RELATIVELY OPTIMAL CONTROL: THE STATIC SOLUTION

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Abstract: A relatively optimal control is a stabilizing controller that, without initialization nor feedforwarding and tracking the optimal trajectory, produces the optimal (constrained) behavior for the nominal initial condition of the plant. In a previous work, a linear *dynamic* relatively optimal control, for discrete–time linear systems, was presented. Here a *static* solution is shown, namely a dead–beat piecewise affine state–feedback controller based on a suitable partition of the state space into polyhedral sets. The vertices of the polyhedrons are the states of the optimal trajectory, hence a bound for the complexity of the controller is known in advance. It is also shown how to obtain a controller that is not dead–beat by removing the zero terminal constraint while guaranteeing stability. Finally, the proposed static compensator is compared with the dynamic one.

Keywords: Optimal control, Linear systems, Discrete–time systems, Invariance

1. INTRODUCTION

It is known that, unless for very special cases, determining an optimal control in a feedback form, under output or input constraints is a computationally hard task. The problem can be addressed in a receding horizon fashion but in this case an optimization problem must be solved on–line at each time interval. Explicit (piecewise affine) solutions exist (Bemporad *et al.*, 2002*b*; Bemporad *et al.*, 2002*a*) but are limited to quadratic or 1–norm cost and linear constraints. However, for those systems which are explicitly built to perform specific operation through a specific trajectory with known initial and final states, the request of optimality from *any* initial state can be relaxed, requiring optimality only from a specific initial condition. The Relatively Optimal Control (ROC) (Blanchini and Pellegrino, 2003) is a stabilizing controller that guarantees optimality of the trajectory and constraint satisfaction from a given (or a set of given) initial condition(s). The ROC does not require any initialization nor the feedforward and tracking of the optimal trajectory. In (Blanchini and Pellegrino, 2003) it has been proved that a controller enjoying these properties is linear dynamic and its order is equal to the length of the optimal trajectory minus the order of the plant. In (Blanchini and Pellegrino, 2004) the zero terminal constraint was removed in order to assign a characteristic polynomial to the closed–loop system and the problem of output feedback was addressed. Here, a *static* ROC is constructed by partitioning the state space into polyhedral

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sets whose vertices are the states of the optimal trajectory and their opposite.

The main contribution of the present paper can be summarized in the following points.

- It is shown that for discrete–time linear systems with convex constraints and cost, it is always possible to construct a static ROC by means of a proper partition of the state space into polyhedral sets (a procedure to construct it is provided).
- If the constraints and/or the cost are not convex, a sufficient condition on the optimal trajectory that guarantees that the static ROC can be constructed is provided.
- The proposed controller is a dead–beat piecewise affine state–feedback controller. The vertices of each of these polyhedral sets are the states of the optimal trajectory and their opposite. The control at each vertex is the corresponding control of the optimal sequence while the control at a generic state is given by a convex combination of the control vectors corresponding to the vertices of the polyhedron the state belongs to.
- An upper bound on the number of polyhedral sets as a function of the order of the system and the length of the optimal trajectory is provided.
- Removing the zero state terminal constraint and requiring the final state of the optimal trajectory to belong to a controlled invariant set it is possible to obtain a non dead–beat controller.

2. PROBLEM STATEMENT

Given the discrete–time reachable system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$ and A, B, C, D are matrices of appropriate dimensions. Consider the locally bounded convex cost functions of the output

$$g(y), \quad l_i(y), \quad i = 1, 2, \ldots, s$$

with assigned initial condition

$$\bar{x} \neq 0$$

and the constraint

$$y(k) \in \mathcal{Y}, \tag{2}$$

where $\mathcal{Y}$ is a convex and closed set. Then consider the following problem (where $k = 1$ is the initial time)

