H_∞ CONTROL FOR NONLINEAR STOCHASTIC SYSTEMS: THE OUTPUT-FEEDBACK CASE

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Abstract: In this paper we develop an H_∞ control theory, from the dissipation point of view, for a large class of time-continuous, stochastic, nonlinear, time-invariant systems with output-feedback. In particular, we introduce a notion of stochastic dissipative systems, analogously to the familiar notion of dissipation associated with deterministic systems and we utilize it as a basis for the development of the theory. In particular, we utilize the stochastic version of what is called Bounded Real Lemma (BRL) to synthesize an output-feedback controller. It is shown that this controller makes the resulting closed-loop system dissipative. Stability, in probability and in the mean square sense, is discussed and sufficient conditions for achieving the stability and the H_∞ performance are introduced.

Keywords: Stochastic systems, Nonlinear systems, Disturbance attenuation, H_∞ control.

1. INTRODUCTION

In recent years there has been a growing interest, as reflects from the various published research works, in the extension of of H_∞ control and estimation theory to accommodate stochastic systems (see e.g. [El Ghaoui, 1995], [Costa and Kubrusly, 1996], [Dragan and Morozan, 1997], [Dragan and Morozan, 1998], [Henriechsen and Pritchard, 1998], [Gershon, *et al.*, 2001], [Gershon, *et al.*, 2002]).

The main thrust for these efforts stems from the attempt to model system uncertainties as a stochastic process, in particular, as a white noise, or formally as a Wiener process. This has led to the development of a H_∞ theory for stochastic linear systems with multiplicative noise.

The present paper is an extension of the state-feedback case ([Berman and Shaked, 2003]) to the case of output-feedback control. In particular, we extend some of the H_∞ theory to nonlinear stochastic systems of the following form.

$$\begin{aligned} dx_t = & f(x_t)dt + g(x_t)u_t dt + g_1(x_t)v_t dt \\ & + \bar{g}(x_t)u_t dW_t + g_2(x_t)v_t dW_t^2 + G(x_t)dW_t^1 \end{aligned} \tag{1}$$

$$dy_t = h_2(x_t)dt + g_3(x_t)v_t dt + G_2(x_t)dW^3 \tag{2}$$

where $\{x_t\}_{t\geq 0}$ is a solution to (1) with the initial condition x_0, an exogenous disturbance $\{v_t\}_{t\geq 0}$, a control signal $\{u_t\}_{t\geq 0}$, and Wiener processes $\{W_t\}_{t\geq 0}$, $\{W_t^1\}_{t\geq 0}$, $\{W_t^2\}_{t\geq 0}$, and $\{W_t^3\}_{t\geq 0}$. Also, y_t is an observations vector in R^p which is corrupted with noise (a Wiener process $\{W_t^3\}_{t\geq 0}$), and contains an uncertain component (a stochastic process $\{v_t\}_{t\geq 0}$). The following will be assumed to hold throughout this work.

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1. Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a filtered probability space where $\{F_t\}_{t\geq 0}$ is the the family of sub σ-algebras generated by $\{W_t\}_{t\geq 0}$, where $\{W_t^1\}_{t\geq 0}$, $\{W_t^2\}_{t\geq 0}$ are taken to be R^l- valued and R^1- valued, respectively.

2. All the functions below are assumed to be continuous on R^n. $f : R^n \to R^n$, $g : R^n \to R^{n\times m}$, $g_1 : R^n \to R^{n\times m_1}$, $g_2 : R^n \to R^{n\times m_2}$, $G : R^n \to R^{n\times l}$. In addition, it is assumed that $f(0) = 0$, $G(0) = 0$, $h_2(0) = 0$ and $G_2(0) = 0$.

3. $\{v_t\}_{t\geq 0}$ is a non-anticipative R^{m_1}-valued stochastic process defined on $(\Omega, F, \{F_t\}_{t\geq 0}, P)$, which satisfies $E\{\int_0^t \|v_s\|^2 ds\} < \infty$ for all $t \in [0, \infty)$, where E stands for the expectation operation, that is if x is a random variable defined on the probability space (Ω, F, P), then $E\{x\} = \int_\Omega x(\omega)dP(\omega)$.

4. $\{u_t\}_{t\geq 0}$ is a non-anticipative R^m-valued stochastic process defined on $(\Omega, F, \{F_t\}_{t\geq 0}, P)$.

5. x_0 is assumed to be F_0-measurable, and to satisfy $E\{\|x_0\|^2\} < \infty$.

