THE CRAMÉR-RAO BOUND FOR ESTIMATION OF CONTINUOUS-TIME ARX PARAMETERS FROM IRREGULARLY SAMPLED DATA

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Abstract: The Cramér-Rao bound for estimation of parameters in continuous-time ARX models from irregularly sampled data is computed. In the derivation, the Slepian-Bangs formula is used together with a state space framework, resulting in a closed form expression for the Cramér-Rao bound.

Keywords: Cramér-Rao bound; Estimation; Continuous-time ARX parameters; Irregular sampling.

1. INTRODUCTION

The continuous-time ARX (CARX) model structure is defined as

$$A(p)y(t) = B(p)u(t) + e(t), \tag{1}$$

where

$$A(p) = p^n + a_1 p^{n-1} + \ldots + a_n,$$
$$B(p) = b_1 p^{n-1} + \ldots + b_n,$$

with p denoting the differentiation operator, $u(t)$ is the input, and where $e(t)$ is continuous-time stationary white process noise, which is assumed to be Gaussian with zero mean and intensity σ_e^2. Note that $y(t)$ can be written as a sum of a deterministic and a stochastic term

$$y(t) = \frac{B(p)}{A(p)}u(t) + \frac{1}{A(p)}e(t) \triangleq y_d(t) + y_s(t), \tag{2}$$

where the stochastic part $y_s(t)$ is well defined, and $n-1$ times differentiable. A CARX process can be very useful in the following cases:

- The data are sampled irregularly.
- It is important to have a model where the parameters have a physical meaning.
- The control design is made in continuous time.

When the data are sampled irregularly, it might be difficult to find a good discrete-time system description involving the shift operator. Even if such a model can be found, it will be time varying and the computational demands for the prediction error method (PEM) will in general increase drastically, (Larsson and Söderström, 2002). A better idea could then be to choose a continuous-time model in which the differentiation operator can be approximated with some difference operator. Note that the situation with unevenly sampled data can also occur when data are sampled regularly, but some samples are missing. The reconstruction of missing values for discrete-time ARX models is discussed in (Isaksson, 1993).

Model parameters with physical interpretations are in many cases an absolute demand for people in chemistry, biology, physics and other sciences. In these areas it is important to have a model derived from physical laws, and stochastic differential equations, (Øksendal, 2003), such as the CARX model (1), are therefore a natural choice. A discrete-time model where the parameters do not have a physical meaning is simply not what these scientists aim for.

Sometimes the control design for stochastic systems is made in continuous-time, see, e.g., (Kushner and Dupuis, 2001; Yong and Zhou, 1999). Just as a discrete-time ARX model is a natural choice for modeling the system that is to be controlled when using a discrete-time setting, a CARX model constitutes a natural choice in a continuous-time framework.

Consider the problem of estimating the CARX parameters

$$\boldsymbol{\theta} = \begin{bmatrix} a_1 & \dots & a_n & b_1 & \dots & b_n & \sigma_e^2 \end{bmatrix}^T \tag{3}$$

from irregularly sampled data $\{y(t_k)\}_{k=1}^N$ and $\{u(t_k)\}_{k=1}^N$, taken at the moments of observations $t_1 < t_2 < \cdots < t_N$. The Cramér-Rao bound (CRB) for the CARX parameter estimation problem from irregularly sampled data is derived in the paper. The derived expression makes it possible to judge the performance of different estimators and to investigate how the achievable accuracy is affected by the sampling scheme. Numerical studies in the paper investigate the influence of the input signal on the CRB for different sampling schemes. The CRBs for continuous-time AR and continuous-time ARMA (CARMA) parameter estimation for the case with irregular sampling are derived in (Larsson and Larsson, 2002) and (Larsson and Larsson, 2004), respectively. The derivation in this paper is based on the same technique used in the latter reference. However, for the derivation of the CRB for CARX parameter estimation from irregularly sampled data, an assumption must be made regarding the behavior of the input signal between the sampling instants. One possibility is to assume that the input signal is constant between the sampling intervals. This would lead to very complex computations when estimating the parameters via the PEM. Instead, in this paper it is assumed that the input signal is given by a CARMA process. An assumption regarding the input signal must also be made when deriving the CRB for the errors-in-variables problem (Karlsson *et al.*, 2000).

A state space description of the CARX process, with the input signal given as a CARMA process, is given in the next section. The description is needed in Section 3, where the CRB is derived using the Slepian-Bangs formula. The derived expression is investigated for different sampling schemes in a numerical study in Section 4. Finally, concluding remarks are given in Section 5. In the paper, $\mathrm{tr}\{\cdot\}$ denotes the trace, $\otimes$ stands for the Kronecker product, $\odot$ denotes element-wise multiplication, and $(\cdot)^c$ stands for the complex conjugate.