$$J_{opt}(\bar{x}) = \min \sum_{k=1}^{N} g(y(k)) \tag{3}$$

$$\text{s.t.} \tag{4}$$

$$x(k+1) = Ax(k) + Bu(k), \; k = 1 \ldots N \tag{5}$$

$$y(k) = Cx(k) + Du(k), \; k = 1 \ldots N \tag{6}$$

$$\sum_{k=1}^{N} l_i(y(k)) \leq \mu_i, \quad i = 1, 2, \ldots, s \tag{7}$$

$$y(k) \in \mathcal{Y}, \quad k = 1, \ldots, N \tag{8}$$

$$x(1) = \bar{x} \tag{9}$$

$$x(N+1) = 0 \tag{10}$$

$$N \geq 0, \quad \text{assigned (or free).} \tag{11}$$

Finding an open–loop solution for the above problem is well–known to be a convex problem which can be solved by means of efficient algorithms. Here, the aim is a feedback static solution. Consider the following problem.

Problem 1. Find a static state–feedback compensator of the form $u = \Phi(x)$ which is (locally) stabilizing and such that for $x(1) = \bar{x}$ the control and state trajectories are the optimal ones.

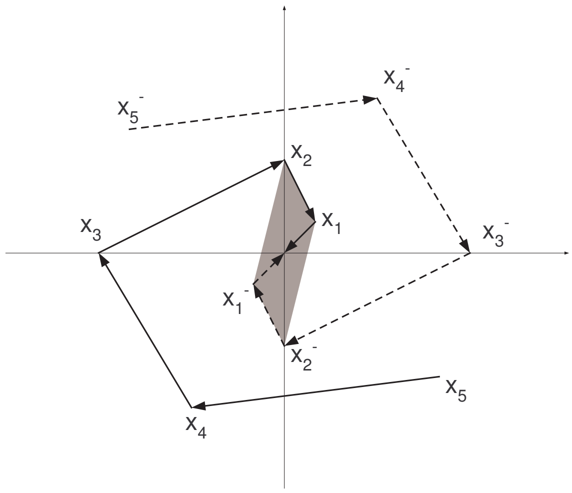

Fig. 1. Example of the set $\mathcal{S}_n$ (gray area) in a two dimensional space. $\mathcal{S}_n$ is the convex hull of the last two states of the optimal trajectory (connected by the continuous line) and their opposite (connected by the dash line).

3. MAIN RESULTS

Denoting by $\bar{X} = [\bar{x}(1) \ldots \bar{x}(N)]$ the optimal state trajectory from the initial condition $\bar{x} = \bar{x}(1)$, the following assumption is introduced (in the following it will be shown how the assumption can be removed).

Assumption 1. The last n states of the optimal trajectory are linearly independent, namely the matrix $S_n = [\bar{x}(N-n+1)\ \bar{x}(N-n+2)\dots\bar{x}(N)]$ is invertible.

Let us consider the polyhedral set $\mathcal{S}_n = \{x : x = S_n\alpha, \|\alpha\|_1 \le 1\}$. Such a set is the convex hull of the last n states of the optimal trajectory and their opposite. It contains the origin in its interior and is zero-symmetric. An example for $n = 2$ is shown in Fig.1. Thanks to Assumption 1 the next lemma holds.

Lemma 3.1. The linear control

$$u(x) = U_n S_n^{-1} x, \qquad (12)$$

where $U_n = [\bar{u}(N-n+1)\ \bar{u}(N-n+2)\dots\bar{u}(N)]$, renders positively invariant the set $\mathcal{S}_n$ satisfying the constraints for all initial conditions inside the set. In particular it is dead–beat and steers the state to zero in at most n steps.

