GENERALIZATION OF CAYLEY-HAMILTON THEOREM FOR N-D POLYNOMIAL MATRICES

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Abstract: The Cayley-Hamilton theorem is extended for real polynomial matrices in n variables. The known extensions of the classical Cayley-Hamilton theorem are particular cases of the proposed extension.

Keywords: Cayley-Hamilton theorem, generalization, n-D polynomial matrix, linear systems, control

1. INTRODUCTION

The classical Cayley-Hamilton theorem (Gantmacher, 1974; Lancaster, 1969) says that every square matrix satisfies its own characteristic equation. The Cayley-Hamilton theorem has been extended for rectangle matrices (Kaczorek, 1995c), block matrices (Kaczorek, 1995b; Victoria, 1982) pairs of matrices (Chang and Chen, 1992; Lewis, 1982; Livsic, 1983; Lewis, 1986; Mertzios and Christodoulous, 1986), pairs of block matrices (Kaczorek, 1998) and standard and singular two-dimensional linear systems (Kaczorek, 1992/93; Kaczorek, 1994; Kaczorek, 1995a; Smart and Barnett, 1989; Theodoru, 1989).
The Cayley-Hamilton theorem and its generalizations have been used in control systems, electrical circuits, systems with delays, singular systems, 2D linear systems, etc., (Kaczorek, 1992/93; Kaczorek, 1995c; Lancaster, 1969).
In this note the Cayley-Hamilton theorem will be extended for n-dimensional (n-D) real polynomial matrices. The known extensions of the classical Cayley-Hamilton theorem are particular cases of the proposed extension.

2. PRELIMINARIES

Let $R^{m\times n}[s_1, s_2, ..., s_n]$ be the set of $m \times n$ real polynomial matrices in n variables $s_1, s_2, ..., s_n$.
Consider an n-dimensional $(n - D)$ polynomial matrix of the form

$$\begin{aligned} &A(s_1, s_2, \ldots, s_n) = \\ &= \sum_{i_1=0}^{q_1} \sum_{i_2=0}^{q_2} \cdots \sum_{i_n=0}^{q_n} A_{i_1 i_2 \ldots i_n} s_1^{i_1} s_2^{i_2} \ldots s_n^{i_n} \quad (1) \\ &\in R^{m\times m}[s_1, s_2, \ldots, s_n] \end{aligned}$$

where $A_{i_1 i_2 \ldots i_n} \in R^{m\times n}$ (the set of $m \times n$ real matrices) and q_k, $k = 1, ..., n$ are nonnegative integers.
It is assumed that the matrix (1) is invertible, i.e. the n-D polynomial

$$\begin{aligned} &\det A(s_1, s_2, ..., s_n) = \\ &= \sum_{j_i=0}^{N_1} \sum_{j_2=0}^{N_2} \cdots \sum_{j_n=0}^{N_n} a_{j_1 j_2 \cdots j_n} s_1^{j_1} s_2^{j_2} \cdots s_n^{j_n} \quad (2) \end{aligned}$$

is nonidentically vanishing.
Let

$$A^{-1}(s_1, s_2, ..., s_n) = \tag{3}$$
$$= \sum_{i_i=-\mu_1}^{\infty} \sum_{i_2=-\mu_2}^{\infty} \cdots \sum_{i_n=-\mu_n}^{\infty} \Phi_{i_1 i_2 \cdots i_n} s_1^{-i_1} s_2^{-i_2} \cdots s_n^{-i_n}$$

where $\Phi_{i_1 i_2 \cdots i_n} \in R^{m \times m}$ and $(\mu_1, \mu_2, ..., \mu_n)$ is the n-D index of the matrix (1).
If the coefficient $a_{N_1 N_2 ... N_n} \neq 0$ then all $\mu_k, k = 1, ..., n$ are finite otherwise some of them may be infinite.

