STABILITY ANALYSIS AND ANTI-WINDUP DESIGN FOR DISCRETE-TIME SYSTEMS BY A SATURATION-DEPENDENT LYAPUNOV FUNCTION APPROACH

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Abstract: This paper deals with the stability analysis and anti-windup design of discrete-time systems subject to actuator saturation. We present a new saturation-dependent Lyapunov function to estimate the domain of attraction, which is then formulated and solved as a constrained LMI optimization problem. Further we propose an anti-windup compensation method to enlarge the domain of attraction in the presence of saturation. Numerical examples are presented to show the effectiveness of the proposed method.

Keywords: Set invariance, Saturation-dependent Lyapunov function, Anti-windup

1. INTRODUCTION

Due to the practical significance and the theoretical challenges, problems for systems subject to saturation have attracted tremendous attention in recent years (Kapoor *et al.*, 1998; Blanchini, 1999; Zaccarian and Teel, 2002; Hu *et al.*, 2002). Generally, the global stability of the systems subject to actuator saturation can be achieved only when the open-loop system is not strictly unstable. With respect to the discrete-time cases, poles must be inside or on the unite circle of the complex plane for the system to be stable (Sussmann *et al.*, 1994; Lin and Saberi, 1995). Therefore there are numerous reports on the local stability and semi-global stability analysis (da Silva Jr. and Tarbouriech, 2001; Cao and Lin, 2003).

One of the main aspects concerning the local stability analysis is the analytical characterization of the domain of attraction. Many methods have been adopted to maximize the invariant set, which has been believed to be very hard except for some special cases (Hu *et al.*, 2001; Blanchini, 1999). The reduction of conservatism and the enlarging of invariant set inside the domain of attraction are the two hot topics, which have drawn much attention. Generally speaking, the existing methods on estimating the stability regions for linear systems with saturating actuators are mainly based on the concept of Lyapunov function level set. One of the most frequently used Lyapunov functions is quadratic Lyapunov function (Hu *et al.*, 2002).

A new sufficient condition for an ellipsoid to be invariant was presented in (Hu *et al.*, 2002) for discrete-time systems subject to actuator saturation, which is less conservative than the traditional circle criterion. The resulting estimate of domain of attraction is a level set of a quadratic Lyapunov function. Other than quadratic Lyapunov function, many other Lyapunov functions, such as piecewise-affine Lyapunov functions, were also adopted to cope with the problem for example (Milani, 2001). More recently, in (Cao and Lin, 2003), the quadratic Lyapunov function approach was extended and a saturation-dependent Lyapunov function was developed to reduce the conservatism in the estimation of domain of attraction. A drawback of the approach is that the estimate of the domain of attraction in (Cao and Lin, 2003) is just an intersection of a set of ellipsoids, which leads to conservatism of this method. In (Hu and Lin, 2003), a new Lyapunov function as so-called composite quadratic Lyapunov function was developed to cope with the stability problem for linear continuous-time systems.

This paper aims at the less conservative saturation-dependent Lyapunov function method to enlarge the domain of attraction for the discrete-time systems subject to actuator saturation. Our goal is to further reduce the conservatism in the estimation of the domain of attraction through a new saturation-dependent Lyapunov function. We also present an anti-windup compensation method to further enlarge the domain of attraction with the iterative approach proposed in (Cao *et al.*, 2002).

2. PRELIMINARIES

Consider the system with actuator saturation.

$$x(k+1) = Ax(k) + B\sigma(u(k)). \tag{1}$$

where $x \in \mathcal{R}^n$ denotes the state vector, $u \in \mathcal{R}^m$ the input vector and A, B are real-valued matrices. The function σ is the standard vector-valued saturation function $\sigma(u) = \left[\, \sigma(u_1) \ \ldots \ \sigma(u_m) \,\right]^T$, where $\sigma(u_i) = \text{sign}(u_i)\min\{1, |u_i|\}$.