Proof The control law $u(x) = U_n S_n^{-1} x$ is a control–at–the–vertices strategy. All $x \in \mathcal{S}_n$ can be written in a unique way as a (convex) combination of the columns of S_n, namely the last n states of the optimal trajectory:

$$x = S_n\alpha.$$

Since S_n is invertible, it follows that

$$\alpha(x) = S_n^{-1} x,$$

hence the control law $u(x) = U_n S_n^{-1} x$ is a linear combination of the control vectors at the vertices of $\mathcal{S}_n$ according to the coefficients $\alpha(x)$. Positive invariance is a consequence of the fact that, by construction, the control at each vertex keeps the state inside the set (Blanchini, 1999). The satisfaction of the constraints is guaranteed for all initial conditions inside the set, being the input and state constraints convex. To prove that the control is dead–beat, let x_i be the state of the optimal trajectory which is i steps far from the origin (i.e. $x_1 = \bar{x}(N)$, $x_2 = \bar{x}(N-1)$ and so on) and u_i the corresponding control. If, at time k,

$$x(k) = x_n\alpha_n + \dots + x_2\alpha_2 + x_1\alpha_1,$$

then the computed control will be

$$u(x(k)) = u_n\alpha_n + \dots + u_2\alpha_2 + u_1\alpha_1.$$

The state at time $k+1$ is, by linearity,

$$x(k+1) = x_{n-1}\alpha_n + \dots + x_1\alpha_2 + 0\alpha_1,$$

and so on. It is immediate to verify that after at most n steps the system will reach the origin. □

Now consider the state $x_{n+1} = \bar{x}(N-n)$ (corresponding to the state x_3 in the example of Fig.1). It can be shown that $x_{n+1} \notin \mathcal{S}_n$: indeed, it is well known that, for convex cost and constraints, the cost–to–go is convex hence every point inside $\mathcal{S}_n$ has a cost–to–go which is less or equal than the cost from each vertex. If $x_{n+1} \in \mathcal{S}_n$, the cost from x_{n+1} would be less than the cost from any of the subsequent points of the optimal trajectory (vertices of $\mathcal{S}_n$), which is impossible. A similar argument holds when the problem (3)-(11) is a minimum time problem, because the origin can be reached from $\mathcal{S}_n$ in at most n steps hence x_{n+1} (which is $n+1$ steps far from the origin along a minimum time trajectory) must be outside $\mathcal{S}_n$.

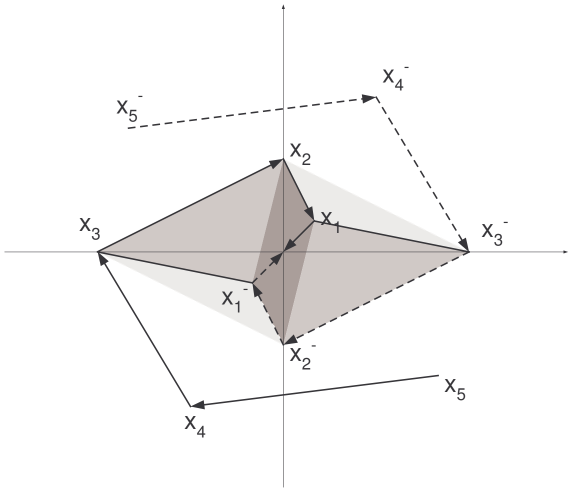

Fig. 2. Considering x_3 and its opposite, four simplices can be constructed. The set $\mathcal{S}_{n+1}$ is the union of $\mathcal{S}_n$ and such simplices (the gray areas)

Since x_{n+1} and its opposite $-x_{n+1}$ are outside $\mathcal{S}_n$, they can be connected to a certain number of vertices of $\mathcal{S}_n$ without crossing such a set, thus constructing some simplices (in the example of Fig. 2, such simplices are the triangles (x_3, x_2, x_1^-) and (x_3, x_1^-, x_2^-) and their symmetric). Denoting with $\mathcal{S}_{n+1}^j$, $j = 1\dots m_{n+1}$ the simplices having x_{n+1} as vertex and with $\mathcal{S}_{n+1}^j$, $j = -m_{n+1}\dots -1$ those having $-x_{n+1}$ as vertex, define the set $\mathcal{S}_{n+1}$ as follows:

$$\mathcal{S}_{n+1} = \bigcup_{j=\pm 1\dots m_{n+1}} \mathcal{S}_{n+1}^j \cup \mathcal{S}_n.$$

The control–at–the–vertices strategy may be extended to the enlarged set $\mathcal{S}_{n+1}$ as follows.

