX_i^k + V_i^kZ_i^k), \quad k = 1, 2, \ldots$$
Thus, $\{f(k)\}$ always converges to some $f^* \geq 2sn$, and if $f^* = 2sn$, the solution obtained from Algorithm RH_∞P is a solution of Theorem 5.

4. NUMERICAL EXAMPLE

In this section, we present a simple numerical example to illustrate the usefulness of the proposed theory. Attention is mainly focused on the design of a robust H_∞ state-feedback controller such that the closed-loop system is robustly mean square stable and has the desired H_∞ performance level over all admissible uncertainties in the mode transition rate matrix. It is assumed that the system under consideration has two operation modes and has uncertainties only in the mode transition rate matrix. The system data of (1) are as follows.
In mode 1:
$$A_1 = \begin{bmatrix} 0.5 & -1 \\ 0 & -1 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.5 & 0 \\ 0.1 & 0.5 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad D_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In mode 2:

$$A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \qquad D_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad D_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The measured mode transition rate matrix and the error bounds are

$$\Pi = \begin{bmatrix} -2 & 2 \\ 10 & -10 \end{bmatrix}, \qquad \varepsilon_{12} = 0.5, \qquad \varepsilon_{21} = 3$$

The nominal system of the uncertain system above is not mean square stable. Suppose that a controller (4) is desired such that the closed-loop system is robustly mean square stable and has guaranteed H_∞ performance level $\gamma_{H_\infty} \triangleq 0.5$ over the admissible uncertainties $\Delta\pi_{12} \in [-1, 1]$ and $\Delta\pi_{21} \in [-6, 6]$.

To compute with Algorithm RH_∞P for this example, it is chosen that $\delta = 10^{-10}$, $N = 100$ and $\beta = 0.01$. One set of solutions is

$$P_1 = \begin{bmatrix} 3.2955 & -2.1522 \\ -2.1522 & 3.0661 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.3038 & -2.2124 \\ -2.2124 & 3.4371 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0.5603 & 0.3933 \\ 0.3933 & 0.6022 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 0.5320 & 0.3425 \\ 0.3425 & 0.5114 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 0.9564 & -0.1518 \\ -0.1518 & 1.1139 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.0287 & -0.0124 \\ -0.0124 & 0.0188 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} 1.0687 & 0.1456 \\ 0.1456 & 0.9176 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} 48.9253 & 32.4144 \\ 32.4144 & 74.7296 \end{bmatrix}$$

$$T_{12} = \begin{bmatrix} 1.8799 & -0.0626 \\ -0.0626 & 0.7438 \end{bmatrix}, \quad T_{21} = \begin{bmatrix} 0.0062 & -0.0342 \\ -0.0342 & 0.2072 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} -0.5027 & -0.0904 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -0.5913 & -0.6436 \end{bmatrix}$$

It can be verified that $\|P_1X_1 - I\|_2 = 5.5375 \times 10^{-12}$, $\|P_2X_2 - I\|_2 = 5.5428 \times 10^{-12}$, $\|V_1Z_1 - I\|_2 = 5.5427 \times 10^{-12}$ and $\|V_2Z_2 - I\|_2 = 5.5407 \times 10^{-12}$. Therefore, the equality constraints in (9) are satisfied. Finally, the state-feedback controller can be obtained from the above solution as

$$K_1 = \begin{bmatrix} -1.4619 & 0.8046 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.5295 & -0.9040 \end{bmatrix}$$

5. CONCLUSION

This paper has proposed a new robust H_∞ control method for a class of Markovian jump linear systems with uncertain switching probabilities. Attention was focused on the design of a robust state-feedback controller such that the closed-loop system is robustly mean square stable and guarantees the desired H_∞ performance over all admissible uncertainties both in the system matrices and in the switching probabilities. An algorithm involving convex optimization was also suggested to construct such controllers effectively. Finally, a numerical example illustrated the developed theory.

REFERENCES

Boukas, E. K., P. Shi and K. Benjelloun (1999). On stabilization of uncertain linear systems with jump parameters. *International Journal of Control* **72**(9), 842–850.

Cao, Y.-Y. and J. Lam (2000). Robust H_∞ control of uncertain Markovian jump systems with time-delay. *IEEE Transactions on Automatic Control* **45**(1), 77–83.

Costa, O. L. V., J. B. R. Val and J. C. Geromel (1999). Continuous-time state-feedback H_2-control of Markovian jump linear system via convex analysis. *Automatica* **35**, 259–268.

de Farias, D. P., J. C. Geromel, J. B. R. do Val and O. L. V. Costa (2000). Output feedback control of Markov jump linear systems in continuous-time. *IEEE Transactions on Automatic Control* **45**(5), 944–949.

do Val, J. B. R., J. C. Geromel and A. P. C. Goncalves (2002). The H_2-control for jump linear systems: cluster observations of the Markov state. *Automatica* **38**, 343–349.

Feng, X., K. A. Loparo, Y. Ji and H. J. Chizeck (1992). Stochastic stability properties of jump linear systems. *IEEE Transactions on Automatic Control* **37**(1), 38–53.

Ghaoui, L. El and M. A. Rami (1996). Robust state-feedback stabilization of jump linear systems via LMIs. *International Journal of Robust and Nonlinear Control* **6**(9-10), 1015–1022.

Ghaoui, L. El, F. Oustry and M. A. Rami (1997). A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control* **42**(8), 1171–1176.

Ji, Y. and H. J. Chizeck (1990). Controllability, stabilizability, and continuous-time Markovian jump linear quadratic control. *IEEE Transactions on Automatic Control* **35**(7), 777–788.

Leibfritz, F. (2001). An LMI-based algorithm for designing suboptimal static H_2/H_∞ output feedback controllers. *SIAM Journal on Control and Optimization* **39**(6), 1711–1735.

Mahmoud, M. S. and P. Shi (2003). Robust stability, stabilization and H_∞ control of time-delay systems with Markovian jump parameters. *International Journal of Robust and Nonlinear Control* **13**, 755–784.

Mariton, M. (1990). *Jump Linear Systems in Automatic Control.* Marcel Dekker. New York.

Xiong, J., J. Lam, H. Gao and D. W.C. Ho (2005). On robust stabilization of Markovian jump systems with uncertain switching probabilities. *Automatica.* to be appear.

Zhang, L., B. Huang and J. Lam (2003). H_∞ model reduction of Markovian jump linear systems. *Systems & Control Letters* **50**, 103–118.