INVARIANT SET-BASED ROBUST SOFTLY SWITCHED MODEL PREDICTIVE CONTROL

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Abstract: It is common that an efficient constrained plant operation under full range of disturbance inputs requires meeting different sets of control objectives. This calls for application of multiple model predictive controllers each of them being best fit into specific operating conditions. It is inevitable then to switch between the controllers during the plant operation. A simple immediate hard switching may introduce unwanted transients and also it may not achieve robustly feasible controller operation. A soft switching strategy distributes the switching process over a certain time period and consequently it smoothes the transient processes and achieves the robust feasibility. The softly switched model predictive control scheme based on the invariant set theory is proposed for the additive set bounded and polytopic models of uncertainty. Simulation examples are given. *Copyright © 2005 IFAC*

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1. INTRODUCTION

The success of Model Predictive Control (MPC) is mainly due to its ability to handle multivariable constrained systems in such a way that constraints can be directly incorporated in the optimization problems. It has been known that the soft constraints method (Scokaert and Rawlings, 1999) can effectively handle constraints for deterministic systems. The safety zone approach (Brdys and Chang, 2002) and control invariant set approach (Kerrigan, 2000; Grieder, *et al.*, 2003) address the state constraint satisfaction for uncertain systems. Although the safety zone approach adapts the feasibility requirements imposed on model based optimisation problem to current and predicted operating conditions at every time step, it incurs tremendous online computation effort.

With the increasing use of MPC for large scale and complex hierarchical systems, the demand to use multiple MPC controllers to meet different control objectives under different operating scenarios arises (Grochowski, *et al.*, 2004). A hard switch is the most direct way to engage a new MPC controller without any intermediate switching process. Although the hard switching is very simple, it does not usually give satisfactory outcome because the switching is in such a sudden way that it can probably cause some unexpected impulsive phenomena such as huge peak overshoot of state/output, abrupt change of control input and even actuator failure if the control objectives of the two MPC controllers greatly differ. These are very likely to occur especially under the scenarios where system constraints significantly change. Moreover, if the loose constraints are to be replaced by the tight ones, a robustly feasible operation of the new MPC controller may be impossible. It is because moving in one control step the system state reached by old MPC into a suitable set, in order to guarantee the robustly feasible operation of the new MPC, may require not available control action. The paper proposes soft switching strategies that achieve new MPC controller fully engaged by distributing the switching process over a number of control steps. The transient processes due to switching are smoothed and after the soft switching has been completed the robustly feasible operation of the new MPC controller will be guaranteed. The resulting overall controller is called the softly switched MPC controller (SS-MPC).

In this paper, invariant set theory (Blanchini, 1999; Dorea and Hennet, 1999) is used to design state constraints for the intermediate controllers during the switching process. Invariant set theory was firstly investigated in the last sixties and it was shown that constraints can be satisfied if and only if the initial state is contained in a positively invariant set or control invariant set for the closed-loop system (Kerrigan, 2000). The invariant set theory was very successful in providing sufficient nominal and robust feasibility conditions in model predictive control (Mayne, 2000; Kerrigan and Maciejowski, 2001; Grieder, *et al.*, 2003).

The paper is organized as follows. The problem of interest is formulated in Section 2. Section 3 recalls necessary background material on invariant set theory. The soft switching algorithms are derived in Section 4 based on the results of Section 3. Section 5 presents some simulation results. Section 6 concludes the paper.

2. PROBLEM FORMULATION

In this paper, two main classes of uncertain linear systems are considered: the system with bounded additive uncertainty and the system with polytopic uncertainty (Bemporad and Morari, 1999).

The discrete linear time-invariant (LTI) system with bounded additive disturbances is formulated as:

$$x_{k+1} = Ax_k + Bu_k + w_k , \qquad (1)$$

where $k \in \mathbb{Z}^+$, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $w_k \in \mathbf{W} \subset \mathbb{R}^d$ are the system time step, system state, control input and unknown disturbance input respectively and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. $\mathbf{W}$ is a closed and bounded polyhedral set.