For each of the simplices $\mathcal{S}_{n+1}^j$:

(1) Order (arbitrarily) the vertices.
(2) Associate a matrix S_{n+1}^j whose columns are the ordered vertices.
(3) Associate a matrix U_{n+1}^j whose columns are the control vectors corresponding to the ordered vertices (if the vertex belongs to the optimal trajectory, take the corresponding control, if it belongs to the opposite of the

optimal trajectory, take the opposite of the corresponding control).

Lemma 3.2. Consider the following control strategy.

Given $x \in \mathcal{S}_{n+1}$,

- if $x \in \mathcal{S}_n$ then $u(x) = U_n S_n^{-1} x$
- otherwise, if $x \in \mathcal{S}_{n+1}^j$ then

$$u(x) = U_{n+1}^j \alpha^j, \tag{13}$$

where α^j is the (unique) vector such that $x = S_{n+1}^j \alpha^j, \quad \sum_k \alpha_k^j = 1$.

Such a control strategy renders positively invariant the set $\mathcal{S}_{n+1}$ and satisfies the constraints. In particular from any point inside $\mathcal{S}_{n+1}$ it steers the system to zero in at most $n+1$ steps.

Proof By construction, each of the simplices has all the vertices but one belonging to $\mathcal{S}_n$. The vertices, which are points of the optimal trajectory or their opposite, are mapped by the chosen control to the subsequent points of the optimal trajectory (or the opposite). Since the vertex that does not belong to $\mathcal{S}_n$ is one step far from $\mathcal{S}_n$ along the optimal trajectory, it follows that the images of the vertices of all the simplices belong to $\mathcal{S}_n$. Hence, any convex combination of them, i.e. any $x = S_{n+1}^j \alpha^j, \quad \sum_k \alpha_k^j = 1$, will be mapped into $\mathcal{S}_n$. $\square$

The procedure outlined above can be extended in order to include all the states of the optimal trajectory.

Procedure 3.1. Given the system (1) and the optimal open–loop trajectory, computed by solving (3)–(11), which satisfies Assumption 1.

(1) Let the set $\mathcal{S}_n = \{x : x = S_n \alpha, \|\alpha\|_1 \leq 1\}$, where $S_n = [x_n \ x_{n-1} \dots x_1]$, be the convex hull of the last n states of the optimal trajectory and their opposite.
(2) Let $U_n = [u_n \ u_{n-1} \dots u_1]$ be the matrix whose columns are the control vectors corresponding to the last n states of the optimal trajectory.
(3) Set $i = n+1$.
(4) Construct the simplices $\mathcal{S}_i^j, \ j = \pm 1 \dots m_i$ by connecting x_i and $-x_i$ to the vertices of $\mathcal{S}_{i-1}$ without crossing such set. This is always possible since $x_i, -x_i \notin \mathcal{S}_{i-1}$.
(5) Let S_i^j be the matrix whose columns are the vertices of $\mathcal{S}_i^j$ in an arbitrary order and U_i^j the control vectors corresponding to the vertices in the same order. For vertices belonging to the opposite of the optimal trajectory, take the opposite of the control.
(6) Let $\mathcal{S}_i = \bigcup_j \mathcal{S}_i^j \cup \mathcal{S}_{i-1}$.
(7) Increase i and go back to step 4 while $i \leq N$.

Note that, by construction, the sets $\mathcal{S}_i, \ i = n, \dots, N$ are convex and zero symmetric and such that $\mathcal{S}_i \subset \mathcal{S}_{i+1}$. Hence the sets $\mathcal{S}_i$ are nested, $\mathcal{S}_N$ being the outermost set. The set $\mathcal{S}_{i+1} \setminus \mathcal{S}_i$, difference between $\mathcal{S}_{i+1}$ and $\mathcal{S}_i$, is composed of simplices S_i^j each of whom has all vertices but one belonging to $\mathcal{S}_i$.

The next theorem is a generalization of Lemma 3.2.

Theorem 3.1. Consider the following control strategy.