Definition 1. The pair $\{u_t, v_t\}_{t\in[(0,\infty)}$, or in short $\{u, v\}$, are said to be admissible if the stochastic differential equation (1) possesses a unique strong solution relative to to the filtered probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ so that $E\{\|x_t\|^2\} < \infty$ for all $t \in [0, \infty)$.

Remark 1. The family of all admissible pairs $\{u, v\}$ will be denoted by $\mathcal{A}$. The notation $\mathcal{A}_u$ will be used for all admissible pairs $\{u, v\}$ with fixed u. We note that $\mathcal{A}_u$ may be empty for some non-anticipative u.

Remark 2. It will be assumed, throughout this paper, that $\{u, v\}$ are admissible. For sufficient conditions which guarantee the existence of a unique strong solution to (1) (see e.g. [Gihman and Skorohod, 1972], [Hasminskii, 1980]).

The problem of H_∞ output-feedback control is formulated in the following obvious way.

Given an output which is to be regulated (entitled, in this paper, controlled output):

$$z(t) = col\{h(x_t, t), u_t\} \tag{3}$$

where $h : R^n \to R^r$, synthesize a controller $u_t = u(Y^t)$, $Y^t = \{y_s : s \leq t\}$, such that , for a given $\gamma > 0$, the following H_∞ criterion is satisfied.

$$E\{\int_0^T \|z_t\|^2 dt\} \leq \gamma^2 E\{\|x_0\|^2 + \int_0^T (\|v_t\|^2)dt\}$$

for all $T \geq 0$ and for all disturbances v_t in $\mathcal{A}_u$ (provided $\mathcal{A}_u$ is nonempty). Whenever the system (1) satisfies the above inequality, it is said to have the L_2-gain property, and we also write L_2-gain$\leq \gamma$.

This problem may be treated within the context of stochastic game theory analogously to the utilization of the game theory in the deterministic H_∞ control and estimation (see, e.g. [Basar and Bernhard, 1995]). In this connection, there is also a recent work ([Charalambous, 2003]) that deals with the problem of stochastic minimax dynamic games using the information state concept. The latter work characterizes the information state by means of a solution to a certain Hamilton-Jacobi equation, and it differ from ours, first in its underlying approach, as our is based on the stochastic dissipation notion of a system, and second, in the representation of the observation model, as Charalambous takes the underlying probability space associated with the measurements to be finitely additive.

As we adopt the approach that is based on the notion of stochastic dissipation, we first introduce the concept of stochastic dissipative systems, and then discuss some properties of these systems. This is done in Section 2. In Section 2 we also state and prove some kind of a bounded real lemma for non linear stochastic systems. In particular, we introduce a certain Hamilton-Jacobi inequality (HJI for short) and we establish necessary and sufficient conditions for the HJI to guarantee a dissipation of the underlying system, which in turn implies the $L_2 - gain$ property of the system.

In Section 3 we develop the H_∞ output feedback control theory for non linear stochastic systems which is based on a consequence of the BRL introduced in Section 2. In particular, we synthesize a controller which results from a solution to a certain algebraic HJI and renders the closed-loop system L_2-gain$\leq \gamma$ We also discuss the stability of the closed-loop system. In particular, we establish sufficient conditions under which stability in probability and in the mean square sense is guaranteed.

2. PRELIMINARIES: DISSIPATIVE STOCHASTIC SYSTEMS, AND THE BRL

This section introduces the concept of a dissipative stochastic system which is to be the basis on which we lay out our H_∞ control theory for nonlinear stochastic systems of the type introduced in Section 1.

The notion of dissipative stochastic systems as introduced in this work is in fact a natural extension of the concept of dissipation introduced by [Willems, 1972]) for deterministic systems; it has been utilized in the development of the H_∞ control and estimation theory for nonlinear deterministic systems by several researchers (see e.g. [Ball and Helton, 1996], [James, 1993], [van der Schaft, 2000]).

The concept of dissipative stochastic system is also related to the notion of passive stochastic systems that has been introduced by ([P.Florchinger, 1999]). It is used there as a basis for the development of a theory for stabilizing, in the probability sense, a class of stochastic nonlinear systems which enjoy this passivity property.

Consider the nonlinear stochastic system of (1) together with the controlled output (3). As in the deterministic case (see, e.g. [van der Schaft, 2000],[Helton and James, 1999]), the notion of supply rate will play a fundamental role in the theory of H_∞ control for nonlinear stochastic systems. Define a function $S : R^m \times R^{r+m_1} \to R$, and call it supply rate.