The LTI system with polytopic uncertainty is:

$$x_{k+1} = \tilde{A}x_k + \tilde{B}u_k \qquad (2a)$$

$$[\tilde{A},\tilde{B}] \in \mathbf{\Omega} \triangleq Co\{[A_1,B_1],\cdots,[A_l,B_l]\} . \qquad (2b)$$

$\tilde{A}$ and $\tilde{B}$ are unknown real plant parameters and Co denotes a convex hull. Since $[\tilde{A},\tilde{B}] \in \mathbf{\Omega}$, there exist l nonnegative coefficients $\mu_1, \cdots \ \mu_l$ such that

$$[\tilde{A},\tilde{B}] = \sum_{j=1}^{l} \mu_j [A_j, B_j], \qquad \sum_{j=1}^{l} \mu_j = 1 \qquad (2c)$$

In order to simplify the notation, a nominal model for the above two kinds of uncertain systems is generally represented as:

$$x_{k+1} = f_m(x_k, u_k) . \qquad (3)$$

The mapping (3) can be obtained by choosing the nominal values of w_k in the set $\mathbf{W}$ for (1) or choosing the nominal values of unknown system parameters in the convex hull $\mathbf{\Omega}$ for (2).

The constraints on states and control inputs are as:

$$x_k \in \mathbf{X} \subset \mathbb{R}^n , \quad u_k \in \mathbf{U} \subset \mathbb{R}^m , \qquad (4)$$

where $\mathbf{X}$ and $\mathbf{U}$ are convex polyhedral sets.

The finite horizon MPC optimisation problem based on a nominal performance index is formulated as:

$$\min_{u_{(\cdot|k)}} J_N(x_k, u_{(\cdot|k)}) = \min_{u_{(\cdot|k)}} \sum_{i=0}^{N-1} \begin{bmatrix} x_{k+i|k}^T Q x_{k+i|k} + \\ u_{k+i|k}^T R u_{k+i|k} \end{bmatrix} + x_{k+N|k}^T P x_{k+N|k} \qquad (5a)$$

subject to: $x_{k|k} = x_k$ (5b)

$$x_{k+i+1|k} = f_m(x_{k+i|k}, u_{k+i|k}), \quad \forall i \in \overline{0:N-1} \qquad (5c)$$

$$x_{k+i|k} \in \mathbf{X} \subseteq \mathbb{R}^n \qquad \forall i \in \overline{1:N} \qquad (5d)$$

$$u_{k+i|k} \in \mathbf{U} \subseteq \mathbb{R}^m \qquad \forall i \in \overline{0:N-1} \qquad (5e)$$

where Q, P are positive semi-definite matrices and R is positive definite matrix. $x_{k+i|k}$ and $u_{k+i|k}$ denote the predicted state and control sequences at time k. At each time step the current state x_k is measured and the MPC action $\kappa(x_k) \triangleq u^*_{k|k}(x_k)$ is computed, where $u^*_{k|k}(x_k)$ is the first element of the solution to the optimal control problem (5) and it depends on the initial state x_k at time k (Bemporad, *et al.*, 2000). Notice that a constrained MPC control law Υ is determined by parameters $N, Q, R, P, \mathbf{X}$ and $\mathbf{U}$.

The problem considered here is how to guarantee robust feasibility under the uncertainty described in (1) and (2) when softly switching from an old MPC controller with $\Upsilon(N_1, Q_1, R_1, P_1, \mathbf{X}_{old}, \mathbf{U}_{old})$ to the new one with $\Upsilon(N_2, Q_2, R_2, P_2, \mathbf{X}_{new}, \mathbf{U}_{new})$. The proposed switching process is performed in such a soft manner that all the parameters of the old MPC controller change gradually to those of the new MPC controller. It should be noticed that the new MPC differs from the old one not only in the parameters of performance indices, but also in the constraints.

3. ROBUST FEASIBILTY BASED ON INVARIANT SETS

In this section, selected necessary MPC invariant set definitions are introduced based on e.g. (Kerrigan and Maciejowski, 2001; Grieder, *et al.*, 2003) and the algorithms to compute these sets are presented.

Definition 1 (Open-loop feasible set X_f).
The open-loop feasible set X_f is the set of all the states x_k for which a feasible control sequence to problem (5) exists, i.e.

$$X_f = \{ x_k \in \mathbb{R}^n \mid \exists U^* = [u^*_{k|k}; \cdots ; u^*_{k+N-1|k}] : \; x_{k+i|k} \in \mathbf{X}, u^*_{k+i-1|k} \in \mathbf{U}, \forall i \in \overline{1:N} \} \qquad (6)$$

Remark 1. Necessary and sufficient condition for an initial state to be feasible for the optimization problem (5) is that it belongs to X_f. However, the set

X_f only addresses the open-loop case and does not take into account the system uncertainty. Hence, meeting the state constraints in the real system is not guaranteed.