Given $x \in \mathcal{S}_N$,

- if $x \in \mathcal{S}_n$ then $u(x) = U_n S_n^{-1} x$
- otherwise, if $x \in \mathcal{S}_i^j$ then

$$u(x) = U_i^j \alpha^j, \tag{14}$$

where α^j is the (unique) vector such that $x = S_i^j \alpha^j, \quad \sum_k \alpha_k^j = 1$.

Such a control strategy renders positively invariant the set $\mathcal{S}_N$ and satisfies the constraints. In particular from any point inside $\mathcal{S}_N$ it steers the system to zero in at most N steps.

Proof It follows by applying recursively the same argument used in the proof of Lemma 3.2: by construction, each point belonging to $\mathcal{S}_i$ is mapped into $\mathcal{S}_{i-1}$ and the control, being a convex combination of admissible control vectors, is admissible. Hence, from $\mathcal{S}_N$, the system reaches the origin in at most N steps. $\square$

For $x \in \mathcal{S}_N$, the controller described above is a solution of Problem 1, hence it is a local stabilizing controller that achieves the optimal trajectory for the given initial condition. The control law is not defined for $x \notin \mathcal{S}_N$. A possible way to extend the control outside $\mathcal{S}_N$ is to "immerse" $\mathcal{S}_N$ in the maximal invariant set $\mathcal{H}_{max}$ (Blanchini, 1999) namely the set of all states which can be steered to the origin in finitely many steps without state or input constraint violations (note that $\mathcal{S}_N \subseteq \mathcal{H}_{max}$). Then, for $x \notin \mathcal{S}_N$, one can apply the control law derived from $\mathcal{H}_{max}$ (many algorithms have been proposed to find $\mathcal{H}_{max}$ and an associated control law, see for example (Gutman and Cwikel, 1987)). Note that such a strategy allows to overcome a limitation of the dynamic ROC, namely the fact that the constraints may be violated for non–nominal initial conditions: on the contrary, for the static ROC extended as shown above, the constraint satisfaction (and the convergence as well) is guaranteed for all $\bar{x} \in \mathcal{H}_{max}$.

If Assumption 1 does not hold, the construction of the regions is basically the same. The only difference is that now the first region to be constructed is $\mathcal{S}_r$, whose vertices are $\bar{x}(N-r+1), \ \bar{x}(N-r+2), \dots, \bar{x}(N)$ where $r < n$ is such that the last r steps of the optimal trajectory are linearly independent while the last $r+1$ are not. Note that $\mathcal{S}_r$ (and, possibly, other subsequent regions) lives in a proper subspace of $\mathbb{R}^n$.

An important question is whether the complexity of the controller (i.e. the number of simplices obtained by partitioning the state space according to Procedure 3.1) is known in advance. Since such simplices form a *triangulation* (DeLoera *et al.*, n.d.) of a point set, their number N_s is bounded according to the following expression (Ziegler, 1994):

$$N_s \leq \begin{pmatrix} 2N+2-\left\lceil \frac{n+1}{2} \right\rceil \\ \left\lfloor \frac{n+1}{2} \right\rfloor \end{pmatrix} + \begin{pmatrix} 2N+1-\left\lceil \frac{n}{2} \right\rceil \\ \left\lfloor \frac{n}{2} \right\rfloor \end{pmatrix} - (n+1). \tag{15}$$

Table 1 reports such an upper bound for some pairs of N and n.

Table 1. Upper bound for the number of simplices given the number of steps of the optimal trajectory (N) and the order of the system (n)

N, n	3	4	8	12	16
4	33	39	-	-	-
8	133	207	1425	-	-
12	297	503	11965	54257	-
16	525	927	47497	592013	$2.1\ 10^6$
20	817	1479	132085	$3.2\ 10^6$	$2.8\ 10^7$

Remark 3.1. As shown above, the convexity of the constraints and the cost, guarantees that

$$x_i \notin \mathcal{S}_{i-1},\ \forall i = n+1, \ldots, N. \tag{16}$$

However, a ROC can be constructed independently of the convexity of the optimization problem provided that (16) hold. In other words, in order to construct the static ROC it is not necessary for the optimization problem to be convex. It is sufficient that each of the points of the optimal trajectory does not belong to the convex hull of the subsequent points and their opposite (condition that is automatically satisfied when the optimization problem is convex). Obviously, the satisfaction of the constraints for all the trajectories originating in $\mathcal{S}_N$, is guaranteed only if the constraints are convex.