2. STATE SPACE DESCRIPTIONS

A state space description of the CARX process (1), needed for the derivation of the CRB in the next section, is given here. However, before writing the CARX process on state space form, an input signal $u(t)$ that is persistently exciting of high enough order must be chosen. The CARMA process

$$\begin{aligned} &\left(p^{m_1} + c_1 p^{m_1-1} + \ldots + c_{m_1}\right) u(t) \\ &\quad = \left(p^{m_2} + d_1 p^{m_2-1} + \ldots + d_{m_2}\right) v(t), \end{aligned} \tag{4}$$

where $v(t)$ is continuous-time stationary white process noise, which is assumed to be Gaussian with zero mean and intensity σ_v^2, and where $m_1 \geq m_2 + n$, is chosen here. The process noises $v(t)$ in (4) and $e(t)$ in (1) are independent. The CARMA process (4) is a general signal description that, by varying $\{c_i\}_{i=1}^{m_1}$ and $\{d_i\}_{i=1}^{m_2}$, has the ability to describe input signals with very desirable properties for identification purposes.

The CARX model (1) can be described in a state-space framework as

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{A}_y \mathbf{x}_1(t) + \mathbf{b}_y u(t) + \mathbf{d}_y e(t), \\ y(t) &= \mathbf{c}_y^T \mathbf{x}_1(t). \end{aligned} \tag{5}$$

In this paper, the observable canonical form is chosen, and hence $\mathbf{A}_y \in \mathbb{R}^{n\times n}$, $\mathbf{b}_y \in \mathbb{R}^{n\times 1}$, $\mathbf{d}_y \in \mathbb{R}^{n\times 1}$ and $\mathbf{c}_y \in \mathbb{R}^{n\times 1}$ have the structures

$$\mathbf{A}_y = \begin{bmatrix} -a_1 & 1 & & 0 \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ -a_n & 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{b}_y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

$$\mathbf{d}_y = \begin{bmatrix} \mathbf{0} & 1 \end{bmatrix}^T, \quad \mathbf{c}_y = \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix}^T.$$

Similarly, the input signal $u(t)$ in (4) is written on observable canonical form

$$\begin{aligned} \dot{\mathbf{x}}_2(t) &= \mathbf{A}_u \mathbf{x}_2(t) + \mathbf{b}_u v(t), \\ u(t) &= \mathbf{c}_u^T \mathbf{x}_2(t), \end{aligned} \tag{6}$$

where $\mathbf{A}_u \in \mathbb{R}^{m_1\times m_1}$, $\mathbf{b}_u \in \mathbb{R}^{m_1\times 1}$ and $\mathbf{c}_u \in \mathbb{R}^{m_1\times 1}$ are defined as

$$\mathbf{A}_u = \begin{bmatrix} -c_1 & 1 & & 0 \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ -c_{m_1} & 0 & \dots & 0 \end{bmatrix}, \mathbf{b}_u = \begin{bmatrix} \mathbf{0} \\ 1 \\ d_1 \\ \vdots \\ d_{m_2} \end{bmatrix}, \mathbf{c}_u = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}.$$

Let

$$\mathbf{x}(t) = \begin{bmatrix}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\end{bmatrix}, \quad \mathbf{w}(t) = \begin{bmatrix}e(t)\\ v(t)\end{bmatrix}, \quad \mathbf{z}(t) = \begin{bmatrix}y(t)\\ u(t)\end{bmatrix},$$
$$\mathbf{\Sigma} = \mathrm{E}\{\mathbf{w}(t)\mathbf{w}^T(t)\} = \begin{bmatrix}\sigma_e^2 & 0\\ 0 & \sigma_v^2\end{bmatrix}.$$

The state-space forms (5) and (6) can then be put together as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{w}(t),\\ \mathbf{z}(t) &= \mathbf{C}^T\mathbf{x}(t),\end{aligned} \tag{7}$$

with $\mathbf{A} \in \mathbb{R}^{\eta\times\eta}$, $\mathbf{B} \in \mathbb{R}^{\eta\times 2}$ and $\mathbf{C} \in \mathbb{R}^{\eta\times 2}$ given as

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_y & \mathbf{b}_y\mathbf{c}_u^T\\ \mathbf{0} & \mathbf{A}_u\end{bmatrix}, \ \mathbf{B} = \begin{bmatrix}\mathbf{d}_y & \mathbf{0}\\ \mathbf{0} & \mathbf{b}_u\end{bmatrix}, \ \mathbf{C} = \begin{bmatrix}\mathbf{c}_y & \mathbf{0}\\ \mathbf{0} & \mathbf{c}_u\end{bmatrix},$$

where $\eta \triangleq n+m_1$ is introduced for notational convenience. Throughout the paper it is assumed that $\mathbf{A}$ is asymptotically stable and, for simplicity, diagonalizable. Extension to the non-diagonalizable case is likely possible, but technically more involved. It is treated for the corresponding problem of deriving the CRB for CARMA parameter estimation from irregularly sampled data in (Larsson, 2004).

