ON CHATTERING-FREE OUTPUT FEEDBACK SLIDING MODE DESIGN FOR MIMO LINEAR SYSTEM

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Abstract: Output feedback sliding mode control has received great research interests, due to it does not require full accessibility of states and increases the ability of practical implementation of sliding mode control. In this paper, a new approach of chattering-free OFSMC design is proposed by incorporating integral sign function control for general multi-input multi-output linear uncertain systems. Compared with the boundary layer type chattering-free designs, the complete robustness to system matched disturbances or uncertainties has not to be sacrificed and the global attractiveness of the sliding surface can be achieved through the controller synthesis.

Key words: chattering, output feedback, sliding-mode control, variable-structure control, linear multivariable systems

1. INTRODUCTION

Sliding mode control (SMC) is a well-known robust control due to its complete insensitivity to the so called matched disturbances or uncertainties. However, the requirement of full accessibility of states either from direct measurement or from observer estimation has brought limitations to practical implementation. To increase its practical usage, SMC designs based on output information were investigated in the last decade and were often termed as output feedback sliding mode control (OFSMC). Various designs were proposed to deal with such problem, for example, a constrained state feedback design (Heck, *et al.*, 1995) was proposed but the effect of system invariant zeros was not specifically addressed (Edwards and Spurgeon, 2000). Some designs based on system input-output relationship (Jiang, *et al.*, 1997; Cunha, *et al.*, 2003) were also proposed, but their internal stability was not discussed. A method (Edwards and Spurgeon, 1995) was proposed where the state dynamics were decomposed into a so called "output feedback canonical form". In this decomposition, it provides a clear structure for output feedback design and system invariant zeros can be proved to be imbedded in the output feedback dynamics. This method provides a useful analysis tool for general OFSMC designs and is also adopted in this research.

The robustness of SMC is provided by a variable structure control used to suppress the effect of system disturbances or uncertainties. In the conventional SMC design, a sign function control acts the variable structure portion. However, a phenomenon called chattering will be generated due to the discontinuity nature of the sign function. It can be considered as the unmodeled high frequency dynamics excited by the discontinuous control action. Hence, chattering becomes a major drawback for SMC and many researches were intended to eliminate this phenomenon. Some techniques such as boundary layer (Kachroo and Tomizuka, 1996), and fuzzy SMC (Lo and Kuo, 1998) etc. were proposed to smooth the sign function. However, in these methods, the complete robustness has to be compromised due to the smoothed sign function control. Chen and Xu (1999) proposed a chattering-free output feedback design based on the output relative degree dynamics. In this method, an integral sign function control was introduced instead of conventional usage. Chattering is eliminated based on the fact that a discontinuous signal will become continuous after the integration action and a similar method (Pan *et al.*, 2000) was also found. These two methods only concentrates on stabilizing the output dynamics and their extensions to multi-input multi-output (MIMO) systems were not fully discussed.

It can be shown that the relative degree based design has the problem of internal stability and its extension to MIMO systems is not straightforward. In this paper, a new approach of OFSMC design incorporating integral sign function control for general MIMO linear uncertain systems of any relative degree is proposed. It will be shown that the properties of chattering-free and complete robustness to matched uncertainty are preserved in this approach. In addition, the problem of "sliding patch" (Edwards, *et al.*, 2001) due to the incompleteness of state information is also overcome in the controller synthesis.

The contents of this paper are organized into following sections: The design based on output relative degree dynamics is briefly introduced and discussed in section 2.1. The problem for MIMO linear uncertain system of any relative degree is formulated in section 2.2. The proposed OFSMC controller synthesis is presented in section 3. A numerical design example and its simulation results are presented in section 4. Section 5 is the conclusions and discussions.

2. PROBLEM FORMULATION

In surveying the integral sign function SMC designs, it is noticed that the sliding surface $\mathbf{s}$ has to be relative degree zero to system input instead of relative degree one in traditional designs. Once the system is in sliding motion, the concept of equivalent control is derived from $\mathbf{s}=0$ instead of $\dot{\mathbf{s}}=0$. This characteristic is also true for the output feedback case. The OFSMC design based on system output relative degree dynamics is briefly introduced in the following section.

2.1. The Introduction of Output Dynamics Designs

Consider the method proposed in Pan, *et al.* (2000) which can be represented by the following SISO linear uncertain system of relative degree r subjected to matched uncertainty f_m.