Remark 3.2. It can be shown that the number of steps required to reach the origin from a specific initial state depends on the simplex the initial state belongs to. More precisely, it is equal to the maximum number of steps among the vertices of simplex.

4. OPTIMAL ARRIVAL TO A TARGET SET

Similarly to (Blanchini and Pellegrino, 2004), the constraint (10) may be relaxed as follows:

$$x(N+1) \in \mathcal{X}_{fin}, \tag{17}$$

where $\mathcal{X}_{fin}$ is a zero-symmetric controlled-invariant polyhedron (that is there exists a local control that renders $\mathcal{X}_{fin}$ positively invariant and such that the constraints are satisfied for all initial conditions inside the set). Then one can construct the ROC by positing $\mathcal{S}_n = \mathcal{X}_{fin}$ and following steps (3)-(7) of Procedure 3.1. As a result, a dual control strategy may be adopted: apply the control–at–the–vertices for $x(k) \notin \mathcal{X}_{fin}$ and switch to the local control as soon as the condition $x(k) \in \mathcal{X}_{fin}$ is satisfied.

5. COMPARISON WITH THE DYNAMIC ROC

Some significant differences between the dynamic Relatively Optimal Control (Blanchini and Pellegrino, 2003; Blanchini and Pellegrino, 2004) and the static one described above are briefly highlighted in the following points.

(1) Since the static ROC is non–linear, the trajectory originating from $\lambda\bar{x}$ is not (in general) proportional to the one originating from $\bar{x}$ as with the dynamic ROC. However, by construction, opposite initial conditions generate opposite trajectories.
(2) The dynamic ROC allows for the optimization from a set of n linearly independent initial conditions while the static version proposed here is thought for a single initial condition. Extending the results to more than one initial conditions for the static ROC is a matter of further investigation.
(3) The dynamic ROC can not guarantee the satisfaction of the constraints for initial conditions different from the nominal one. Hence it is only suitable for dealing with *soft* constraints. On the contrary, by immersing the set $\mathcal{S}_N$ in the maximal invariant set as shown in Section 3, the control proposed here can deal effectively with *hard* constraints.

6. EXAMPLE

Consider the double integrator:

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$

under the constraints $|x(k)| \leq 5, |u(k)| \leq 3$. Given the initial state $x(1) = [-2 \quad 5]^T$, the horizon $N = 5$, the final state $x(N+1) = 0$, and the cost function $J = \sum_{i=1}^{N} u(k)^2$, the

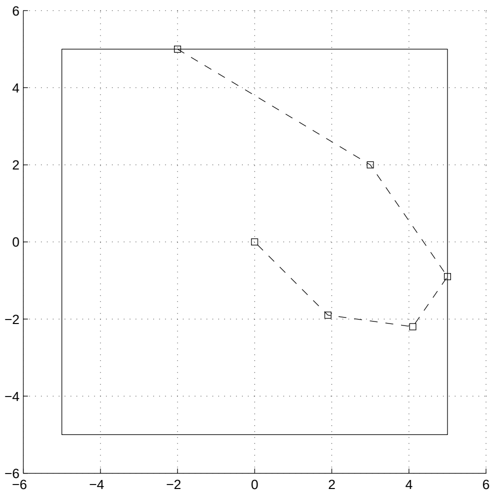

Fig. 3. The optimal trajectory

optimal (open–loop) control and trajectory, found by solving a quadratic–programming problem, are respectively:

$$\bar{U} = \begin{bmatrix} -3 & -2.9 & -1.3 & 0.3 & 1.9 \end{bmatrix}$$

and

$$\bar{X} = \begin{bmatrix} -2 & 3 & 5 & 4.1 & 1.9 \\ 5 & 2 & -0.9 & -2.2 & -1.9 \end{bmatrix}.$$