Definition 2 (Closed-loop feasible set $\tilde{X}_f$).
The closed-loop feasible set $\tilde{X}_f$ is the set of all the states x_k from which the evolution of (1) or (2) under the control law $\kappa(x_k) = u^*_{k|k}(x_k)$ is always feasible for the model based optimization problem (5) and also for the real system. That is for uncertain system described by (1), $\tilde{X}_f$ is such a set that:

$$\tilde{X}_f \triangleq \{ x_k \in X_f \cap \mathbf{X} \mid (Ax_{k+i} + B\kappa(x_{k+i}) + w_{k+i}) \in X_f \cap \mathbf{X} \quad \forall w_{k+i} \in \mathbf{W},\ \forall i \in \overline{0:\infty} \} \tag{7}$$

For uncertain system described by (2) $\tilde{X}_f$ is such a set that:

$$\tilde{X}_f \triangleq \{ x_k \in X_f \cap \mathbf{X} \mid (\bar{A}x_{k+i} + \bar{B}\kappa(x_{k+i})) \in X_f \cap \mathbf{X} \quad \forall (\bar{A}, \bar{B}) \in \mathbf{\Omega} \wedge \forall i \in \overline{0:\infty} \} \tag{8}$$

Remark 2. The closed-loop feasible set $\tilde{X}_f$ is the maximal control invariant set by the definition. It is the subset of the set $X_f \cap \mathbf{X}$, i.e. $\tilde{X}_f \subseteq X_f \cap \mathbf{X}$. $\tilde{X}_f$ includes all the initial states x_k which can guarantee feasibility of the optimization problem (5) for all the time along the system evolution of (1) and (2) from x_k. In other words, once $x_k \in \tilde{X}_f$, the optimization problem (5) will be feasible all the time in the future for the systems (1) and (2).

It was shown in (Bemporad, *et al*., 2000) that the expression of the explicit solutions to the MPC control problem (5) with polyhedral constraints is piecewise affine (PWA), i.e.

$$\kappa(x_k) = F_r x_k + G_r \quad \text{if}$$
$$x_k \in \Pi_r = \{ x \in \mathbb{R}^n \mid H_r x \le M_r \},\ r = 1, \cdots, R$$

The union of all Π_r is equal to the feasible set:

$$X_f = \bigcup_1^R \Pi_r . \tag{9}$$

The closed-loop feasible set $\tilde{X}_f$ of the associated MPC for the uncertain system (1) and (2) can be computed separately by using the following algorithms similar to (Grieder, *et al*., 2003):

Algorithm 1 (bounded additive uncertainty).

1) Let $\tilde{X}_f = \mathbf{X}$.
2) Add an extra constraint to the optimization problem (5) to let it incorporate uncertainty:
$$x_{k+1|k} \in \tilde{X}_f \sim \mathbf{W} .$$
3) Solve the optimization problem (5) with the constraint in 2) by using mp-QP technique (Bemporad, *et al*., 2000) to obtain the union of the state-space partitions $\tilde{X}_f = \bigcup_1^R \Pi_r$.
4) Repeat steps 2) and 3) until $\tilde{X}_f$ stops shrinking.

where $\sim$ represents the Pontryagin set difference, i.e.

$$\mathbf{Y} \sim \mathbf{Z} = \{ m \in \mathbb{R}^n : m + z \in \mathbf{Y}, \forall z \in \mathbf{Z} \}$$

Algorithm 2 (polytopic uncertainty).

1) Let $\tilde{X}_f = \mathbf{X}$.
2) Add extra constraints to the optimization problem (5) to let it incorporate uncertainty by checking each vertex of the polytope:
$$x^j_{k+1|k} = A_j x_{k|k} + B_j u_{k|k} \quad \forall j \in \overline{1:l}$$
$$x^j_{k+1|k} \in \tilde{X}_f \cap \mathbf{X},\ \forall j \in \overline{1:l} .$$
3) Solve the optimization problem (5) with the constraint in 2) by using mp-QP technique to obtain the union of the state-space partitions:
$$\tilde{X}_f = \bigcup_1^R \Pi_r .$$
4) Repeat steps 2) and 3) until $\tilde{X}_f$ stops shrinking.