Consider the following linear state feedback law

$$u(k) = Fx(k). \tag{2}$$

We want to know how the closed-loop system behaves in the presence of saturation, in particular, to what extent the stability is preserved. In the first step, we aim at obtaining an estimate of the domain of attraction of the origin of the closed-loop system.

$$x(k+1) = Ax(k) + B\sigma(Fx(k)). \tag{3}$$

Let f_i be the i-th row of the matrix F. We define the symmetric polyhedron $\mathcal{L}(F) = \{x \in \mathcal{R}^n : |f_i x| \leq 1, i = 1, 2, \ldots, m\}$.

For $x(0) = x_0 \in \mathcal{R}^n$, denote the state trajectory of the system (3) as $\psi(k, x_0)$ at time k. Then the domain of attraction of the origin is $S := \{x_0 \in \mathcal{R}^n : \lim_{k\to\infty}\psi(k, x_0) = 0\}$. A set is said to be invariant if all the trajectories starting from it will remain in it (Blanchini, 1999).

Let $P \in \mathcal{R}^{n\times n}$ be a positive-definite matrix. For a number $\rho > 0$, an ellipsoid $\Omega(P, \rho)$ is defined as $\Omega(P, \rho) = \left\{x \in \mathcal{R}^n : x^T P x \leq \rho\right\}$.

Let $\mathcal{V}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathcal{V}$. Suppose that each element of $\mathcal{V}$ is labeled as $E_i, i = 1, 2, \ldots, 2^m$, and denotes $E_i^- = I - E_i$. Clearly, $E_i^- \in \mathcal{V}$ if $E_i \in \mathcal{V}$.

Lemma 1. (Hu et al.*, 2002)* Let $F, H \in \mathcal{R}^{m\times n}$ be given. For an $x \in \mathcal{R}^n$, if $x \in \mathcal{L}(H)$, then

$$\sigma(Fx) \in \text{co}\left\{E_i Fx + E_i^- Hx : i \in [1, 2^m]\right\},$$

where $co\{\cdot\}$ denotes the convex hull of a set. Consequently, $\sigma(Fx)$ can be expressed as

$$\sigma(Fx) = \sum_{i=1}^{2^m} \eta_i (E_i F + E_i^- H)x, \tag{4}$$

where $\eta_i(k)$ is a parameter related to the severity of saturation and $\sum_{i=1}^{2^m} \eta_i(k) = 1, 0 \leq \eta_i(k) \leq 1$.

Note that one of the main advantages of the above lemma is that $\sigma(Fx(k))$ can be represented as a convex hull of a group of linear feedback, which will be seen in the following sections.

3. A SATURATION-DEPENDENT LYAPUNOV FUNCTION

In this section, we will introduce a new saturation-dependent Lyapunov function to estimate the domain of attraction for the saturated system (3) by the invariant set approach.

To clearly present the problem, we denote $\hat{A}_i = A + B(E_i F + E_i^- H)$, where $H \in \mathcal{R}^{m\times n}$ satisfies $||Hx||_\infty \leq 1$. Following Lemma 1, we can rewrite the closed-loop system as follows

$$x(k+1) = \hat{A}(\eta(k))x(k), \quad \forall x \in \mathcal{L}(H), \tag{5}$$

where $\hat{A}(\eta(k)) := \sum_{i=1}^{2^m} \eta_i(k)\hat{A}_i$

$$= \sum_{i=1}^{2^m} \eta_i(k)(A + B(E_i F + E_i^- H)) \tag{6}$$

and $\eta(k) = \left[\, \eta_1(k) \ \eta_2(k) \ \ldots \ \eta_{2^m}(k) \,\right]$ is a time-varying parameter dependent on $x(k)$ and $0 \leq \eta_i(k) \leq 1$, $\sum_{i=1}^{2^m} \eta_i(k) = 1$. It is easy to see that parameter $\eta(k)$ depends on amplitude of the saturation (Cao and Lin, 2003).