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}+\mathbf{B}(u+f_m)\\ y &= \mathbf{C}\mathbf{x}\end{aligned} \tag{1}$$

where $\mathbf{x}\in\Re^n$, $\mathbf{u}\in\Re$, $\mathbf{y}\in\Re$, $0<r\le n$ and $f_m\in\Re$ represents the matched uncertainty. Then the r output derivatives can be written as

$$\begin{aligned} y &= \mathbf{C}\mathbf{x}\\ \dot{y} &= \mathbf{C}\mathbf{A}\mathbf{x}\\ &\vdots\\ y^{(r)} &= \mathbf{C}\mathbf{A}^r\mathbf{x}+\mathbf{C}\mathbf{A}^{r-1}\mathbf{B}(u+f_m)\end{aligned} \tag{2}$$

To incorporate the integral sign function control, the sliding surface is defined to be

$$s=c_1y+c_2\dot{y}+\cdots+c_ry^{(r-1)}+y^{(r)}\equiv\mathbf{F}\mathbf{Y} \tag{3}$$

where $c_i,(i=1,2,\cdots,r)$ are the scalar design parameters and $\mathbf{F}=[c_1\ \cdots\ c_r\ \ 1]$, $\mathbf{Y}=[y\ \cdots\ y^{(r-1)}\ \ y^{(r)}]^{\mathbf{T}}$. The design criterion is on the selection of F such that the following polynomial is stable

$$c_1+c_2\xi+\cdots+c_r\xi^{r-1}+\xi^r=0 \tag{4}$$

where ξ denotes the Laplace operator. The controller of following type

$$u=-\gamma\int_{t_0}^{t} sgn(s)dt \tag{5}$$

can drive the system into sliding mode if γ is large enough. Due to the nature of relative zero of the sliding surface, once system is in sliding motion, its dynamics can be derived from the concept of equivalent control at $s=0$ which can be computed and denoted as

$$u_{eq}=-(\mathbf{C}\mathbf{A}^{r-1}\mathbf{B})^{-1}\mathbf{F}\begin{bmatrix}\mathbf{C}\\ \vdots\\ \mathbf{C}\mathbf{A}^r\end{bmatrix}\mathbf{x}-f_m\equiv\mathbf{\Gamma}\mathbf{x}-f_m \tag{6}$$

The closed-loop system becomes a state feedback dynamics $\dot{\mathbf{x}}=(\mathbf{A}-\mathbf{B}\mathbf{\Gamma})\mathbf{x}$ where the matrix $\mathbf{A}-\mathbf{B}\mathbf{\Gamma}$ should be stable for internal stability. However, three important characteristics can be found in this design:

1. stable output dynamics does not imply stable internal dynamics.
2. open-loop eigenvalues can be affected by output feedback.
3. system invariant zero is imbedded in the output feedback dynamics.

These important characteristics were not specifically addressed in the methods based on output relative degree and its extension to general MIMO system will not be so straightforward. Hence, this paper is to investigate these problems for the MIMO system design.

2.2 Formulation of MIMO system

Consider a MIMO linear uncertain system with m inputs, p outputs and relative degree r which is denoted by a triple $(\mathbf{A},\mathbf{B},\mathbf{C})$ in the form of eq.(1) except $\mathbf{u}\in\Re^m$, $\mathbf{y}\in\Re^p$ and $\mathbf{f_m}\in\Re^m$. $\mathbf{f_m}$ is assumed to be time differentiable and its derivative is norm bounded. Define $\mathbf{C}_k=\mathbf{C}\mathbf{A}^k, k=0,1,2,...,r$, then the r derivatives of output in eq.(2) represented by $\mathbf{C}_k$ becomes

$$
\begin{aligned}
&\dot{\mathbf{y}} = \mathbf{C}_1\mathbf{x} \\
&\vdots \\
&\mathbf{y}^{(r-1)} = \mathbf{C}_{r-1}\mathbf{x} \\
&\mathbf{y}^{(r)} = \mathbf{C}_r\mathbf{x} + \mathbf{C}_{r-1}\mathbf{B}(\mathbf{u}+\mathbf{f_m})
\end{aligned} \tag{7}
$$

Consider the new system $(\mathbf{A},\mathbf{B},\mathbf{C}_{r-1})$ instead of original $(\mathbf{A},\mathbf{B},\mathbf{C})$ and assume it satisfies the following conditions:

1. $p \ge m$
2. $rank(\mathbf{C}_{r-1}\mathbf{B}) = \mathbf{m}$
3. $(\mathbf{A},\mathbf{B},\mathbf{C}_{r-1})$ is minimum phase