Remark 3. The sets $\tilde{X}_f$ computed by using either *Algorithm 1* or *Algorithm 2* is an invariant set if it is finitely determined, because in *Algorithm 1*, $\forall k \in \mathbb{Z}^+$

$$\forall x_k \in \tilde{X}_f \Rightarrow x_{k|k} \in \tilde{X}_f \Rightarrow x_{k+1|k} \in \tilde{X}_f \sim \mathbf{W}$$
$$\Rightarrow x_{k+1} \in \tilde{X}_f .$$

In *Algorithm 2*, $\forall k \in \mathbb{Z}^+$

$$\forall x_k \in \tilde{X}_f \Rightarrow x_{k|k} \in \tilde{X}_f \Rightarrow x^j_{k+1|k} \in \tilde{X}_f .$$

Due to the fact that the convex combination of all the vertices of the polytope still remains in that polytope, the following relation holds:

$$x^j_{k+1|k} \in \tilde{X}_f \Rightarrow x_{k+1|k} \in \tilde{X}_f \Rightarrow x_{k+1} \in \tilde{X}_f .$$

In general, *Algorithm 1* and *Algorithm 2* cannot be ensured to terminate in finite time, but confining the iteration times and defining precision degree are usual ways to obtain an approximate subset of $\tilde{X}_f$.

4. SOFTLY SWITCHED MPC

In the proposed SS-MPC, a sequence of intermediate combined MPC controllers linking the old and the new MPC controllers $\Upsilon(N_1, Q_1, R_1, P_1, \mathbf{X}_{old}, \mathbf{U}_{old})$ and $\Upsilon(N_2, Q_2, R_2, P_2, \mathbf{X}_{new}, \mathbf{U}_{new})$ are applied to the plant during the soft switching process. At each new time step during the soft switching a different combined controller is applied to the plant with different performance indices, input and state constraints. For simplicity, the old and the new MPC controllers are assumed to have the same length of prediction horizon, that is $N \triangleq N_1 = N_2$.

4.1 Design of combined MPC controller

The fundamental idea behind the combined MPC controller design is to use a sort of combination of the old MPC controller and the new one, which is actually made up in two aspects. The first is to build the performance index of the combined controller by combining the performance indices of the two controllers (Grochowski, 2003; Grochowski, *et al.*, 2004; Wang, *et al.*, 2005). The second is to properly design the constraints of the combined MPC.

Performance index of the combined MPC controller is designed as follows:

$$J_{Combined}(x_k, u_{(\cdot|k)}) = \sum_{i=0}^{N-1} \begin{Bmatrix} x_{k+i|k}^T [w_1^k(i)Q_1 + w_2^k(i)Q_2] x_{k+i|k} + \\ u_{k+i|k}^T [w_1^k(i)R_1 + w_2^k(i)R_2] u_{k+i|k} \end{Bmatrix} + x_{k+N|k}^T [w_1^k(N)P_1 + w_2^k(N)P_2] x_{k+N|k} \quad (10)$$

where w_1^k and w_2^k are called *weighting vectors*. Let the switching starts at $k = k_s$, the values of w_1^k and w_2^k can be determined by the following algorithm (Wang, *et al.*, 2005):

Algorithm 3.

- If $k+i < N+k_s$, $w_1^k(i) = \lambda^{k-k_s+i}$, $\forall i \in \overline{0:N-1}$ (11a)
- If $k+i \geq N+k_s$, $w_1^k(i) = 0$, $\forall i \in \overline{0:N-1}$ (11b)
- $w_1^k(N) = w_1^k(N-1)$ (11c)
- $w_2^k(i) = 1 - w_1^k(i)$ $\quad \forall i \in \overline{0:N}$ (11d)

where λ is a tuning knob whose value is between 0 and 1.

The most direct way to design constraints for the combined MPC is to combine the old constraints and new constraints in a convex manner:

$$\mathbf{X}_{combined} \triangleq (1-\alpha)\mathbf{X}_{old} + \alpha\mathbf{X}_{new} \quad (12a)$$

$$\mathbf{U}_{combined} \triangleq (1-\alpha)\mathbf{U}_{old} + \alpha\mathbf{U}_{new}, \quad (12b)$$

where $0 \leq \alpha \leq 1$ is a tuning knob. During the switching process α varies from 0 to 1 moving the MPC optimisation problem (5) from the old MPC form to the new one.