With $P > 0$, a quadratic Lyapunov function can be defined as $V(x(k)) = x^T P x$. For a $\rho > 0$, a level set of $V(\cdot)$, denoted $L_V(\rho)$, is

$$L_V(\rho) := \{x \in \mathcal{R}^n : V(x(k)) \leq \rho\} = \Omega(P, \rho).$$

It is noticed that the unknown but measurable time-varying parameters $\eta(k)$ can provide real-time information on the variations of the saturation. To reduce the conservatism in analyzing the stability of the saturated system (3), it is desirable to use this information. Based on this idea, (Cao and Lin, 2003) introduced the following saturation-dependent Lyapunov function

$$V(x(k)) = x^T(k)P(\eta(k))x(k), \tag{7}$$

where $P(\eta(k)) = \sum_{i=1}^{2^m} \eta_i(k)P_i$, $P_i > 0$. Then the estimation of the domain of attraction is obtained by the Lyapunov level set $\Omega_\eta(P(\eta), \rho) = \left\{x \in \mathcal{R}^n : x^T P(\eta)x \leq \rho\right\}$. However, it is easy to see that the region $\Omega_\eta(P(\eta), \rho)$ is the intersection of the ellipsoids $\Omega(P_i, \rho)$, *i.e.*

$$\Omega_\eta(P(\eta), \rho) = \cap_{i=1}^{2^m} \Omega(P_i, \rho).$$

Because of the characteristic of the intersection of sets, it is easy to see that the above method may still be conservative. Now we introduce a new saturation-dependent Lyapunov function for the discrete-time system (3) similar to the composite Lyapunov function described in (Hu and Lin, 2003). Let $Q_i = (P_i/\rho)^{-1}, i \in [1, 2^m]$. We change the definition of $P(\eta(k))$ in (7) as follows

$$P(\eta(x(k))) := \rho Q(\eta(x(k)))^{-1}, \tag{8}$$

$$Q(\eta(x(k))) := \sum_{i=1}^{2^m} \eta(x(k)) Q_i, \tag{9}$$

If we set $Q_i = Q_1$ for all i, $V(x(k))$ will become the common quadratic Lyapunov function. This new saturation-dependent Lyapunov function has a very desirable property, which will be presented in what follows. In (Hu and Lin, 2003), the authors proposed a composite Lyapunov function

$$V_c(x) = \min_{\gamma \in \Gamma} x^T P(\gamma) x, \tag{10}$$

where $P(\gamma) = Q^{-1}(\gamma)$, $Q(\gamma) = \sum_{i=1}^{N} \gamma_i Q_i$ and $\Gamma = \left\{ \gamma \in \mathcal{R}^N : \sum_{i=1}^{N} \gamma_i = 1, 0 \le \gamma_i \le 1 \right\}$. For this Lyapunov function, the level set $L_V(\rho) = \mathrm{co}\{\Omega(P_i, \rho), i \in [1, N]\} = \cup_{\gamma \in \Gamma} \Omega(P(\gamma), \rho)$. Obviously, our saturation-dependent Lyapunov function (7) is different to the composite Lyapunov function (10) although it uses the same structure, see (8) and (9). In what follows, we will show that we can use $\cup_{i=1}^{2^m} \Omega(P_i, \rho)$ to estimate $L_V(\rho)$, while we can only use $\cap_{i=1}^{2^m} \Omega(P_i, \rho)$ to estimate $L_V(\rho)$ in (Cao and Lin, 2003).

The closed-loop system (3) is asymptotically stable at the origin with the level set $L_V(\rho)$ contained in the domain of attraction if $L_V(\rho) \subset \mathcal{L}(H)$ and

$$\triangle V(x(k)) = x^T(k) \left[(\sum_{i=1}^{2^m} \eta_i(k) \hat{A}_i^T) P(\eta(k+1)) \right.$$
$$\left. (\sum_{i=1}^{2^m} \eta_i(k) \hat{A}_i) - P(\eta(k)) \right] x(k) < 0$$

for any $x(k) \in L_V(\rho) \setminus \{0\}$. In what follows, a condition under which $\triangle V(x(k)) < 0$ holds will be given for the general $P(\eta(k))$.