Based on the output canonical decomposition (Edwards and Spurgeon, 1995), then $(\mathbf{A},\mathbf{B},\mathbf{C}_{r-1})$ in the transformed coordinate can be written as

$$
\begin{bmatrix} \dot{\bar{\mathbf{x}}}_1 \\ \dot{\bar{\mathbf{x}}}_2 \\ \dot{\bar{\mathbf{x}}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \\ \bar{\mathbf{x}}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} (\mathbf{u}+\mathbf{f_m})
\qquad
\mathbf{y} = [\mathbf{0} \quad \mathbf{T}]\bar{\mathbf{x}} \equiv \mathbf{T}\begin{bmatrix} \bar{\mathbf{x}}_2 \\ \bar{\mathbf{x}}_3 \end{bmatrix} \tag{8}
$$

where $\bar{\mathbf{x}}_1 \in \Re^{n-p}$, $\bar{\mathbf{x}}_2 \in \Re^{p-m}$, $\bar{\mathbf{x}}_3 \in \Re^{m}$ and $\mathbf{A}_{ij},(i,j=1,2,3)$ are system submatrices partitioned accordingly to the dimensions of $\bar{\mathbf{x}}_1$, $\bar{\mathbf{x}}_2$ and $\bar{\mathbf{x}}_3$. Matrix $\mathbf{T} \in \Re^{p\times p}$ is orthogonal and $\mathbf{B}_2 \in \Re^{m\times \mathbf{m}}$ is square of rank m. In addition, once the invariant zeros of $(\mathbf{A},\mathbf{B},\mathbf{C}_{r-1})$ exist, then with another transformation, submatrices $\mathbf{A}_{11}$ and $\mathbf{A}_{21}$ will have a particular structure written as

$$
\mathbf{A}_{11} = \begin{bmatrix} \mathbf{A}_{11}^{\circ} & \mathbf{A}_{12}^{\circ} \\ 0 & \mathbf{A}_{22}^{\circ} \end{bmatrix}
\qquad
\mathbf{A}_{21} = \begin{bmatrix} 0 & \mathbf{A}_{21}^{\circ} \end{bmatrix} \tag{9}
$$

where $\mathbf{A}_{11}^{\circ} \in \Re^{q\times q}$, $\mathbf{A}_{22}^{\circ} \in \Re^{(n-p-q)\times(n-p-q)}$, $\mathbf{A}_{21}^{\circ} \in \Re^{(p-m)\times(n-p-q)}$ with $(\mathbf{A}_{22}^{\circ},\mathbf{A}_{21}^{\circ})$ is a observable pair. The number q denotes the number of invariant zeros of $(\mathbf{A},\mathbf{B},\mathbf{C}_{r-1})$ which can be stated by the following lemma:

Lemma: The eigenvalues of $\mathbf{A}_{11}^{\circ}$ *are the invariant zeros of* $(\mathbf{A},\mathbf{B},\mathbf{C}_{r-1})$.
proof: see Edwards and Spurgeon (1995).

To proceed the proposed design with integral sign function control, an augmented system $(\mathbf{A_a},\mathbf{B_a},\mathbf{C_a})$ is constructed by augmenting the new m states of $\dot{\bar{\mathbf{x}}}_3$ into eq.(8). By defining new state $\bar{\mathbf{x}}_4 = \dot{\bar{\mathbf{x}}}_3$ the augmentation can be obtained as

$$
\begin{bmatrix} \dot{\bar{\mathbf{x}}}_1 \\ \dot{\bar{\mathbf{x}}}_2 \\ \dot{\bar{\mathbf{x}}}_3 \\ \dot{\bar{\mathbf{x}}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & 0 \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & 0 \\ 0 & 0 & 0 & \mathbf{I_m} \\ \mathbf{A}_{41} & \mathbf{A}_{42} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \\ \bar{\mathbf{x}}_3 \\ \bar{\mathbf{x}}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbf{B}_2 \end{bmatrix} (\dot{\mathbf{u}}+\dot{\mathbf{f}}_\mathbf{m}) \tag{10}
$$

where

$$
\begin{aligned}
\mathbf{A}_{41} &= \mathbf{A}_{31}\mathbf{A}_{11} + \mathbf{A}_{32}\mathbf{A}_{21} \\
\mathbf{A}_{42} &= \mathbf{A}_{31}\mathbf{A}_{12} + \mathbf{A}_{32}\mathbf{A}_{22} \\
\mathbf{A}_{43} &= \mathbf{A}_{31}\mathbf{A}_{13} + \mathbf{A}_{32}\mathbf{A}_{23} \\
\mathbf{A}_{44} &= \mathbf{A}_{33}
\end{aligned} \tag{11}
$$