Thus, at time step k during the soft switching process, the combined MPC optimisation problem reads:

$$\min_{u_{(\cdot|k)}} J_{Combined}(x_k, u_{(\cdot|k)}) \quad (13a)$$

subject to:

$$x_{k|k} = x_k \quad (13b)$$

$$x_{k+i+1|k} = f_m(x_{k+i|k}, u_{k+i|k}) \quad \forall i \in \overline{0:N-1} \quad (13c)$$

$$x_{k+i|k} \in \mathbf{X}_{combined} \quad \forall i \in \overline{0:N} \quad (13d)$$

$$u_{k+i|k} \in \mathbf{U}_{combined} \quad \forall i \in \overline{0:N-1} \quad (13e)$$

4.2 Design of soft switching process

By ensuring robust feasibility of a whole switching process it is meant that optimisation problems of all the combined controllers have feasible solutions and the combined state constraints in the real system are satisfied. It has been shown in Section 3 that this can be achieved by maintaining for each combined MPC controller the initial state in the corresponding invariant set $\tilde{X}_f$. The control invariant set based design is illustrated in Fig. 1.

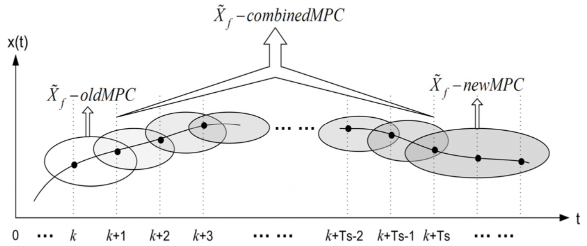

Fig. 1. Soft switching design scheme guaranteeing robust feasibility of the switching process: the ellipsoids from left to right separately represent the corresponding closed-loop feasible sets of the old MPC controller, combined MPC controllers and new MPC controller.

In Fig. 1 the switching starts at time k. In order for the first combined MPC to be feasible, an effective method is to force the initial state at $k+1$ to enter the intersection of the closed-loop feasible sets of the old MPC and the first combined MPC controller.

For bounded additive uncertainty, an extra constraint (14f) is added to the old MPC optimisation problem to produce a feasible initial state for the first combined MPC controller:

$$\min_{u_{(\cdot|k)}} J_{oldMPC}(x_k, u_{(\cdot|k)}) = \min_{u_{(\cdot|k)}} \sum_{i=0}^{N-1} \begin{bmatrix} x_{k+i|k}^T Q_1 x_{k+i|k} + \\ u_{k+i|k}^T R_1 u_{k+i|k} \end{bmatrix} + x_{k+N|k}^T P_1 x_{k+N|k} \quad (14a)$$

subject to:

$$x_{k|k} = x_k \quad (14b)$$

$$x_{k+i+1|k} = f_m(x_{k+i|k}, u_{k+i|k}) \quad \forall i \in \overline{0:N-1} \quad (14c)$$

$$x_{k+i|k} \in \mathbf{X}_{old} \quad \forall i \in \overline{0:N} \quad (14d)$$

$$u_{k+i|k} \in \mathbf{U}_{old} \quad \forall i \in \overline{0:N-1} \quad (14e)$$

$$x_{k+1|k} \in \tilde{X}_f^{CombinedMPC(1)} \sim \mathbf{W} \quad (14f)$$

where $\tilde{X}_f^{CombinedMPC(1)}$ represents the closed-loop feasible set of the first combined MPC controller.

In the case of polytopic uncertainty, the constraints in the optimization problem (14) are modified as:

$$x_{k|k} = x_k \quad (15a)$$

$$x_{k+i+1|k} = f_m(x_{k+i|k}, u_{k+i|k}) \quad \forall i \in \overline{0:N-1} \quad (15b)$$

$$x_{k+i|k} \in \mathbf{X}_{old} \quad \forall i \in \overline{0:N} \quad (15c)$$

$$u_{k+i|k} \in \mathbf{U}_{old} \quad \forall i \in \overline{0:N-1} \quad (15d)$$

$$x_{k+1|k}^j = A_j x_{k|k} + B_j u_{k|k} \quad \forall j \in \overline{1:l} \quad (15e)$$

$$x_{k+1|k}^j \in \tilde{X}_f^{CombinedMPC(1)} \cap \mathbf{X}_{old} \quad \forall j \in \overline{1:l} \quad (15f)$$