Theorem 2. Consider the closed-loop system (3). If there exist matrices $X, H, Q_i > 0$, $i = 1, 2, \ldots, 2^m$, such that

$$\begin{bmatrix} X + X^T - Q_i & X^T \hat{A}_i^T \\ \hat{A}_i X & Q_j \end{bmatrix} > 0, \ i, j \in [1, 2^m] \tag{11}$$

and $L_V(\rho) \subset \mathcal{L}(H)$ with $P_i = \rho Q_i^{-1}$, then the closed-loop system (3) is asymptotically stable at the origin with the level set $L_V(\rho)$ contained in the domain of attraction.

Proof. As mentioned above, for any $x \in L_V(\rho) \subset \mathcal{L}(H)$, $\triangle V(x(k)) < 0$ for any $x(k) \in L_V(\rho) \setminus \{0\}$, if $\triangle V(x(k)) = \hat{A}^T(\eta(k)) P(\eta(k+1)) \hat{A}(\eta(k)) - P(\eta(k)) < 0$. Based on the Schur complement, it is equivalent to

$$\begin{bmatrix} P(\eta(k)) & \hat{A}^T(\eta(x(k))) \\ \hat{A}(x(k)) & P^{-1}(\eta(k+1)) \end{bmatrix} > 0,$$

which is equivalent to

$$\begin{bmatrix} X^T Q^{-1}(\eta(k)) X & X^T \hat{A}^T(\eta(x(k))) \\ \hat{A}(x(k)) X & Q(\eta(k+1)) \end{bmatrix} > 0 \tag{12}$$

for any matrix $X \in \mathcal{R}^{n \times n}$. Note that $(X - Q(\eta))^T Q^{-1}(\eta)(X - Q(\eta)) \ge 0$, we have

$$X^T Q^{-1}(\eta(k)) X \ge X + X^T - Q(\eta),$$

which implies that (12) holds if

$$\begin{bmatrix} X + X^T - Q(\eta) & X^T \hat{A}^T(\eta(x(k))) \\ \hat{A}(x(k)) X & P^{-1}(\eta(k+1)) \end{bmatrix} > 0$$

The left side can be rewritten as

$$\begin{bmatrix} X + X^T - \sum_{i=1}^{2^m} \eta_i(k) Q_i & * \\ \left(\sum_{i=1}^{2^m} \eta_i(k) A_i \right) X & \sum_{j=1}^{2^m} \eta_j(k+1) Q_j \end{bmatrix}$$
$$= \sum_{i=1}^{2^m} \eta_i(k) \sum_{j=1}^{2^m} \eta_j(k+1) \begin{bmatrix} X + X^T - Q_i & X^T \hat{A}_i^T \\ \hat{A}_i X & Q_j \end{bmatrix}$$

where $*$ represents blocks that are readily inferred by symmetry. Hence, we have $\triangle V(x(k)) < 0$, $\forall x(k) \in L_V(\rho) \setminus \{0\}$, if (11) holds for all $i, j \in [1, 2^m]$. And then we can conclude system (3) is asymptotically stable at the origin with $L_V(\rho)$ contained in the domain of attraction. ∎

Note that Theorem 2 holds for any $\eta \in \Gamma$. It is easy to see that Theorem 2 is irrelevant with η. Moreover, if we set $Q_i = Q = X$, $\forall i \in [1, 2^m]$, inequality (11) becomes

$$\begin{bmatrix} Q & Q \hat{A}_i^T \\ \hat{A}_i Q & Q \end{bmatrix} > 0. \tag{13}$$

Let $P = Q^{-1}$, inequality (13) is equivalent to

$$\begin{bmatrix} P & \hat{A}_i^T P \\ P \hat{A}_i & P \end{bmatrix} > 0, \quad \forall j \in [1, 2^m].$$

Thus we recover Theorem 1 of (Hu *et al.*, 2002). It is easy to see that when the condition of Thereom 2 holds, we then have

$$\begin{bmatrix} X^T Q_i^{-1} X & X^T \hat{A}_i^T \\ \hat{A}_i X & Q_j \end{bmatrix} > 0 \Rightarrow \begin{bmatrix} Q_i^{-1} & \hat{A}_i^T \\ \hat{A}_i & Q_j \end{bmatrix} > 0,$$
$$\Rightarrow \hat{A}_i^T P_j \hat{A}_i - P_i < 0, \quad \forall i, j \in [1, 2^m].$$

We have $\hat{A}_i^T P_i \hat{A}_i - P_i < 0, \quad \forall i \in [1, 2^m]$.