Remark 3. Dealing with H_∞ control, we will be concerned exclusively, in this work with the particular supply rate defined by $S(v,z) = \gamma^2||v||^2 - ||z||^2$, where $(v,z) \in R^m \times R^{r+m_1}$ Using the notion of supply rate, we have now the following definition of dissipative stochastic systems.

Definition 2. Consider the system (1) together with the controlled output as defined by (3), and let S be a supply rate as defined above. Let u be such that $\mathcal{A}_u$ is nonempty. Then, the system (1) is said to be dissipative with respect to the supply rate S if there is a function $V : R^n \to R$, with $V(x) \geq 0$ for all $x \in R^n$, so that $V(0) = 0$ satisfies $E\{V(x_t)\} < \infty$ for all $t \geq 0$ whenever $\{x_t\}_{t\geq 0}$ is a strong solution to (1), and

$$E\{V(x_t)\} \leq E\{V(x_s)\} + E\{\int_s^t (\gamma^2\|v_\sigma\|^2 - \|z_\sigma\|^2)d\sigma\}$$

for all $t \geq s \geq 0$ and for all admissible disturbances $\{v_t\}_{t\geq 0}$ in $\mathcal{A}_u$. V is then called the storage function of the system (1).

Similar to the deterministic theory of dissipative systems, the theorem below (the proof of which may be found in [Berman and Shaked 2003]) establishes conditions under which the system (1) possesses a storage function. First we introduce a candidate for a storage function.

Definition 3. Consider the system (1). Let $t \in [0,\infty)$ and let x_t be an R^n valued random variable defined on the probability space (Ω, F, P). Assume also that x_t is F_t measurable. Let u be such that $\mathcal{A}_u$ is nonempty. Define

$$V_a(x_t) = \sup_{\substack{T \geq t \\ v \in \mathcal{A}_u}} [-E\{[\int_t^T S(v_s, z_s)ds]/x_t\}] \quad (4)$$

Remark 4. We note that in the case where x is deterministic, V_a assumes the following form:

$$V_a(x_t) = \sup_{T \geq t,\ v \in \mathcal{A}_u} [-E\{\int_t^T S(v_s, z_s)ds\}] \quad (5)$$

Theorem 1. The function V_a of the above definition is a storage function for the system (1) (or equivalently, the system (1) is dissipative with respect to the supply rate S) iff $E\{V_a(x_t)\}$ is finite for all $t \in [0,\infty)$. The proofs of the next lemma and the following theorem (entitled Bounded Real Lemma or BRL, in short) may be found in [Berman and Shaked, 2003] and therefore are omitted.

Lemma 1. Suppose there is a controller $u_t = u(x_t, t)$ such that the system (1) is dissipative with respect to the supply rate $S(v,z) = \gamma^2\|v\|^2 - \|z\|^2$ and assume that the associated storage function satisfies $E\{V(x_0,t)\} \leq \gamma^2 E\|x_0\|^2$ for all $t \geq 0$. Then, the closed-loop system (1) has an L_2-gain$\leq \gamma$.

Utilizing now the of stochastic dissipation concept, we prove the following:

Theorem 2. Consider the system described by (1) with the controlled output of (3), and the supply rate $S(v,z) = \gamma^2\|v\|^2 - \|z\|^2$. Then the following hold:

A. Suppose there is a positive function $V(x,t) \in C^2$. Let $V(x)$ satisfy $\gamma^2 I - \frac{1}{2}U(x) \geq \alpha I$ for some $\alpha > 0$, and for all x, , where $U(x)$ is defined by

$$U(x) = [g_2(x)]^T V_{xx}(x) g_2(x) \quad (6)$$

Assume the following HJI is satisfied for all $x \in R^n$.

$$\begin{aligned} &V_x(x)f(x) - \frac{1}{4}V_x(x)g(x)[I + \frac{1}{2}\bar{g}^T(x)V_{xx}(x)\bar{g}(x)]^{-1} \\ &g^T(x)V_x^T(x) + \frac{1}{4}V_x(x)g_1(x)[\gamma^2 I - \frac{1}{2}U(x)]^{-1}g_1^T(x) \quad (7) \\ &V_x(x) + \frac{1}{2}trace\{G^T(x)V_{xx}(x)G(x)\} + h^T(x)h(x) \leq 0 \end{aligned}$$

Then, for $u(x) = -\frac{1}{2}[I + \frac{1}{2}\bar{g}^T(x)V_{xx}\bar{g}(x)]^{-1}g^T(x)V_x^T(x)$ the system (1) is dissipative with respect to the supply rate $S(v,z)$ (provided $\mathcal{A}_u$ is nonempty).