Applying the same idea to ensure robust feasibility of the second combined MPC, the optimisation problem at time $k+1$ is formulated as follows for bounded additive uncertainty:

$$\min_{u_{(\cdot|k+1)}} J_{CombinedMPC}(x_{k+1}, u_{(\cdot|k+1)}) = \min_{u_{(\cdot|k+1)}} \sum_{i=0}^{N-1} \begin{bmatrix} x_{k+i+1|k+1}^T Q_{CombinedMPC(1)} x_{k+i+1|k+1} \\ + u_{k+i+1|k+1}^T R_{CombinedMPC(1)} u_{k+i+1|k+1} \end{bmatrix}$$

$$+x^T_{k+N+1|k+1} P_{CombinedMPC(1)} x_{k+N+1|k+1} \quad (16a)$$

subject to: $x_{k+1|k+1} = x_{k+1}$ (16b)

$$x_{k+i+2|k+1} = f_m(x_{k+i+1|k+1}, u_{k+i+1|k+1}) \ \forall i \in \overline{0:N-1} \quad (16c)$$

$$x_{k+i+1|k+1} \in \mathbf{X}_{CombinedMPC(1)} \ \forall i \in \overline{0:N} \quad (16d)$$

$$u_{k+i+1|k+1} \in \mathbf{U}_{CombinedMPC(1)} \ \forall i \in \overline{0:N-1} \quad (16e)$$

$$x_{k+2|k+1} \in \tilde{X}_f^{CombinedMPC(2)} \sim \mathbf{W} \quad (16f)$$

where $Q_{CombinedMPC(1)}$, $R_{CombinedMPC(1)}$ and $P_{CombinedMPC(1)}$ are the weights on state, input and terminal state for the first combined MPC. $\mathbf{X}_{CombinedMPC(1)}$ and $\mathbf{U}_{CombinedMPC(1)}$ are the designed state and input constraints (see (12)).

In the case of polytopic uncertainty, the constraints in the optimization problem (16) are modified as:

$$x_{k+1|k+1} = x_{k+1} \quad (17a)$$

$$x_{k+i+2|k+1} = f_m(x_{k+i+1|k+1}, u_{k+i+1|k+1}) \ \forall i \in \overline{0:N-1} \quad (17b)$$

$$x_{k+i+1|k+1} \in \mathbf{X}_{CombinedMPC(1)} \ \forall i \in \overline{0:N} \quad (17c)$$

$$u_{k+i+1|k+1} \in \mathbf{U}_{CombinedMPC(1)} \ \forall i \in \overline{0:N-1} \quad (17d)$$

$$x^j_{k+2|k+1} = A_j x_{k+1|k+1} + B_j u_{k+1|k+1} \quad \forall j \in \overline{1:l} \quad (17e)$$

$$x^j_{k+2|k+1} \in \tilde{X}_f^{CombinedMPC(2)} \cap \mathbf{X}_{CombinedMPC(1)} \quad (17f)$$

Remark 4. Note that the computation of the sets is carried out offline. The combined MPC controllers (13) are actually modified since extra constraints are added. Hence, one needs to compute the closed-loop feasible sets of these modified combined controllers, e.g. (16) and (17). In order to obtain $\tilde{X}_f^{CombinedMPC(1)}$, one needs to compute $\tilde{X}_f^{CombinedMPC(2)}$ in advance since the optimization problem of the first combined MPC controller has already included the information about the second combined MPC controller. Hence, the closed-loop feasible sets of the combined controllers have to be computed backwards in time from the last controller to the first one.

Suppose the duration time of the switching process is *Ts* time steps. Before the switching process ends, in order to have a feasible initial state for the new MPC controller, the optimization problem at $k + Ts$-1 is:

$$\min_{u_{(\cdot|k+Ts-1)}} J_{CombinedMPC}(x_{k+Ts-1}, u_{(\cdot|k+Ts-1)}) =$$

$$\min_{u_{(\cdot|k+Ts-1)}} \sum_{i=0}^{N-1} \begin{bmatrix} x^T_{k+i+Ts-1|k+Ts-1} Q_{CombinedMPC(Ts-1)} x_{k+i+Ts-1|k+Ts-1} \\ +u^T_{k+i+Ts-1|k+Ts-1} R_{CombinedMPC(Ts-1)} u_{k+i+Ts-1|k+Ts-1} \end{bmatrix}$$

$$+x^T_{k+N+Ts-1|k+Ts-1} P_{CombinedMPC(Ts-1)} x_{k+N+Ts-1|k+Ts-1} \quad (18a)$$

subject to: $x_{k+Ts-1|k+Ts-1} = x_{k+Ts-1}$ (18b)