If we constrain $\Omega(P_i,\rho) \subset \mathcal{L}(H)$, then $\Omega(P_i,\rho)$ is an invariant set of the closed-loop system (3) (Hu *et al.*, 2002). So we have the following corollary.

Corollary 3. Consider the system (3) under a given state feedback control matrix F. If there exist matrices $X \in \mathcal{R}^{n\times n}, H \in \mathcal{R}^{m\times n}, Q_i \in \mathcal{R}^{n\times n}$, and $Q_i > 0, i = 1,2,\ldots,2^m$, such that

$$\begin{bmatrix} X + X^T - Q_i & X^T\hat{A}_i^T \\ \hat{A}_i X & Q_j \end{bmatrix} > 0, \quad \forall i,j \in [1,2^m],$$

and $\cup_{i=1}^{2^m}\Omega(P_i,\rho) \subset \mathcal{L}(H)$ with $P_i = \rho Q_i$, then (3) is asymptotically stable at the origin with $\cup_{i=1}^{2^m}\Omega(P_i,\rho)$ contained in the domain of attraction.

4. ESTIMATION OF DOMAIN OF ATTRACTION

Among all the level sets that satisfy the conditions of Theorem 2, it is natural to choose the largest one to obtain the least conservative estimate of the domain of attraction. Generally, a shape reference set, such as a polyhedron or ellipsoid, is adopted to measure the size of the domain of attraction (Hu *et al.*, 2002). Let $\mathcal{X}_R \subset \mathcal{R}^n$ be a prescribed bounded convex set containing the origin. For a set $\mathcal{L} \subset \mathcal{R}^n$ which contains the origin, define $\alpha\mathcal{X}_R(\mathcal{L}) := \sup\{\alpha > 0 : \alpha\mathcal{X}_R \subset \mathcal{L}\}$. We choose $\mathcal{X}_R$ to be a polyhedron defined as

$$\mathcal{X}_R = \mathrm{co}\{x_1, x_2, \ldots, x_l\}. \tag{14}$$

Another frequently used reference set is ellipsoid $\mathcal{X}_R = \{x \in \mathcal{R}^n : x^TRx \le 1, \quad R > 0\}$.

Theorem 2 gives a condition for the level set $L_V(\rho)$ to be inside the domain of attraction. With the above shape reference sets, we can choose from all the $L_V(\rho)$'s that satisfy the condition of Theorem 2 such that the quantity $\alpha\mathcal{X}_R$ is maximized. This problem can be formulated as the following constrained optimization problem:

$$\begin{aligned} &\min_{Q_i>0,X,H} \alpha, \\ \text{s.t. } &(a)\ \ \alpha\mathcal{X}_R \subset L_V(\rho), \\ &(b)\ \ \text{inequalities (11)}, \\ &(c)\ \ |h_i x| \le 1 \quad \forall x \in L_V(\rho). \end{aligned} \tag{15}$$

where h_i denotes the ith row of H.