Lemma *The matrices $\Phi_{i_1 i_2 \cdots i_n}$ satisfy the equalities*

$$\sum_{i_i=0}^{q_1} \sum_{i_2=0}^{q_2} \cdots \sum_{i_n=0}^{q_n} A_{i_1 i_2 \cdots i_n} \Phi_{i_1+k_1, i_2+k_2, \cdots, i_n+k_n} =$$
$$\sum_{i_i=0}^{q_1} \sum_{i_2=0}^{q_2} \cdots \sum_{i_n=0}^{q_n} \Phi_{i_1+k_1, i_2+k_2, \cdots, i_n+k_n} A_{i_1 i_2 \cdots i_n} \tag{4}$$
$$= \begin{cases} I_m \ \ for\ k_1 = k_2 = \cdots k_n = 0 \\ 0 \ \ for\ k_1 \neq 0\ or/and\ k_2 \neq 0 \\ \quad or/and \ldots k_n \neq 0 \end{cases}$$

Proof. From definition of the inverse matrix and (1) and (3) we have

$$A(s_1, s_2, ..., s_n) A^{-1}(s_1, s_2, ..., s_n) =$$
$$= A^{-1}(s_1, s_2, ..., s_n) A(s_1, s_2, ..., s_n) =$$
$$\left(\sum_{i_i=0}^{q_1} \sum_{i_2=0}^{q_2} \cdots \sum_{i_n=0}^{q_n} A_{i_1 i_2 \cdots i_n} s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n} \right) \times$$
$$\left(\sum_{i_i=-\mu_1}^{\infty} \sum_{i_2=-\mu_2}^{\infty} \cdots \sum_{i_n=-\mu_n}^{\infty} \Phi_{i_1 i_2 \cdots i_n} s_1^{-i_1} s_2^{-i_2} \cdots s_n^{-i_n} \right) = \tag{5}$$
$$\left(\sum_{i_i=-\mu_1}^{\infty} \sum_{i_2=-\mu_2}^{\infty} \cdots \sum_{i_n=-\mu_n}^{\infty} \Phi_{i_1 i_2 \cdots i_n} s_1^{-i_1} s_2^{-i_2} \cdots s_n^{-i_n} \right) \times$$
$$\left(\sum_{i_i=0}^{q_1} \sum_{i_2=0}^{q_2} \cdots \sum_{i_n=0}^{q_n} A_{i_1 i_2 \cdots i_n} s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n} \right) = I_n$$

Comparison of the matrix coefficients of the same powers of $s_k^{i_k}$ for $i_k = 0, \pm1, \pm2, ...; k = 1, ..., n$ in (5) yields (4). □

3. MAIN RESULT

Theorem. *Let the n-D polynomial matrix (1) be invertible and let (3) hold. Then*

$$\sum_{i_i=0}^{N_1} \sum_{i_2=0}^{N_2} \cdots \sum_{i_n=0}^{N_n} a_{i_1 i_2 \cdots i_n} \Phi_{i_1+k_1, i_2+k_2, \cdots, i_n+k_n} = 0 \tag{6}$$
$$for\ k_j = 1, 2, ...\ \ (j = 1, ..., n)$$

where $a_{i_1 i_2 ... i_n}$ are the real coefficients of the n-D polynomial (2) and N_k $(k = 1, ..., n)$ are nonnegative integers.

Proof. Taking into account that $Adj A(s_1, s_2, ..., s_n) = A^{-1}(s_1, s_2, ..., s_n) \det A(s_1, s_2, ..., s_n)$ ($Adj A$ denotes the adjoint matrix) and using (2) and (3) we may write

$$Adj(s_1, s_2, ..., s_n) =$$
$$= \sum_{k_1=0}^{M_1} \sum_{k_2=0}^{M_2} ... \sum_{k_n=0}^{M_n} K_{k_1 k_2 \cdots k_n} s_1^{k_1} s_2^{k_2} ... s_n^{k_n} =$$
$$\left(\sum_{i_i=-\mu_1}^{\infty} \sum_{i_2=-\mu_2}^{\infty} ... \sum_{i_n=-\mu_n}^{\infty} \Phi_{i_1 i_2 ... i_n} s_1^{-i_1} s_2^{-i_2} ... s_n^{-i_n} \right) \times \tag{7}$$
$$\left(\sum_{j_i=0}^{N_1} \sum_{j_2=0}^{N_2} ... \sum_{j_n=0}^{N_n} a_{j_1 j_2 ... j_n} s_1^{j_1} s_2^{j_2} ... s_n^{j_n} \right)$$

where $M_k (k = 1, ...n)$ are nonnegative integers.
Comparison of the matrix coefficients at the same powers of s_k^k for $i_k = -1, -2, ...; k = 1, ..., n$ in (7) yields (6).□