Theorem 1: $(\mathbf{A_a},\mathbf{B_a},\mathbf{C_a})$ *is controllable and observable if and only if* $(\mathbf{A},\mathbf{B},\mathbf{C}_{r-1})$ *is controllable and observable***.**
proof:
This result can be derived by applying the P.B.H. rank test which writes: For controllability test, utilizing the fact of $rank(\mathbf{B}_2) = \mathbf{m}$ which implies

$$
\mathbf{rank}[\mathbf{zI}-\mathbf{A} \quad \mathbf{B}] = \mathbf{rank}\begin{bmatrix} \mathbf{zI}-\mathbf{A}_{11} & -\mathbf{A}_{12} & -\mathbf{A}_{13} \\ -\mathbf{A}_{21} & \mathbf{zI}-\mathbf{A}_{22} & -\mathbf{A}_{23} \end{bmatrix} + \mathbf{m}
$$

and

$$
\mathbf{rank}[\mathbf{zI}-\mathbf{A_a} \quad \mathbf{B_a}] = \mathbf{rank}\begin{bmatrix} \mathbf{zI}-\mathbf{A}_{11} & -\mathbf{A}_{12} & -\mathbf{A}_{13} \\ -\mathbf{A}_{21} & \mathbf{zI}-\mathbf{A}_{22} & -\mathbf{A}_{23} \end{bmatrix} + 2\mathbf{m}
$$

Hence $(\mathbf{A_a},\mathbf{B_a})$ to be controllable $\Leftrightarrow$

$$
\mathbf{rank}\begin{bmatrix} \mathbf{zI}-\mathbf{A}_{11} & -\mathbf{A}_{12} & -\mathbf{A}_{13} \\ -\mathbf{A}_{21} & \mathbf{zI}-\mathbf{A}_{22} & -\mathbf{A}_{23} \end{bmatrix} = \mathbf{n}-\mathbf{m} \quad \Leftrightarrow
$$

$(\mathbf{A},\mathbf{B})$ to be controllable.
Similar procedure can be used for observability test, then after some derivation which can be stated as

$$
(\mathbf{A_a},\mathbf{C_a}) \;\Leftrightarrow\; \mathbf{rank}\begin{bmatrix} \mathbf{zI}-\mathbf{A}_{11} \\ -\mathbf{A}_{21} \\ -\mathbf{A}_{31} \end{bmatrix} = \mathbf{n}-\mathbf{p} \;\Leftrightarrow\; (\mathbf{A},\mathbf{C}_{r-1})
$$

to be observable. Q.E.D.

In this section, general MIMO linear uncertain system of relative degree r is formulated into an augmented system of relative degree one. The augmented system has a particular structure, which clearly specifies the system invariant zeros and accessible states, makes it suitable for output feedback design purpose. A controller design based on this formulation will be proposed and presented in the following section.

3. THE PROPOSED OFSMC DESIGN

The design method is separated into two steps. One is the selection of sliding surface and its corresponding sliding motion dynamics. The other is the controller synthesis.

The Sliding Surface Design; The main aim of the proposed OFSMC design is to construct a sliding surface based on the augmented system of eq.(10) by using the accessible information of $\bar{\mathbf{x}}_2$, $\bar{\mathbf{x}}_3$ and $\bar{\mathbf{x}}_4$. Under such circumstance, the sliding surface $\mathbf{s} \in \Re^{\mathbf{m}}$ can be defined to be

$$
\mathbf{s} = \mathbf{F}_1\bar{\mathbf{x}}_4 + \mathbf{F}_2\bar{\mathbf{x}}_2 + \mathbf{F}_3\bar{\mathbf{x}}_3 \tag{12}
$$

where $\mathbf{F}_1 \in \Re^{m\times m}$ is an invertible square matrix and

$\mathbf{F}_2 \in \Re^{m\times(p-m)}$, $\mathbf{F}_3 \in \Re^{m\times m}$ are two matrices to be determined. For systematic analysis and synthesis purposes, a nonsingular transformation $\mathbf{T}_s$ is introduced and by defining $\mathbf{K}_2 = \mathbf{F}_1^{-1}\mathbf{F}_2$ and $\mathbf{K}_3 = \mathbf{F}_1^{-1}\mathbf{F}_3$, then the state equation of eq.(10) in the new coordinate becomes

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_1 \\ \dot{\bar{\mathbf{x}}}_2 \\ \dot{\bar{\mathbf{x}}}_3 \\ \dot{\mathbf{s}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & 0 \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & 0 \\ 0 & -\mathbf{K}_2 & -\mathbf{K}_3 & \mathbf{F}_1^{-1} \\ \mathbf{A}_{41}^* & \mathbf{A}_{42}^* & \mathbf{A}_{43}^* & \mathbf{A}_{44}^* \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \\ \bar{\mathbf{x}}_3 \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbf{F}_1\mathbf{B}_2 \end{bmatrix} \left(\dot{\mathbf{u}} + \dot{\mathbf{f}}_\mathbf{m}\right) \quad (13)$$