By Corollary 3, we can substitute $L_V(\rho)$ with $\cup_{i=1}^{2^m}\Omega(P_i,\rho)$. Problem (15) can be reduced to the following constrained optimization problem.

$$\max_{Q_i>0,H,X} \alpha \quad \text{s.t.} \tag{16}$$

(a) $\alpha\mathcal{X}_R \subset \Omega(P_i,\rho), \quad \forall i \in [1,2^m]$

(b) inequalities (11),

(c) $|h_i x| \le 1, \forall x \in \Omega(P_j,\rho), \forall j \in [1,2^m], i \in [1,m]$

Optimization problem (16) can be reduced to an LMI optimization problem. Without loss of generality, we will let $\rho = 1$ in what follows. First note that Constraint (a) of (15) is equivalent to

$$\alpha^2 x_j^T(P(\eta))x_j \le 1 \Leftrightarrow \begin{bmatrix} \alpha^{-2} & x_j^T \\ x_j & Q_i \end{bmatrix} \ge 0$$

Condition (b) is equivalent to

$$\begin{bmatrix} X + X^T - Q_i & * \\ AX + B(E_iFX + E_i^-HX) & Q_j \end{bmatrix} > 0,$$

for $\forall i,j \in [1,2^m]$, i.e.

$$\begin{bmatrix} X + X^T - Q_i & * \\ AX + B(E_iFX + E_i^-Z) & Q_j \end{bmatrix} > 0,\ \forall i,j \in [1,2^m],$$

where $Z = HX$. Let $z_j = h_jX$. Condition (c) is equivalent to

$$\begin{aligned} h_jP^{-1}(\eta)h_j^T \le 1 \Leftrightarrow &\begin{bmatrix} 1 & h_j \\ h_j^T & Q_i^{-1} \end{bmatrix} \ge 0, \\ \Leftarrow &\begin{bmatrix} 1 & z_j \\ z_j^T & X + X^T - Q_i \end{bmatrix} \ge 0, \end{aligned} \tag{17}$$

Based on the description above, the problem of enlarging the domain of attraction can be reduced to an LMI optimization problem defined as follows.

$$\min_{Q_i>0,X,Z} \gamma, \quad \text{s.t.} \tag{18}$$

(a) $\begin{bmatrix} \gamma & x_j^T \\ x_j & Q_i \end{bmatrix} \ge 0, \quad \forall j \in [1,l], i \in [1,2^m],$

(b) $\begin{bmatrix} X + X^T - Q_i & * \\ AX + B(E_iFX + E_i^-HX) & Q_j \end{bmatrix} > 0,$ $\forall i,j \in [1,2^m]$

(c) $\begin{bmatrix} 1 & z_j \\ z_j^T & X + X^T - Q_i \end{bmatrix} \ge 0, j \in [1,m], i \in [1,2^m]$

where $\gamma = \alpha^{-2}$.

It is clear that constraints (b) and (c) ensure that the level set $L_V(\rho)$ is contained in the domain of attraction. Using the LMI tool to solve this optimization problem, we can obtain a set of P_i. By Corollary 3, the obtained estimation is the union of this set of ellipsoids. This property can generally reduce the conservatism in the estimation of the domain of attraction.

5. DESIGN OF ANTI-WINDUP COMPENSATION GAIN

Consider the system subject to input saturation

$$x(k+1) = Ax(k) + B\sigma(u(k)), \tag{19}$$
$$y(k) = Cx(k). \tag{20}$$

We assume a dynamic compensator of the form

$$x_c(k+1) = A_c x_c(k) + B_c y(k), \quad x_c(0) = 0,$$
$$u(k) = C_c x_c(k) + D_c y(k),$$

where $x_c(k) \in \mathcal{R}^{n_c}$, has been designed. This compensator is designed to stabilize (19) and to meet the required performance specifications in the absence of actuator saturation .

A typical anti-windup compensator involves adding a correction term of the form $E_c(\sigma(u(k)) - u(k))$ with the modified compensator

$$x_c(k+1) = A_c x_c(k) + B_c y(k) + E_c(\sigma(u(k)) - u(k))$$
$$u(k) = C_c x_c(k) + D_c y(k),$$

with $x_c(0) = 0$. Under the compensated controller, the closed-loop system can be written as

$$\tilde{x}_c(k+1) = \tilde{A}_c \tilde{x}_c(k) + \tilde{B}_c(\sigma(u(k)) - u(k)) \quad (21)$$
$$u(k) = F\tilde{x}(k), \quad (22)$$

where

$$\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_C \end{bmatrix},$$
$$\tilde{B} = \begin{bmatrix} B \\ E_c \end{bmatrix}, \quad F = \begin{bmatrix} D_cC & C_c \end{bmatrix}.$$