$$x_{k+i+Ts|k+Ts-1} = f_m(x_{k+i+Ts-1|k+Ts-1}, u_{k+i+Ts-1|k+Ts-1}) \quad (18c)$$

$$x_{k+i+Ts-1|k+Ts-1} \in \mathbf{X}_{CombinedMPC(Ts-1)} \ \forall i \in \overline{0:N} \quad (18d)$$

$$u_{k+i+Ts-1|k+Ts-1} \in \mathbf{U}_{CombinedMPC(Ts-1)} \ \forall i \in \overline{0:N-1} \quad (18e)$$

$$x_{k+Ts|k+Ts-1} \in \tilde{X}_f^{newMPC} \sim \mathbf{W} \quad (18f)$$

In the case of polytopic uncertainty, the constraints in the optimization problem (18) are modified as:

subject to: $x_{k+Ts-1|k+Ts-1} = x_{k+Ts-1}$ (19a)

$$x_{k+i+Ts|k+Ts-1} = f_m(x_{k+i+Ts-1|k+Ts-1}, u_{k+i+Ts-1|k+Ts-1}) \quad (19b)$$

$$x_{k+i+Ts-1|k+Ts-1} \in \mathbf{X}_{CombinedMPC(Ts-1)} \ \forall i \in \overline{0:N} \quad (19c)$$

$$u_{k+i+Ts-1|k+Ts-1} \in \mathbf{U}_{CombinedMPC(Ts-1)} \ \forall i \in \overline{0:N-1} \quad (19d)$$

$$x^j_{k+Ts|k+Ts-1} = A_j x_{k+Ts-1|k+Ts-1} + B_j u_{k+Ts-1|k+Ts-1} \quad (19e)$$

$$x^j_{k+Ts|k+Ts-1} \in \tilde{X}_f^{newMPC} \cap \mathbf{X}_{CombinedMPC(Ts-1)} \quad (19f)$$

After the soft switching process has been completed, the new MPC will be robustly feasible since the initial state of the new controller has entered its closed-loop feasible invariant set. Notice that if $T_s = 1$ then robustly feasible hard switching is performed. However, if the new and old state constraints significantly differ and the new control constraints are tight then the sets (18) or (19) with $T_s = 1$ are likely to be empty. Moving system state reached by the old MPC into a closed-loop feasible set of the new MPC, in order to guarantee robustly feasible operation of the new MPC, in one control step may be then impossible. The proposed soft switching strategy achieves this by distributing the state transfer over a number of control steps. Tuning the parameter α should ensures that the feasible sets of two neighbouring MPC controllers have nonempty intersection; hence the robustly feasible state transfer is possible.

5. NUMERICAL EXAMPLES

In this section, numerical examples of the above designed SS-MPC scheme are presented by using the Multi-Parametric Toolbox (MPT) provided by the Automatic Control Laboratory at the Swiss Federal Institute of Technology.

Consider a second order discrete LTI system:

$$A = \begin{bmatrix} 0.7326 & -0.1722 \\ 0.0861 & 0.9909 \end{bmatrix} B = \begin{bmatrix} 0.0861 \\ 0.0045 \end{bmatrix} C = [0 \ \ 1] \ \ D = 0$$

The control problem is to regulate the system from the old control objective to the new control objective by switching from the old MPC controller to the new MPC controller.

The prediction horizons of the MPC controllers are both equal to $N = 5$. The old controller parameters are $Q = 10I$, $R = 1$ and $P = 5000$. The target set-point is 6. The constraints related to the old controller are as follows:

output constraint : $-7 \le y \le 7$ and $-1 \le \Delta y \le 1$
input constraint: $-10 \le u \le 10$ and $-0.8 \le \Delta u \le 0.8$

The new MPC controller parameters are $Q = 1I$, $R = 1$ and $P = 500$. The new set point is 3 and the new constraints are as follows:

output constraint : $-4 \le y \le 4$ and $-0.5 \le \Delta y \le 0.5$
input constraint: $-10 \le u \le 10$ and $-0.4 \le \Delta u \le 0.4$

The tuning parameter $\lambda = 2/3$ and the weighting vectors w_1^k and w_2^k are shifted one bit leftward at each new time step. Hence, it takes 5 time steps to complete the soft switching process. Values of the tuning parameter α for the combined MPC controllers are $[0/5, 1/5, 2/5, 3/5, 4/5]$ in the sequence.