B. Assume that for some control $u(x) = l(x)$ the system (1) is dissipative with respect to the supply rate $S(v,z)$ for some storage function $V \in C^2$ which is assumed to satisfy $2\gamma^2 I - U(x) \geq \alpha I$ for all x. Assume also that $v(x) = [2\gamma^2 I - U(x)]^{-1}g_1^T(x)V_x^T(x) \in \mathcal{A}_u$. Then $V(x)$ satisfies the HJI for all $x \in R^n$.

3. STOCHASTIC H_∞ CONTROL: THE OUTPUT FEEDBACK CASE

We consider now the system (1), together with the observations (2) and the controlled output (3). Since the state x_t of the plant is not available, we follow the common practice (the certainty equivalence approach) of replacing the the state that is to be processed by the controller, with the estimator output. A natural choice of an estimator (see, e.g. [Isidori, 1994] for the deterministic case) is:

$$\begin{aligned} d\hat{x}_t = & f(\hat{x}_t)dt + g(\hat{x}_t)u_t^*(\hat{x}_t)dt + g_1(\hat{x}_t)v_t^*(\hat{x}_t) \\ & + K(\hat{x}_t)(dy_t - h_2(\hat{x}_t)dt - g_3(\hat{x}_t)v_t^*(\hat{x}_t)dt) \end{aligned} \tag{8}$$

where $K(\hat{x}_t)$ is the estimator gain, an $n \times r$ matrix, $u_t^*(\hat{x}_t) = -\frac{1}{2}[I + \frac{1}{2}\bar{g}^T(\hat{x}_t)V_{xx}(\hat{x}_t)\bar{g}(\hat{x}_t)]^{-1} g^T(\hat{x}_t)V_x^T(\hat{x}_t)$ and $v_t^*(\hat{x}_t) = \frac{1}{2}[\gamma^2 I - \frac{1}{2}U(\hat{x}_t)]^{-1} g_1^T(\hat{x}_t)V_x^T(\hat{x}_t)$.

Using now y_t of the observations equation (2) in (8), we arrive at the following augmented system.

$$\begin{aligned} dx_t^e = & f^e(x_t^e, K)dt + g_1^e(x_t^e, K)[v_t - v_t^*(x_t)]dt \\ & + g_2^e(x_t)[v_t - v_t^*(x_t)]dW_t^2 + G^e(x_t^e)d\bar{W}_t^1 \end{aligned} \tag{9}$$

where

$$x_t^e = col\{x_t, \hat{x}_t\}, \quad \bar{W}_t^1 = col\{W_t, W_t^1, W_t^2, W_t^3\},$$

$$f^e(x_t^e) = \begin{bmatrix} f(x_t) \\ f(\hat{x}_t) + g(\hat{x}_t)u_t^*(\hat{x}_t) \\ +g_1(x_t,t)v_t^*(x_t) + g(x_t)u_t^*(\hat{x}_t) \\ +g_1(\hat{x}_t)v_t^*(\hat{x}_t) + K(\hat{x}_t)(\tilde{h}_2(x_t) - \tilde{h}_2(\hat{x}_t)) \end{bmatrix}$$

$$g_1^e(x_t^e, t) = col\{g_1(x_t), K(\hat{x}_t)g_3(x_t)\}$$
$$g_2^e(x_t, t) = col\{g_2(x_t, t), 0\}$$

$$G^e(x_t^e, v_t) = \begin{bmatrix} \bar{g}(x)u^*(\hat{x}) & G(x_t) & g_2(x_t)v_t^*(x_t) & 0 \\ 0 & 0 & 0 & K(\hat{x}_t)G_2(x_t) \end{bmatrix}$$

and

$$\tilde{h}_2(x) = h_2(x) - g_3(x)v_t^*(x_t)$$
$$\tilde{h}_2(\hat{x}) = h_2(\hat{x}) - g_3(\hat{x})v_t^*(\hat{x}_t).$$

We now have the following theorem.

Theorem 3. Consider the stochastic system (1) together with the augmented system (9) and the controlled output (2). Assume there is a positive function $V : R^n \to R^+$, with $V \in C^2$ so that it satisfies the HJI (7) of Theorem 2. Assume also that there are: a positive function $\tilde{W} : R^{2n} \to R^+$ and a matrix $K(\hat{x})$, which satisfy the following HJI for some $\gamma > 0$, and that $\gamma^2 I - \frac{1}{2}[g_2^e(x^e)]^T \tilde{W}_{x^e x^e}(x^e) g_2^e(x^e) \geq \alpha I$ for some positive number α, and for all $x^e \in R^{2n}$.