where

$$\begin{aligned} \mathbf{A}_{41}^* &= \mathbf{F}_2\mathbf{A}_{21} + \mathbf{F}_1\mathbf{A}_{41} \\ \mathbf{A}_{42}^* &= \mathbf{F}_2\mathbf{A}_{22} + \mathbf{F}_1\mathbf{A}_{42} - \mathbf{F}_3\mathbf{K}_2 - \mathbf{F}_1\mathbf{A}_{44}\mathbf{K}_2 \\ \mathbf{A}_{43}^* &= \mathbf{F}_2\mathbf{A}_{23} + \mathbf{F}_1\mathbf{A}_{43} - \mathbf{F}_3\mathbf{K}_3 - \mathbf{F}_1\mathbf{A}_{44}\mathbf{K}_3 \\ \mathbf{A}_{44}^* &= \mathbf{F}_3\mathbf{F}_1^{-1} + \mathbf{F}_1\mathbf{A}_{44}\mathbf{F}_1^{-1} \end{aligned} \quad (14)$$

Once the sliding motion is attained or $\mathbf{s} = \dot{\mathbf{s}} = 0$, then the sliding motion dynamics become a reduced system of order n as

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_1 \\ \dot{\bar{\mathbf{x}}}_2 \\ \dot{\bar{\mathbf{x}}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ 0 & -\mathbf{K}_2 & -\mathbf{K}_3 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \\ \bar{\mathbf{x}}_3 \end{bmatrix} \quad (15)$$

which can be viewed as a static output feedback problem of $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}}$ with matrices

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{I}_\mathbf{m} \end{bmatrix} \quad (16)$$
$$\tilde{\mathbf{C}} = \begin{bmatrix} 0 & \mathbf{I}_\mathbf{p} \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} \mathbf{K}_2 & \mathbf{K}_3 \end{bmatrix}$$

Once $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ is static output feedback stabilizable, then the standard output feedback design techniques can be utilized in designing the feedback gain matrix **K**. It is often required that system is controllable and observable in order to apply the standard design techniques. The controllability of $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ can be easily seen from the proof of theorem 1. The observability of $(\tilde{\mathbf{A}}, \tilde{\mathbf{C}})$ can be assured from the P.B.H. rank test of these two matrices in eq.(16) and the structure of eq.(9) together with the fact of observable pair $(\mathbf{A}_{22}^\circ, \mathbf{A}_{21}^\circ)$. By defining new matrices $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ to be

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{22}^\mathbf{o} & \mathbf{A}_{122} & \mathbf{A}_{132} \\ \mathbf{A}_{21}^\mathbf{o} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ 0 & 0 & 0 \end{bmatrix} \in \Re^{(n-q)\times(n-q)}$$
$$\hat{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{I}_m \end{bmatrix} \in \Re^{(n-q)\times m}, \quad \hat{\mathbf{C}} = \begin{bmatrix} 0 & \mathbf{I}_p \end{bmatrix} \in \Re^{p\times(n-q)} \quad (17)$$

where $\mathbf{A}_{122}$ and $\mathbf{A}_{132}$ are the submatrices of $\mathbf{A}_{12}$ and $\mathbf{A}_{13}$, respectively. It can be shown that the q invariant zeros in the submatrix $\mathbf{A}_{11}^\circ$ cannot be affected by the static output feedback which can be stated in the following theorem.

Theorem 2:The spectrum of $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}}$ *is the spectrum of* $\lambda(\mathbf{A}_{11}^\mathbf{o}) \cup \lambda(\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{K}\hat{\mathbf{C}})$.

proof:

Based on eq.(9) and (17) and the matrix K in eq.(16), the closed-loop matrix $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}}$ can be rearranged as

$$\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{A}_{11}^\circ & \begin{bmatrix} \mathbf{A}_{12}^\circ & \mathbf{A}_{121} & \mathbf{A}_{131} \end{bmatrix} \\ 0 & \begin{bmatrix} \hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{K}\hat{\mathbf{C}} \end{bmatrix} \end{bmatrix}$$

which directly implies

$\lambda(\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}}) = \lambda(\mathbf{A}_{11}^\circ) \cup \lambda(\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{K}\hat{\mathbf{C}})$. Q.E.D.

If the so called Kimula-Davison condition (Syrmos, *et al.*, 1997) of $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ is satisfied, which in this case the dimensionality satisfies $m + p > n - r + 1$, then the eigenvalues of $\hat{\mathbf{A}} - \hat{\mathbf{B}}\mathbf{K}\hat{\mathbf{C}}$ can be placed as close as possible to the desired values.