Remark. If instead of (3) we use the expansion

$$A^{-1}(s_1, s_2, ..., s_n) =$$
$$\sum_{i_i=-\mu_1}^{\infty} \sum_{i_2=-\mu_2}^{\infty} ... \sum_{i_2=-\mu_n}^{\infty} \bar{\Phi}_{i_1 i_2 ... i_n} \times \tag{8}$$
$$s_1^{-(i_1+1)} s_2^{-(i_2+1)} ... s_n^{-(i_n+1)}$$

then the equalities (6) take the form

$$\sum_{i_i=0}^{N_1} \sum_{i_2=0}^{N_2} ... \sum_{i_n=0}^{N_n} a_{i_1 i_2 ... i_n} \Phi_{i_1+k_1, i_2+k_2, ..., i_n+k_n} = 0 \tag{9}$$
$$for\ k_j = 0, 1, 2, ...\ \ (j = 1, ..., n)$$

Note that the well-known extensions of the classical Cayley-Hamilton theorem (Chang and Chen, 1992; Kaczorek, 1994; Kaczorek, 1995a; Lewis, 1982; Livsic, 1983; Lewis, 1986; Mertzios and Christodoulous, 1986; Smart and Barnett, 1989) are particular cases of the proposed extension.
For $A(s) = [Is - A]$ the proposed theorem is equivalent to the classical Cayley-Hamilton theorem for the matrix A. If $A(s) = [Es - A]$ $(E, A \in R^{n \times n})$ we obtain the generalization of the theorem for singular systems and for

$$A(z_1, z_2) = \begin{bmatrix} I_{n_1} z_1 - A_1 & -A_2 \\ -A_3 & I_{n_2} z_2 - A_4 \end{bmatrix}$$
$$A_1 \in R^{n_1 \times n_1} \quad A_4 \in R^{n_2 \times n_2}$$

we obtain the generalization of the theorem for 2D linear systems described by the Roesser model

(Kaczorek, 1992/93).

Examples. Three examples of n-D invertible polynomial matrices for $n = 1, 2, 3$ will be considered.

The inverse matrix of the 1-D polynomial matrix

$$A(s) = \begin{bmatrix} 1+s & -s \\ s^2 & 2s \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \tag{10}$$

$$= A_2 s^2 + A_1 s + A_0$$

has the form

$$A^{-1}(s) =$$
$$= \Bigg[\begin{matrix} 2s^{-2} - 4s^{-3} + 4s^{-4} - 8s^{-6} + ... \\ -2s^{-1} + 4s^{-2} - 4s^{-3} + 8s^{-5} - 16s^{-6} + ... \end{matrix}$$

$$\begin{matrix} s^{-2} - 2s^{-3} + 2s^{-4} - 4s^{-6} + ... \\ s^{-2} - s^{-3} + 2s^{-5} - 4s^{-6} + ... \end{matrix} \Bigg] =$$

$$= \Phi_1 s^{-1} + \Phi_2 s^{-2} + \Phi_3 s^{-3} + \Phi_4 s^{-4} +$$
$$+ \Phi_5 s^{-5} + \Phi_6 s^{-6} + ...$$

where

$$\begin{aligned} \Phi_1 &= \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}, \; \Phi_2 = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}, \\ \Phi_3 &= \begin{bmatrix} -4 & -2 \\ -4 & -1 \end{bmatrix}, \Phi_4 = \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}, \\ \Phi_5 &= \begin{bmatrix} 0 & 0 \\ 8 & 2 \end{bmatrix}, \Phi_6 = \begin{bmatrix} -8 & -4 \\ -16 & -4 \end{bmatrix} \end{aligned} \tag{11}$$