First, an additive bounded disturbance was added to the above system with

$$\mathbf{W} = \left\{ w \in \mathbb{R}^d \;\middle|\; \|w\|_\infty \le 0.1 \right\}.$$

The closed-loop feasible invariant sets containing the real system trajectory generated during the switching process are illustrated in Fig. 2.

Figure 3 illustrates application results of the SS-MPC developed in the paper to the example of polytopic uncertainty with $l = 2$ and $\mu = 0.1$:

$$[A_1, B_1] = [(1-\mu)A, B], [A_2, B_2] = [(1+\mu)A, B]$$

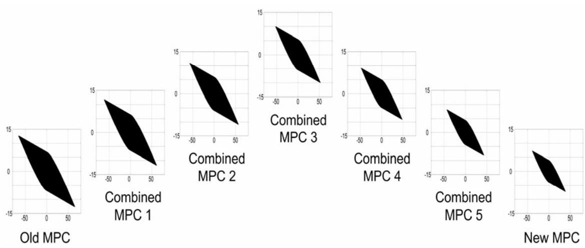

Fig. 2. Softly switched model predictive control with bounded additive uncertainty: asymptotic change of the invariant closed-loop feasible set

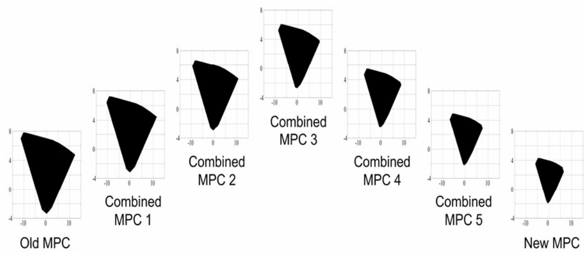

Fig. 3. Softly switched model predictive control with polytopic uncertainty: asymptotic change of the invariant closed-loop feasible set

6. CONCLUSIONS

This paper has considered soft switching between multiple MPC controllers. The soft switching rather than simple hard switching is required in order to avoid unwanted transients and to achieve robustly feasible operation. An invariant set theory has been applied to design a robustly feasible softly switched MPC. Online tuning of the soft switching mechanism is under research.

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REFERENCES

Bemporad, A. and M. Morari (1999). Robust model predictive control: a survey. *Robustness in Identification and Control*, **245**, 207–226.

Bemporad, A., M. Morari, V. Dua and E.N. Pistikopoulos (2000). The explicit solution of model predictive control via multiparametric quadratic programming. *Proc. of the Amecican Control Conference, Chicago*.

Blanchini, F. (1999). Set Invariance in control. *Automatica*, **35** (11) 1747–1767.

Brdys, M.A. and T. Chang (2002). Robust model predictive control under output constraints. *Proc. of the 15th IFAC World Congress*, *Barcelona*.

Dorea, C. and J.C. Hennet (1999). (A,B) – invariant polyhedral sets of linear discrete-time systems. *Journal of Optimization Theory and Application*. **103**(3), 521–542.

Grieder, P., P.A. Parrilo, and M. Morari (2003). Robust receding horizon control – analysis and synthesis. *Proc. of 42th IEEE Conference on Decision and Control*, *Hawaii*.

Grochowski, M. (2003). Intelligent control of integrated wastewater treatment system under full range of operating conditions. PhD thesis, Gdansk University of Technology, Poland.

Grochowski, M., M.A. Brdys, T. Gmiński and P. Deinrych (2004). Softly switched model predictive control for control of integrated wastewater treatment systems at medium time scale. *10th IFAC/IFORS/IMACS/IFIP Symposium on Large Scale Complex Systems, Osaka, Japan*.

Kerrigan, E.C. (2000). Robust constraints satisfaction: invariant sets and predictive control. PhD thesis, The University of Cambridge, UK.

Kerrigan, E.C. and J.M. Maciejowski (2001). Robust feasibility in model predictive control: necessary and sufficient conditions. *Proc. of 40th IEEE Conference on Decision and Control*, *Florida*.

Mayne, D.Q., J.B. Rawlings, C.V. Rao and P.O.M. Scokaert (2000). Constrained model predictive control: stability and optimality. *Automatica*, **36**(6), 789–814.

Scokaert, P.O.M. and J.B. Rawlings (1999). Feasibility issues in linear model predictive control. *AIChE Journal*, **45**(8), 1649–1659.

Wang, J., M. Grochowski and M.A. Brdys (2005). Analysis and design of softly switched model predictive control. *16th IFAC World Congress*, *Prague*, *Czech Republic*, to be published.