The Controller Synthesis; By defining $\dot{\mathbf{u}} = \mathbf{v}_l + \mathbf{v}_n$ in eq.(13) where $\mathbf{v}_l$ and $\mathbf{v}_n$ denote the linear and nonlinear controls respectively, a controller can be synthesized similar to the conventional design procedure. However, it notes that only the accessible information of $\bar{\mathbf{x}}_2$, $\bar{\mathbf{x}}_3$ and $\mathbf{s}$ in eq.(13) can be used in the synthesis. By simply assigning the linear and the nonlinear portions to be

$$\mathbf{v}_\mathbf{l} = -(\mathbf{F}_1\mathbf{B}_2)^{-1}\left(\mathbf{A}_{42}^*\bar{\mathbf{x}}_2 + \mathbf{A}_{43}^*\bar{\mathbf{x}}_3 + \mathbf{A}_{44}^*\mathbf{s}\right) \quad (18)$$

$$\mathbf{v}_\mathbf{sw} = -\gamma(\mathbf{F}_1\mathbf{B}_2)^{-1} sgn(\mathbf{s}) \quad (19)$$

where $\gamma > 0$ is a scalar parameter and the function $sgn(\bullet)$ is a unit vector function defined as

$$sgn(\mathbf{s}) = \begin{cases} \dfrac{\mathbf{s}}{\|\mathbf{s}\|} & ,\text{if } \mathbf{s} \neq 0 \\ 0 & ,\text{otherwise} \end{cases} \quad (20)$$

The closed-loop dynamics can be obtained as

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_1 \\ \dot{\bar{\mathbf{x}}}_2 \\ \dot{\bar{\mathbf{x}}}_3 \\ \dot{\mathbf{s}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & 0 \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & 0 \\ 0 & -\mathbf{K}_2 & -\mathbf{K}_3 & \mathbf{F}_1^{-1} \\ \mathbf{A}_{41}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \\ \bar{\mathbf{x}}_3 \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbf{F}_1\mathbf{B}_2\dot{\mathbf{f}}_\mathbf{m} - \gamma sgn(\mathbf{s}) \end{bmatrix} \quad (21)$$

To guarantee the existence of sliding motion, the so called approaching condition of the sliding surface must be satisfied which is referred to the derivative of energy function $\mathbf{V}(\mathbf{s}) = 1/2\,\mathbf{s}^\mathbf{T}\mathbf{s}$ being negative. From the energy function, the approaching condition can be written as

$$\begin{aligned} \dot{\mathbf{V}}(\mathbf{s}) &= \mathbf{s}^\mathbf{T}\dot{\mathbf{s}} = \mathbf{s}^\mathbf{T}\left[\mathbf{A}_{41}^*\bar{\mathbf{x}}_1 + \mathbf{F}_1\mathbf{B}_2\dot{\mathbf{f}}_\mathbf{m} - \gamma\, sgn(\mathbf{s})\right] \\ &\leq \|\mathbf{s}\|\left(\|\mathbf{A}_{41}^*\|\|\bar{\mathbf{x}}_1\| + \|\mathbf{F}_1\mathbf{B}_2\|\|\dot{\mathbf{f}}_\mathbf{m}\| - \gamma\right) \end{aligned} \quad (22)$$

If γ is selected to be

$$\gamma = \|\mathbf{A}_{41}^*\|\|\bar{\mathbf{x}}_1\| + \|\mathbf{F}_1\mathbf{B}_2\|\|\dot{\mathbf{f}}_\mathbf{m}\| + \eta \quad (23)$$

where $\eta > 0$ is a scalar, then the sliding surface satisfies the finite time approaching condition.

In eq.(23) the exact value of $\|\bar{\mathbf{x}}_1\|$ cannot be known since the information of $\bar{\mathbf{x}}_1$ is not accessible and it will result a problem called sliding patch as mentioned in the introduction. If there exists the sliding patch then the sliding surface will not be global attractive. However, this problem can be avoided by the estimation of $\|\bar{\mathbf{x}}_1\|$ with a slight modification of the approach in Kwan (2001). Consider the dynamics of the first three states in eq.(21) which can be concisely denoted by

$$\dot{\bar{\mathbf{x}}} = \left(\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}}\right)\bar{\mathbf{x}} + \tilde{\mathbf{F}}\mathbf{s} \tag{24}$$

where $\tilde{\mathbf{F}} = \begin{bmatrix} 0 & 0 & \mathbf{F}_1^{-1} \end{bmatrix}^{\mathbf{T}}$. Assume the closed-loop matrix $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}}$ is stable and has the spectrum of $\{-\lambda_1, -\lambda_2, \cdots, -\lambda_{\mathbf{n}}\}$ and define $\lambda_{min} = \min_{\mathbf{k}=1\sim\mathbf{n}} \mathrm{Re}(\lambda_{\mathrm{k}})$. Then there exists a scalar time differentiable variable $\omega(\mathrm{t})$ satisfying