Taking into account that $\det A(s) = s^3 + 2s^2 + 2s$ $(a_0 = 0,\ a_1 = a_2 = 2,\ a_3 = 1)$ and using (6) and (11) we obtain:
for $k_1 = 1$

$$a_0\Phi_1 + a_1\Phi_2 + a_2\Phi_3 + a_3\Phi_4 =$$
$$= 2\begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} + 2\begin{bmatrix} -4 & -2 \\ -4 & -1 \end{bmatrix} +$$
$$+ \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for $k_1 = 2$

$$a_0\Phi_2 + a_1\Phi_3 + a_2\Phi_4 + a_3\Phi_5 =$$
$$= 2\begin{bmatrix} -4 & -2 \\ -4 & -1 \end{bmatrix} + 2\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} +$$
$$+ \begin{bmatrix} 0 & 0 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and for $k_1 = 3$

$$a_0\Phi_3 + a_1\Phi_4 + a_2\Phi_5 + a_3\Phi_6 =$$
$$= 2\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 8 & 2 \end{bmatrix} +$$
$$+ \begin{bmatrix} -8 & -4 \\ -16 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The inverse matrix of the 2-D polynomial matrix

$$A(s_1, s_2) = \begin{bmatrix} 1 + s_1 s_2 & s_2 \\ -s_2 & s_1 s_2 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s_1 s_2 + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} s_2 + + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \tag{12}$$
$$= A_{11} s_1 s_2 + A_{01} s_2 + A_{00}$$

has the form

$$A^{-1}(s_1, s_2) =$$

$$= \Bigg[\begin{matrix} s_1^{-1}s_2^{-1} - s_1^{-2}s_2^{-2} + s_1^{-3}s_2^{-3} - s_1^{-4}s_2^{-4} + ... \\ s_1^{-2}s_2^{-1} - s_1^{-3}s_2^{-2} - s_1^{-4}s_2^{-1} - s_1^{-5}s_2^{-2} + ... \end{matrix}$$

$$\begin{matrix} -s_1^{-2}s_2^{-1} + s_1^{-3}s_2^{-2} + s_1^{-4}s_2^{-1} - s_1^{-5}s_2^{-2} + ... \\ s_1^{-1}s_2^{-1} + s_1^{-3}s_2^{-3} - s_1^{-2}s_2^{-2} - s_1^{-4}s_2^{-4} - s_1^{-4}s_2^{-4} + \end{matrix}$$

$$\begin{matrix} ... \\ -s_1^{-4}s_2^{-2} + s_1^{-5}s_2^{-2} + s_1^{-6}s_2^{-2} + ... \end{matrix} \Bigg] =$$

$$= \Phi_{11}s_1^{-1}s_2^{-1} + \Phi_{12}s_1^{-1}s_2^{-2} + \Phi_{21}s_1^{-2}s_2^{-1} +$$
$$+ \Phi_{22}s_1^{-2}s_2^{-2} + \Phi_{13}s_1^{-1}s_2^{-3} + \Phi_{32}s_1^{-3}s_2^{-2} +$$
$$+ \Phi_{33}s_1^{-3}s_2^{-3} + \Phi_{24}s_1^{-2}s_2^{-4} + \Phi_{44}s_1^{-4}s_2^{-4} + \ldots$$

where

$$\begin{aligned} \Phi_{11} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Phi_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \Phi_{21} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \Phi_{22} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \Phi_{13} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \Phi_{32} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\ \Phi_{33} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Phi_{24} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \Phi_{44} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{13}$$

Taking into account that

$$\det A(s_1, s_2) = s_1^2 s_2^2 + s_1 s_2 + s_2^2$$
$$(a_{22} = a_{11} = a_{02} = 1)$$

and using (6) and (13) we obtain for $k_1 = k_2 = 1$

$$a_{02}\Phi_{13} + a_{11}\Phi_{22} + a_{22}\Phi_{33} =$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and for $k_1 = k_2 = 2$

$$a_{02}\Phi_{24} + a_{11}\Phi_{33} + a_{22}\Phi_{44} =$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The inverse matrix of 3-D polynomial matrix

$$A^{-1}(s_1,s_2,s_3)=\begin{bmatrix}1+s_1s_2s_3\ ,\ s_1s_2s_3\\ s_1s_2s_3\ ,\ s_1s_2s_3\end{bmatrix}= \\ =\begin{bmatrix}1\ 1\\1\ 1\end{bmatrix}s_1s_2s_3+\begin{bmatrix}1\ 0\\0\ 0\end{bmatrix}=A_{111}s_1s_2s_3+A_{000} \tag{14}$$