$$\begin{aligned} & \tilde{W}_{x^e}(x^e)f^e(x^e) + h^e(x^e)^T h^e(x^e) + \frac{1}{4}\tilde{W}_{x^e}(x^e) \\ & g_1^e(x^e)[\gamma^2 I - \frac{1}{2}\bar{U}(x^e)]^{-1} g_1^e(x^e)^T \tilde{W}_{x^e}^T(x^e) \\ & + \frac{1}{2} trace\left\{(G^e)^T \tilde{W}_{x^e x^e}(x^e) G^e(x^e)\right\} \leq 0 \end{aligned} \tag{10}$$

where

$$\bar{U}(x^e) = [g_2^e(x^e)]^T \tilde{W}_{x^e x^e}(x^e) g_2^e(x^e),$$
$$h^e(x^e) = u^*(x) - u^*(\hat{x}), \text{ and } r = v - v^*(x).$$

Then, the closed-loop system with the control $u^*(\hat{x}) = -\frac{1}{2}[I + \frac{1}{2}\bar{g}^T(\hat{x})V_{xx}(\hat{x})\bar{g}(\hat{x})]^{-1} g^T(\hat{x})V_{xx}(\hat{x})$ is dissipative with respect to the supply rate $\gamma^2||v||^2 - ||z||^2$, it possesses a storage function defined as $S(x^e) = V(x,t) + \tilde{W}(x^e)$, and has an L_2-gain $\leq \gamma$.

Proof. Application of the HJI (7) yields

$$\begin{aligned} & V_x[f(x)) + g_1(x)v + g(x)u^*(\hat{x})] + \|z\|^2 - \gamma^2 \|v\|^2 \\ & + \frac{1}{2} trace\{G^T(x)V_{xx}(x)G(x)\} \\ & + \frac{1}{2}u^{*T}(\hat{x})\bar{g}^T(x)V_{xx}(x)\bar{g}(x)u^*(\hat{x})\} \leq \|\frac{1}{2}[I + \\ & \frac{1}{2}\bar{g}^T(x)V_{xx}(x)\bar{g}(x)]^{-1} g^T(x)V_x^T(x) + u^*(\hat{x})\|^2 \\ & - \|v - \frac{1}{2\gamma^2}g_1^T(x)V_x^T(x)\|^2 = ||u^*(x) - u^*(\hat{x})||^2 \\ & - \gamma^2||v - v^*(x)||^2 = ||h^e(x^e)||^2 - \gamma^2||r||^2 \end{aligned} \tag{11}$$

Define $S(x^e) = V(x) + \tilde{W}(x^e)$. Thus S is positive definite and satisfies: $S(0) = 0$. Obviously, the infinitesimal generator of the augmented system satisfies

$$L\{S(x^e)\} = L\{V(x)\} + L\{\tilde{W}(x^e)\}$$

where

$$\begin{aligned} & L\{V(x)\} = V_x(x)\{f(x) + g_1(x)v + g(x)u^*(\hat{x})\} \\ & + \frac{1}{2} trace\left\{G^T(x)V_{xx}(x)G(x)\right\} \\ & + \frac{1}{2}u^{*T}(\hat{x})\bar{g}^T(x)V_{xx}(x)\bar{g}(x)u^*(\hat{x}) \\ & L\left\{\tilde{W}(x^e)\right\} = \tilde{W}_{x^e}[f^e(x^e) + g_1^e(x^e)r] + \\ & \frac{1}{2} trace\left\{(G(x^e))^T \tilde{W}_{x^e x^e} G^e(x^e)\right\} \end{aligned}$$

Recall (11), that is:

$$L\{V(x)\} + ||z||^2 - \gamma^2||v||^2 \leq ||h^e(x^e)||^2 - \gamma^2||r||^2$$

By the HJI (10) it follows that

$$L\left\{\tilde{W}(x^e)\right\} + ||h^e(x^e)||^2 - \gamma^2||r||^2 \leq 0$$

Therefore

$$\begin{aligned} & L\{S(x^e)\} + ||z||^2 - \gamma^2||v||^2 = L\{V(x^e)\} + ||z||^2 \\ & - \gamma^2||v||^2 + L\left\{\tilde{W}(x^e)\right\} \leq 0 \end{aligned}$$

This implies that $S(x^e)$ is a storage function for the closed loop system with the supply rate $||z||^2 - \gamma^2||v||^2$, which implies that the closed loop system is L_2-gain$\leq \gamma$.