$$\dot{\omega} = -\lambda_{min}\omega + \left\|\mathbf{F}_1^{-1}\mathbf{s}\right\| \tag{25}$$

with initial value $\omega(0) > \kappa\|\bar{\mathbf{x}}(0)\| > 0$ where κ is a positive constant such that $\omega(\mathrm{t}) \geq \|\bar{\mathbf{x}}\| \geq \|\bar{\mathbf{x}}_1\|$. Then γ in eq.(23) can be replaced by

$$\gamma = \left\|\mathbf{A}_{41}^*\right\|\omega(\mathbf{t}) + \left\|\mathbf{F}_1\mathbf{B}_2\right\|\left\|\dot{\mathbf{f}}_{\mathbf{m}}\right\| + \eta \tag{26}$$

which results a stronger finite time approaching condition and a global attractive sliding surface. Based on the controls of eq.(18) and (19), the actual control u is the integration of v which written as

$$\mathbf{u} = -(\mathbf{F}_1\mathbf{B}_2)^{-1}\left\{\int_0^{\mathbf{t}}[\mathbf{v}_1 + \gamma\, sgn(\mathbf{s})]\mathrm{dt}\right\} + \mathbf{u}(0) \tag{27}$$

with $\mathbf{u}(0)$ being an integration constant.

The Controller in Original Coordinate; From eq.(8) the system output can be written as

$$\mathbf{y} = [\mathbf{0} \quad \mathbf{T}]\bar{\mathbf{x}} \equiv \mathbf{T}\begin{bmatrix} \bar{\mathbf{x}}_2 \\ \bar{\mathbf{x}}_3 \end{bmatrix} \tag{28}$$

Since $\mathbf{T}$ is orthogonal, then $\mathbf{T}^{-1} = \mathbf{T}^{\mathbf{T}}$ and partition the matrix $\mathbf{T}^{\mathbf{T}}$ to be

$$\mathbf{T}^{\mathbf{T}} = \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix} \tag{29}$$

where $\mathbf{T}_1 \in \Re^{(\mathbf{p}-\mathbf{m})\times\mathbf{p}}$ and $\mathbf{T}_2 \in \Re^{\mathbf{m}\times\mathbf{p}}$. The information of $\bar{\mathbf{x}}_4$ can be obtained from system output by

$$\bar{\mathbf{x}}_4 = \dot{\bar{\mathbf{x}}}_3 = \mathbf{T}_2\dot{\mathbf{y}} \tag{30}$$

Then the sliding surface in eq.(12) and the linear control in eq.(18) in original coordinate becomes

$$\mathbf{s} = \mathbf{F}_1\mathbf{T}_2\dot{\mathbf{y}} + \mathbf{F}_2\mathbf{T}_1\mathbf{y} + \mathbf{F}_3\mathbf{T}_2\mathbf{y} \tag{31}$$

$$\mathbf{v}_1 = -(\mathbf{F}_1\mathbf{B}_2)^{-1}\left(\mathbf{A}_{42}^*\mathbf{T}_1\mathbf{y} + \mathbf{A}_{43}^*\mathbf{T}_2\mathbf{y} + \mathbf{A}_{44}^*\mathbf{s}\right) \tag{32}$$

4. CASE STUDY

Consider a 3-state system subject to a sinusoidal disturbances in the following

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1/3 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}(\mathbf{u} + 10\, sin(2\pi\mathbf{t}))$$

$$\mathbf{y} = \begin{bmatrix} 1 & 8/3 & 1 \\ 4 & 2/3 & -2 \end{bmatrix}\mathbf{x}$$

which is a system of 2 outputs, 1 input, relative degree 1 and no invariant zero. The augmented system $(\mathbf{A_a}, \mathbf{B_a}, \mathbf{C_a})$ in eq.(10) can be computed as

$$\mathbf{A_a} = \begin{bmatrix} -1.5816 & 0.0192 & 0.1457 & 0 \\ 1.4071 & 0.3845 & -1.7080 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0113 & 0.1364 & -0.5376 & 0.1971 \end{bmatrix}$$

$$\mathbf{B_a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.9016 \end{bmatrix}, \quad \mathbf{C_a} = \begin{bmatrix} 0 & 0.3417 & -0.9398 & 0 \\ 0 & 0.9398 & 0.3417 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Due to no invariant zero exists in the system, the system $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ in eq.(16) is used for static output feedback design where they are:

$$\tilde{\mathbf{A}} = \begin{bmatrix} -1.5816 & 0.0192 & 0.1457 \\ 1.4071 & 0.3845 & -1.7080 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\tilde{\mathbf{C}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Select the matrix gain to be $\mathbf{K} = [-9.5 \quad 10]$, then $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}}$ possesses stable eigenvalues at $\{-1, -2.1608, -8.0363\}$. For simplicity, chose $\mathbf{F}_1 = 1$ then its corresponding sliding surface in original coordinate of eq.(31) can be calculated to be

$$\mathbf{s} = [-0.9398 \quad 0.3417]\dot{\mathbf{y}} + [-12.6445 \quad -5.5106]\mathbf{y}$$

and the controller in original coordinate of eq.(32) is

$$\mathrm{u} = 0.2563\int_0^{\mathbf{t}}\{[1129912 \quad 58.2490]\mathbf{y} + 10.1971\mathbf{s} + \gamma\, sgn(\mathbf{s})\}\mathrm{dt}$$

where $\gamma = 13.3563\omega(\mathrm{t}) + 78.032\pi + 5$ and $\omega(\mathrm{t})$ satisfying the dynamic equation of $\dot{\omega}(\mathrm{t}) = -\omega(\mathrm{t}) + |\mathbf{s}|$. In the simulations, the initial values of $\mathbf{x}(0)$ and $\mathbf{u}(0)$ are selected to be $[1 \quad 1 \quad 1]^{\mathrm{T}}$ and 0 respectively. $\omega(0)$ is selected to be 2 which satisfies the condition of eq.(25). The sliding surface response $\mathbf{s}$ is shown in Fig.1 which demonstrates the global attractiveness of the proposed control. The state reaches sliding surface $\mathbf{s} = 0$ at approximately t=0.4 sec and stays at it for the subsequent time. The results of state response and output response are shown in Fig.2 and Fig.3 respectively. These two figures display the asymptotical stability of the system despite the existence of external disturbance. The control force is shown in Fig.4 which is a smooth control without chattering.

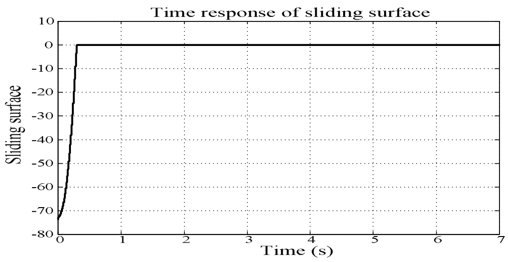

Fig. 1. The evolution of sliding surface.

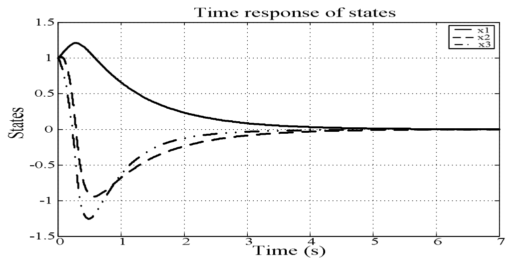

Fig. 2. The evolution of states.

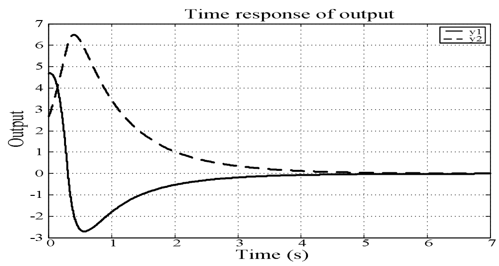

Fig. 3. The evolution of outputs.

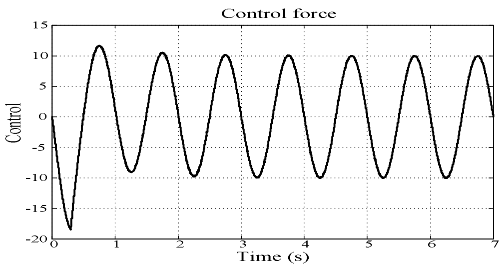

Fig. 4. The control force

5. CONCLUSIONS

In this paper, a chattering-free OFSMC design for general MIMO linear uncertain systems of any relative degree is studied. An integral sign function control is used to avoid the chattering phenomenon. To incorporate the integral sign function and preserve the robustness of SMC, the construction of the sliding surface must be relative degree zero to system input which results in the requirement of higher derivative of the output. To deal with such problem, an augmented system formulation is proposed which provides a clear feature for system invariant zeros. In addition, the accessible states for output feedback design are also clearly sorted. An OFSMC design based on this formulation is proposed where the sliding patch problem is avoided. A numerical example with an external disturbance is given to illustrate the desired chattering-free and global attractiveness property of this approach.

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