has the form

$$A^{-1}(s_1,s_2,s_3)=\begin{bmatrix}1 & -1\\ -1 & 1+s_1^{-1}s_2^{-1}s_3^{-1}\end{bmatrix}= \\ =\Phi_{111}s_1^{-1}s_2^{-1}s_3^{-1}+\Phi_{000}$$

where

$$\Phi_{111}=\begin{bmatrix}0\ 0\\0\ 1\end{bmatrix},\ \Phi_{000}=\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \tag{15}$$

Taking into account that

$$\det A(s_1,s_2,s_3)=s_1s_2s_3 \\ \left(a_{ijk}=\begin{cases}1 & for\ i=j=k=1\\ 0 & otherwise\end{cases}\right)$$

and using (6) and (15) we obtain

$$a_{ijk}\Phi_{i+k_1,j+k_2,k+k_3}=0$$

$$\text{for}\quad \begin{matrix}i,j,k\in\{0,1,...\}\\ k_1,k_2,k_3\in\{0,1,...\}\end{matrix}$$

$$\text{and}\ \ k_1+k_2+k_3\geq 1$$

4. APPLICATION

Application of the Theorem will be illustrated by the following two examples.

Consider the discrete-time linear system with h ($h\geq 1$)delays described by the equation

$$x_{i+1}=A_0x_i+A_1x_{i-1}+A_2x_{i-2}+... \\ ...+A_hx_{i-h}+Bu_i \tag{16}$$

where $x_i\in R^n$ is the state vector, $u_i\in R^m$ is the input vector and $A_k\in R^{n\times n}$, $k=0,1,...,h$, $B\in R^{n\times m}$.

The characteristic equation of (16) can be written in the form

$$det[Iz-A_0-A_1z^{-1}-A_2z^{-2}-...-A_hz^{-h}]= \\ =z^h det[Iz^{h+1}-A_0z^h-A_1z^{h-1}-...-A_h]= \\ =z^{-h}(z^N-a_{N-1}z^{N-1}-a_{N-2}z^{N-2}+... \\ ...-a_1z-a_0) \\ N=n(h+1) \tag{17}$$

The matrix

$$A(z)=[Iz^{h+1}-A_0z^h-A_1z^{h-1}-...-A_h]$$

is invertible.

Let

$$A^{-1}(z)=\sum_{i=0}^{\infty}\Phi_iz^{-i} \tag{18}$$

From Theorem and (17), (18) we obtain

$$\sum_{i=0}^{N}a_1\Phi_{i+k}\quad for\ k=0,1,2,... \tag{19}$$

Example. Consider the 2D linear system described by the equation

$$x_{i+1,j+1}=A_0x_{ij}+A_1x_{i+1,j}+A_2x_{i,j+1}+Bu_{ij} \\ i,j\in Z_+=\{0,1,2,...\} \tag{20}$$

where $x_{ij}\in R^n$ is the state vector, $u_{ij}\in R^m$ is the input vector and $A_k\in R^{n\times n}$, $k=0,1,2$, $B\in R^{n\times m}$.

The characteristic equation of (20) has the form

$$det[Iz_1z_2-A_0-A_1z_1-A_2z_2]= \\ =\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}z_1^iz_2^j \tag{21}$$

The matrix

$$A(z_1,z_2)=[Iz_1z_2-A_0-A_1z_1-A_2z_2]$$

is invertible. Let

$$A^{-1}(z_1,z_2)=\sum_{i=0}^{\infty}\sum_{j=0}^{n}\Phi_{ij}z_1^{-i}z_2^{-j} \tag{22}$$

From Theorem and (21), (22) we obtain

$$\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}\Phi_{i+k_1,j+k_2}=0 \\ for\ k_1,k_2=0,1,2,.. \tag{23}$$

5. CONCLUSION

The classical Cayley-Hamilton theorem has been extended for real polynomial matrices in n variables. It has been shown that the known extensions of the Cayley-Hamilton theorem are articular cases of the proposed extension. Application of the proposed extension has been illustrated by examples.

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