With the feedback control law (22), the system (21) can be rewritten as

$$\tilde{x}(k+1) = (\tilde{A} - \tilde{B}F)\tilde{x}(k) + \tilde{B}\sigma(F\tilde{x}(k))). \quad (23)$$

For the closed-loop system (23), we can directly apply the method presented in Section 4 to obtain an estimation of domain of attraction. Substitute A, and B in (18) with $\tilde{A} - \tilde{B}F$, and $\tilde{B}$, we obtain

$$\min_{Q_i>0,X,Z} \gamma, \quad \text{s.t.} \quad (24)$$
$$(a) \begin{bmatrix} \gamma & x_j^T \\ x_j & Q_i \end{bmatrix} \geq 0, \quad \forall j \in [1, l], i \in [1, 2^m],$$
$$(b) \begin{bmatrix} X + X^T - Q_i & * \\ \tilde{A}X - \tilde{B}FX + \tilde{B}(E_iFX + E_i^- Z) & Q_j \end{bmatrix} > 0, \quad \forall i, j \in [1, 2^m]$$
$$(c) \begin{bmatrix} 1 & z_j \\ z_j^T & X + X^T - Q_i \end{bmatrix} \geq 0, j \in [1, m], i \in [1, 2^m]$$

where $\gamma = \alpha^{-2}$.

In (24), if E_c is prefixed, then it is an LMI optimization problem to estimate the domain of attraction of the closed-loop system in the presence of actuator saturation. As introduced in (Cao *et al.*, 2002), we can use E_c as a free design parameter to enlarge the domain of attraction and hence improve the stability of the closed-loop system in the presence of actuator saturation. Similar to the method used in (Cao *et al.*, 2002), we will present an iterative design approach to obtain the anti-windup compensation gain such that the domain of the attraction may be as large as possible.

Note that constraint (b) in optimization problem (24) is not linear in E_c, Q_i, and Z simultaneously. This implies we may not solve the optimal anti-windup compensation gain by directly solving an constrained LMI optimization problem. In what follows, an iterative algorithm will be presented to compute the anti-windup compensation gain.

Denote $\tilde{B}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\tilde{B}_2 = \begin{bmatrix} 0 \\ E_c \end{bmatrix}$, Note that, if H and X are fixed, constraints (b) in (24) can be rewritten to

$$\begin{bmatrix} X^T + X - Q_i & * \\ (\tilde{A} + \tilde{B}_1 M_i)X + (\tilde{B}_2 M_i)X & Q_j \end{bmatrix}, \quad (25)$$

where $M_i = -F + E_iF + E_i^- H$. Obviously with fixed H and X, (25) is linear in E_c and Q_i. Hence, we can formulate a constrained optimization problem to design E_c to make the domain of attraction as large as possible.

$$\min_{Q_i>0,\tilde{B}_2} \gamma, \text{ s.t.} \quad (26)$$
$$(a) \begin{bmatrix} \gamma & x_j^T \\ x_j & Q_i \end{bmatrix} \geq 0, \quad \forall j \in [1, l], i \in [1, 2^m],$$
$$(b, c) \text{ inequality (25), and (17).}$$

Based on the above derivation, we present an iterative LMI approach to design E_c such that the closed-loop system has a domain of attraction as large as possible.

Iterative Algorithm for Determination of Anti-windup Compensation Gain E_c:

Step 1) Given reference set $\mathcal{X}_R$ and $E_c = 0$, solve the optimization problem (24). Denote the solution as γ_0, X_0, Q_i, and Z_0. Set $\mathcal{X}_R = \gamma_0^{-1/2}\mathcal{X}_R$.

Step 2) Set E_c with an initial value. Also set $i = 1$ and $\gamma_{\text{opt}} = 1$.