Remark 5. As in the deterministic case, it is difficult to establish, in general, conditions under which there exists a matrix $K(\hat{x}, t)$ so that the HJI (10) is satisfied. The part of the latter inequality that contains K is given by:

$$\begin{aligned} & \Gamma(K) = \tilde{W}_{\hat{x}}(x^e)K(\hat{x}_t)(\tilde{h}_2(x_t) - \tilde{h}_2(\hat{x}_t)) \\ & + \frac{1}{4}(\tilde{W}_{\hat{x}}(x^e)K(\hat{x}_t)g_3(x_t) + \tilde{W}_x(x^e)g_1(x_t)) \\ & (\gamma^2 I - \frac{1}{2}\tilde{U}(x^e))^{-1}(\tilde{W}_{\hat{x}}(x^e)K(\hat{x}_t)g_3(x_t) + \tilde{W}_x(x^e) \\ & g_1(x_t))^T = \frac{1}{4\gamma^2}[\tilde{W}_{\hat{x}}(x^e)K(\hat{x}_t) - \Psi(x^e)]R_a(x_t) \\ & \left[K^T(\hat{x}_t)\tilde{W}_{\hat{x}}^T(x^e) - \Psi^T(x^e)\right] - \frac{1}{4\gamma^2}[\Psi(x^e)R_a(x_t) \\ & \Psi^T(x^e) - \tilde{W}_x(x^e)g_1(x_t)(I - \frac{1}{2\gamma^2}\tilde{U}(x^e))^{-1}g_1^T(x_t)\tilde{W}_x^T(x^e)] \end{aligned}$$

where

$$\Psi(x^e) = -[2\gamma^2(\tilde{h}_2(x_t) - \tilde{h}_2(\hat{x}_t))^T + \tilde{W}_x(x^e)g_1(x_t)$$
$$(I - \frac{1}{2\gamma^2}g_2^T(x_t)\tilde{W}_{xx}(x^e)g_2(x_t))^{-1}g_3^T(x_t)]R_a^{-1}(x_t)$$

and

$$R_a(x_t) = g_3(x_t)(I - \frac{1}{2\gamma^2}g_2^T(x_t)\tilde{W}_{xx}(x^e)g_2(x_t))^{-1}g_3^T(x_t)$$

The gain matrix $K(\hat{x}_t)$ that minimizes $\Gamma(K)$, and thus leads to a minimum left hand side in (10), is clearly one that satisfies $\tilde{W}_{\hat{x}}(x^e)K(\hat{x}_t) = \Psi(x^e)$. Unfortunately, the latter equation may not possess a solution for K which depends only on $\hat{x}_t$. One way to circumvent this difficulty is to choose $K(\hat{x}_t)$ s.t.

$$\tilde{W}_{\hat{x}}(x^e)K(\hat{x}_t) = \Psi(x^e) + \Phi(x^e) \qquad (12)$$

where $\Phi(x^e)$ is a function that allows a solution K^* for (12) that is independent of x_t. For this choice of $K^*(\hat{x}_t)$ the above $\Gamma(K)$ becomes the following.

$$\Gamma(K^*) = \frac{1}{4\gamma^2}[\Phi(x^e)R_a(x_t)\Phi^T(x^e) - \Psi(x^e)R_a(x_t)\Psi^T(x^e)$$
$$+\tilde{W}_x(x^e)g_1(x_t)(I - \frac{1}{2\gamma^2}\tilde{U}(x^e))^{-1}g_1^T(x_t)\tilde{W}_x^T(x^e)]$$

Assuming an existence of a solution $K(\hat{x}_t)$ to (12) for some function $\Phi(x^e)$, we have established the following theorem.

Theorem 4. Consider the stochastic system (1) together with the augmented system (9) and the controlled output (2). Assume there is a positive function $V : R^n \to R^+$, with $V \in C^2$so that it satisfies the HJI (7) of Theorem 2. Assume also that there are: a positive function $\tilde{W} : R^{2n} \to R^+$ in C^2 and a matrix $K(\hat{x})$, which satisfy (12). In addition, let $\tilde{W}$ satisfy the following HJI.