The optimal trajectory is reported in Fig.3. By means of Procedure 3.1, the triangulation reported in Fig.4 is obtained; the number of triangles is 12 (including the four triangles in which the darkest region, i.e. $\mathcal{S}_2$, can be split). The piecewise affine control law obtained by applying a control–at–the–vertices strategy inside each of the triangles, as stated above, is relatively optimal, hence is optimal for the nominal initial condition and guarantees convergence and constraint satisfaction for the other initial conditions. In Fig. 4, the trajectories for three non–nominal initial conditions are reported. Note that the number of steps required to reach the origin depends on the triangle the initial state belongs to. Note also that the control law is not defined outside the convex hull of the points of the optimal trajectory and their opposite. There, a control derived by the maximal invariant set $\mathcal{H}_{max} \supseteq \mathcal{S}_N$ (Blanchini, 1999; Gutman and Cwikel, 1987) may be used.

7. CONCLUSIONS

In this paper, a static version of the Relatively Optimal Control (Blanchini and Pellegrino, 2003; Blanchini and Pellegrino, 2004) is proposed. The controller is based on a triangulation of the points of the optimal trajectory and their opposite (an upper bound on the number of simplices is provided). The proposed control can deal effectively with hard constraints (a significant advantage with respect to the dynamic one previously introduced). Further work includes extending the

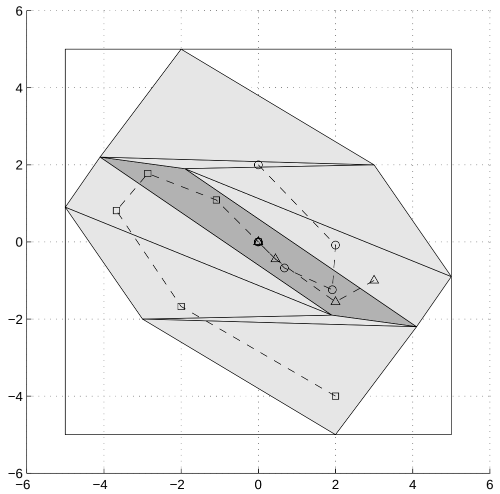

Fig. 4. The triangulation induced by the optimal trajectory and the trajectories from three non–nominal initial conditions

results to more than one initial condition and exploiting the particular structure of the triangulation in order to obtain a tighter bound on the number of simplices.

REFERENCES

Bemporad, A., F. Borrelli and M. Morari (2002*a*). Model predictive control based on linear programming—the explicit solution. *IEEE Trans. on Autom. Contr.* **47**, 1974–1985.

Bemporad, A., M. Morari, V. Dua and E.N. Pistikopoulos (2002*b*). The explicit linear quadratic regulator for constrained systems. *Automatica* **38**, 3–20.

Blanchini, F. (1999). Set invariance in control – a survey. *Automatica* **35**, 1747–1767.

Blanchini, F. and F.A. Pellegrino (2003). Relatively optimal control and its linear implementation. *IEEE Trans. on Autom. Contr.* **48**, 2151–2162.

Blanchini, F. and F.A. Pellegrino (2004). Relatively optimal control with characteristic polynomial assignment and output feedback. *Proceedings of the 43rd IEEE Conference on Decision and Control, Paradise Island, Bahamas.*

DeLoera, J.A., J. Rambau and F. Santos (n.d.). *Triangulations Of Point Sets*. monograph in preparation.

Gutman, P.O. and M. Cwikel (1987). An algorithm to find maximal state constraint sets for discrete–time linear dynamical systems with bounded control and states. *IEEE Trans. on Autom. Contr.* **32**, 251–254.

Ziegler, G.M. (1994). *Lectures on Polytopes*. Springer-Verlag.