Step 3) Solve the optimization problem (24) for γ, Q_i, X, and Z. Denote the solution as γ_i, X, Q_i, and Z, respectively.

Step 4) Let $\gamma_{\text{opt}} = \gamma_i\gamma_{\text{opt}}$, $\mathcal{X}_R = \gamma_i^{-1/2}\mathcal{X}_R, H = ZX^{-1}$.

Step 5) If $|\gamma_{\text{opt}} - \gamma_i| < \sigma$, a pre-determined tolerance, stop; Else goto Step 6).

Step 6) Solve the optimization problem (26) for E_c with the X and H determined in Steps 3) and 4). Set $i = i + 1$ and goto Step 3).

6. NUMERICAL EXAMPLES

Example 1. First, we will present an example to illustrate the effectiveness of our new saturation-dependent Lyapunov function in estimation of

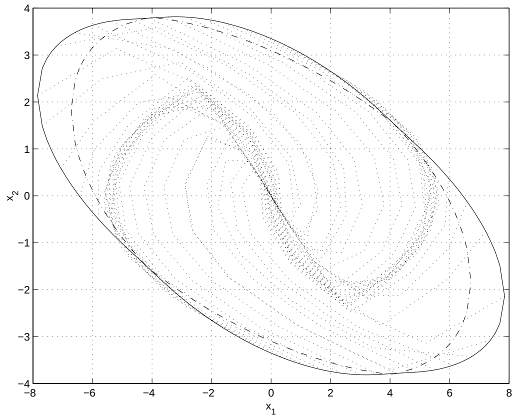

Fig. 1. Domain of attraction at $\theta = 0.4\pi$: approach of this paper – solid region; approach of (Cao and Lin, 2003) – dotted region

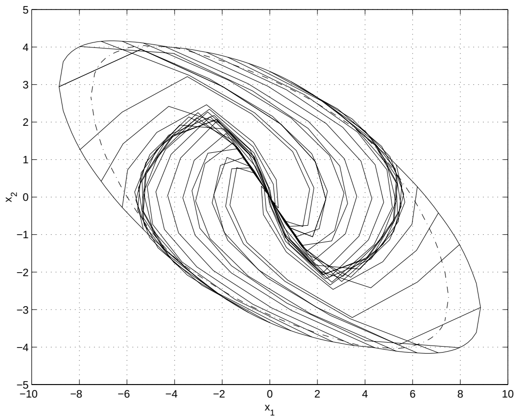

Fig. 2. Domain of attraction varying $\theta \in [0,\ 2\pi]$: approach of this paper – solid region; approach of (Cao and Lin, 2003) – dotted region

domain of attraction. Considering the following closed-loop system with (Cao and Lin, 2003)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix}.$$

We design the state feedback control law by the LQR approach with $Q = I$ and $R = 0.1$. For the above system, we obtain the following controller, $F = \begin{bmatrix} -0.6167 & -1.2703 \end{bmatrix}$. As in (Cao and Lin, 2003), we use the shape reference set of the form $\mathcal{X}_R = \left\{ \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} \right\}$, $\theta \in [0, 2\pi]$.

For this example, when $\theta = 0.4\pi$, we have $\alpha_c = 4.5235$, which is same as that obtained in (Cao and Lin, 2003). But we obtain a region which is the union of a set of ellipsoids, which is larger than that obtained in (Cao and Lin, 2003). Fig. 1 shows these estimates. The dotted curves to the origin are trajectories starting from the bound of the estimate. Fig. 2 shows the region obtained by varying θ from 0 to 2π, i.e., $\eta \in \Gamma$. The out curve is the borderline of the union of the ellipsoids. Obviously, the region obtained by the approach proposed in this paper is much larger than that obtained in (Cao and Lin, 2003).

7. CONCLUSIONS

We considered linear discrete-time systems subject to actuator saturation. A new saturation-dependent Lyapunov function was presented to reduce the conservatism when estimating the domain of attraction. An iterative algorithm was formulated to design the anti-windup compensation gain to enlarge the domain of attraction.

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