$$\tilde{W}_t(x^e) + \tilde{W}_x(x^e)[f(x_t) + g_1(x_t)v_t^*(x_t) + g(x_t)u_t^*(\hat{x}_t)]$$
$$+\tilde{W}_{\hat{x}}(x^e)[f(\hat{x}_t) + g_1(\hat{x}_t)v_t^*(\hat{x}_t) + g(\hat{x}_t)u^*(\hat{x}_t)] +$$
$$\frac{1}{2}trace\left\{G(x)^T\tilde{W}_{xx}(x^e)G(x)\right\} + \frac{1}{2}u^{*T}(\hat{x}_t)\bar{g}^T(x_t)\tilde{W}_{xx}(x^e)$$
$$\bar{g}(x_t)u^*(\hat{x}_t) + \frac{1}{2}v^{*T}(x_t)g_2^T(x_t)\tilde{W}_{xx}(x^e)g_2(x_t)v^*(x_t)$$
$$+h^e(x^e)^T h^e(x^e) + \frac{1}{4\gamma^2}[\Phi(x^e)R_a(x_t)\Phi^T(x^e) - \Psi(x^e)$$
$$R_a(x_t)\Psi^T(x^e) + \tilde{W}_x(x^e)g_1(x_t)(I - \frac{1}{2\gamma^2}g_2^T(x_t)\tilde{W}_{xx}(x^e)$$
$$g_2(x_t))^{-1}g_1^T(x_t)\tilde{W}_x^T(x^e)] \le 0 \quad \forall x^e \in R^{2n}$$

Then, the closed-loop system is dissipative with respect to the supply rate $\gamma^2||v||^2 - ||z||^2$, with the storage function defined as $S(x^e) = V(x) + \tilde{W}(x^e)$, and therefore has an L_2-gain $\le \gamma$.

3.1 *Stability*

Various types of asymptotic stability may be considered. We consider here global asymptotic stability in probability and mean square sense. A comprehensive account of stochastic systems' stability may be found in [Hasminskii, 1980]. We recall first some sufficient conditions for global asymptotic stability of the stochastic system given by

$$dx_t = f(x_t)dt + G(x_t)dW_t \qquad (13)$$

with $f(0) = G(0) = 0$, and assume that f, G satisfy conditions that guarantee a unique strong solution relative to the filtered probability space $(\Omega, F, \{F_t\}_{t\ge 0}, P)$.

Sufficient conditions for a global stability in probability and in the mean square sense are summarized in the following two theorems.

Theorem 5. ([Hasminskii, 1980])Assume there exists a positive function $V(x) \in C^2$, with $V(0) = 0$, so that $(LV)(x) < 0$ for all $x \in R^n$. Assume also that $V(x) \to \infty$ as $\|x\| \to \infty$. Then, the system of (13) is globally asymptotically stable in probability.

Theorem 6. ([Hasminskii, 1980])Assume there exists a positive function $V(x) \in C^2$, with $V(0) = 0$. Then the system of (13) is globally exponentially stable if there are positive numbers k_1, k_2, k_3 such that the following hold.

$$k_1||x||^2 \le V(x) \le k_2||x||^2$$
$$(LV)(x) \le -k_3||x||^2$$

As a consequence of the last two theorems we have the following results.

Lemma 3. Assume there exists a positive function $V(x) \in C^2$ such that $V(x) \to \infty$ as $\|x\| \to \infty$, satisfying the HJI (7) with $h_2^T(x)h_2^T(x) > 0$ for all x. Assume also that there is a positive function $\tilde{W} \in C^2$, $\tilde{W} : R^{2n} \to R^+$, satisfying the HJI (10) with a strict inequality, so that $\tilde{W}(x^e) \to \infty$ as $||x^e|| \to \infty$. Then, the closed loop system is internally globally asymptotically stable in probability.

Lemma 4. Assume there exists a positive function $V(x) \in C^2$, with $V(0) = 0$ which satisfies the HJI of (7) for some $\gamma > 0$. In addition, let V satisfy

$$k_1||x||^2 \le V(x) \le k_2||x||^2$$

for some positive numbers k_1, k_2. Furthermore, assume that for some k_3, $h_2^T(x)h_2(x) \ge k_3||x||^2$ for all $x \in R^n$. Assume also that there is a positive function $\tilde{W} \in C^2$. and $\tilde{W} : R^{2n} \to R^+$ with

$$k_4||\hat{x}||^2 \le \tilde{W}(x^e) \le k_5||x^e||^2 \quad x^e \in R^{2n}$$

which satisfies the following algebraic HJI:

$$\tilde{W}_{x^e}(x^e)f^e(x^e) + \frac{1}{2}trace\left\{(G^e)^T\tilde{W}_{x^e x^e}(x^e)G^e(x^e)\right\}$$
$$+(h^e(x^e)^T h^e(x^e) \le -Q(x^e) \quad \forall x^e \in R^{2n}$$

for some positive function $Q(x^e)$ with the property that $(h^e(x^e,t)^T h^e(x^e,t) + Q(x^e) \ge k_6||\hat{x}||^2$ for all

$x^e \in R^{2n}$, and for some $k_6 > 0$. Then the closed-loop system (9) with $v = 0$ and $u = -\frac{1}{2}g^T(x)V_x^T(\hat{x})$ is exponentially stable in the mean square sense, and has the property of $L_2 - gain \leq \gamma$, that is

$$E\{\int_0^\infty \|z_t\|^2 \, dt\} \leq \gamma^2 E\{\|x_0\|^2 + \int_0^\infty (\|v_t\|^2) dt\}$$

for all non-anticipative stochastic processes v that satisfy $E\{\int_0^\infty (\|v_t\|^2) dt\} < \infty$, and whenever x_0 satisfies $E\{V(x_0)\} \leq \gamma^2 E\{||x_0||^2\}$.

4. CONCLUSIONS

A comprehensive treatment of output-feedback control for nonlinear stochastic systems is introduced. Conditions are found for the existence of stabilizing controllers that satisfy prescribed H_∞ performance requirements. The theory developed can also be used to derive nonlinear estimators that achieve a prescribed H_∞ bound on the estimation accuracy.

REFERENCES

Boyd, S., L. El Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequality in Systems and Control Theory*, SIAM Frontier Series.

El Ghaoui, L. (1995). State-feedback control of systems with multiplicative noise via linear matrix inequalities, *Systems and Control Letters* **24**, 223-228.

Costa, O.L.V and C.S. Kubrusly (1996). State feedback H_∞ -control for discrete-time infinite dimensional stochastic bilinear systems, *Journal of Math. Systems, Estimation and Control*. **6**, 1-32.

Dragan, V. and T. Morozan (1997). Mixed input-output optimization for time-varying Ito systems with state dependent noise, *Dynamics of continuous, discrete and impulsive systems*, **3**, 317-333.

Dragan, V. and A. Stoica (1998). A γ attenuation problem for discrete-time time-varying stochastic systems with multiplicative noise, *Reprint Series of the Institute of Mathematics of the Romanian Academy*. No 10/1998.

Hinriechsen, D. and A.J. Pritchard (1998). Stochastic H_∞. *SIAM J. of Contr. Optimiz.*, **36**, 1504-1538.

Gershon, E., U. Shaked and I. Yaesh (2001). H_∞ control and filtering of discrete-time stochastic systems with multiplicative noise.*Automatica*. 37 409-417.

Gershon, E., U. Shaked and I. Yaesh (2002). Robust H_∞ estimation of stationary discrete-time linear processes with stochastic uncertainties, *Systems & Control Letters*, **45**, 257-269.

Gihman,I.I. and A.V. Skorohod (1972). *Stochastic Differential Equations*, Springer Verlag.

Has'minskii, R.H (1980). *Stochastic Stability of Differential Equation*, Sijthoff &Noordhoff.

Helton J.W. and M.R. James (1999). Extending H_∞ control to nonlinear systems: Control of nonlinear systems to achieve performance objec tives, *SIAM Advances in Design and Control series*.

van der Schaft, A.J. (2000). L_2- *Gain and Passivity Techniques in Nonlinear Control*, Communication and Control Engineering, Springer-Verlag.

James, M.R. (1993). A partial differential inequality for dissipative nonlinear systems, *Systems Control Letters*, **21**, 315-320.

Ball J.A. and J.W. Helton (1996). Viscosity solutions of Hamilton-Jacobi Equations arising in nonlinear H_∞ control, *Journal of Mathematical Systems, Estimation and Control*, **6**, 1-22.

Willems, J.C. (1972). Dissipative dynamical systems-part I: general theory", *Arch. Rational Mech. Anal.*, **45**, 321-351.

Florchinger, P. (1999). A Passive System Approach to Feedback Stabilization of Nonlinear Control Stochastic Systems, *SIAM J. of Contr. Optimiz.*, **37(6)**, 1848-1864.

Basar, T. and P. Bernhard (1995). H_∞ *Optimal Control and Related Minmax Design Problems: a Dynamic Game approach*, Birkhauser.

Petersen,I.R., V. Ugronovskii and A.V. Savkin (2000). *Robust control design using* H_∞ *methods*, Springer Verlag.

Charalambous, C.D. (2003). Stochastic nonlinear minmax dynamic games with noisy measurements, *IEEE Trans. Automat. Contr.*, **48**, 261-266.

Berman, N. and U. Shaked (2003). Stochastic H_∞ control for nonlinear stochastic systems, in *Proc. Conf. Decision Control*, Maui, Hawaii.

Isidori, A, (1994). H_∞ control via measurement feedback for affine nonlinear systems, *Int. J. Robust & Nonlinear Contr.*, **4**, 553-574.