3. THE CRB

Let the CARX parameters $\boldsymbol{\theta}$ in (3), together with the CARMA parameters

$$\boldsymbol{\nu} = \begin{bmatrix}c_1 & \dots & c_{m_1} & d_1 & \dots & d_{m_2} & \sigma_v^2\end{bmatrix}^T, \tag{8}$$

define the augmented parameter vector

$$\begin{aligned}\boldsymbol{\psi} &= [\boldsymbol{\theta}^T \ \boldsymbol{\nu}^T]^T\\ &= \begin{bmatrix}a_1 & \dots & a_n & b_1 & \dots & b_n & \sigma_e^2\\ & & & c_1 & \dots & c_{m_1} & d_1 \ \dots \ d_{m_2} \ \sigma_v^2\end{bmatrix}^T.\end{aligned} \tag{9}$$

The CRB for the estimation of $\boldsymbol{\psi}$ given the samples $\{y(t_1), u(t_1), \dots, y(t_N), u(t_N)\}$ is given by $\mathrm{CRB} = \mathbf{J}^{-1}$, where $\mathbf{J}$ is the Fisher information matrix (FIM). The (k,l)th element of the FIM is given by the Slepian-Bangs formula (Slepian, 1954; Bangs, 1971)

$$[\mathbf{J}]_{k,l} = \frac{1}{2}\,\mathrm{tr}\{\mathbf{R}^{-1}\mathbf{R}_k'\mathbf{R}^{-1}\mathbf{R}_l'\}, \tag{10}$$

where

$$\mathbf{R} = \mathrm{E}\left\{\begin{bmatrix}\mathbf{z}(t_1)\\ \vdots\\ \mathbf{z}(t_N)\end{bmatrix}\begin{bmatrix}\mathbf{z}^T(t_1) & \dots & \mathbf{z}^T(t_N)\end{bmatrix}\right\} \tag{11}$$

is the covariance matrix of the sampled data, and where $\mathbf{R}_k' = \partial\mathbf{R}/\partial\psi_k$, with ψ_k denoting the kth element of $\boldsymbol{\psi}$. The CRB for $\boldsymbol{\theta}$, which is of primary interest here, is found from the upper left block of $\mathbf{J}^{-1}$. In order to obtain the upper left block of $\mathbf{J}^{-1}$, all elements of $\mathbf{J}$ must be evaluated. It can therefore be interpreted as if the input signal parameters $\boldsymbol{\nu}$ enter the computations as nuisance parameters. Next, compact closed-form expressions for $\mathbf{R}$ and its derivatives $\mathbf{R}_k'$ are derived.

To find an expression for $\mathbf{R}$, it is first noted that the covariance function of $\mathbf{z}(t)$ can be written as (Åström, 1970; Söderström, 2002)

$$\mathbf{r}(\tau) = \mathrm{E}\{\mathbf{z}(t)\mathbf{z}^T(t-\tau)\} = \mathbf{C}^T e^{\mathbf{A}\tau}\mathbf{P}\mathbf{C}, \tag{12}$$

for $\tau \geq 0$. In (12), $e^{\mathbf{A}\tau}$ is the standard matrix exponential, which has the spectral representation

$$e^{\mathbf{A}\tau} = \sum_{k=1}^{\eta} \boldsymbol{\rho}_k \boldsymbol{\xi}_k^H e^{\lambda_k\tau}, \tag{13}$$

where $\boldsymbol{\rho}_i$ and $\boldsymbol{\xi}_i^H$ are the right and left eigenvectors of $\mathbf{A}$, respectively (normalized such that $\boldsymbol{\rho}_i^H\boldsymbol{\xi}_i = 1$), and λ_i are the eigenvalues of $\mathbf{A}$. Also, in (12), $\mathbf{P}$ is the unique and nonnegative definite solution to the continuous-time Lyapunov equation

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T + \mathbf{B}\mathbf{\Sigma}\mathbf{B}^T = 0. \tag{14}$$

This is a linear system of equations for the entries of $\mathbf{P}$, and hence easily solved numerically in a standard manner. Next, by observing that $\mathbf{r}(\tau)$ is defined for $\tau \geq 0$, we find that $\mathbf{R}$ in (11) can be written as:

$$\mathbf{R} = \tilde{\mathbf{R}} + \mathbf{U}^T, \tag{15}$$

where

$$\begin{aligned}\tilde{\mathbf{R}} &\triangleq \mathbf{I} \otimes \mathbf{r}(0) + \mathbf{U},\\ \mathbf{U} &\triangleq \begin{bmatrix}\mathbf{0} & \dots & \dots & \mathbf{0}\\ \mathbf{r}(t_2-t_1) & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \vdots\\ \mathbf{r}(t_N-t_1) & \dots & \mathbf{r}(t_N-t_{N-1}) & \mathbf{0}\end{bmatrix},\end{aligned}$$

and $\mathbf{I}$ is the identity matrix of dimension N. It turns out that $\tilde{\mathbf{R}}$ can be computed by a compact formula, as shown below. Once $\tilde{\mathbf{R}}$ is known, $\mathbf{U}$ is obtained from the lower left part of $\tilde{\mathbf{R}}$ and $\mathbf{R}$ is found via (15). Note also that all elements of $\mathbf{U}$ are 2×2 matrices.

From (12), (13) and the solution to (14), it follows that $\tilde{\mathbf{R}}$ can be computed as

$$\tilde{\mathbf{R}} = \sum_{k=1}^{\eta} \mathbf{\Gamma}(\lambda_k) \otimes (\boldsymbol{\gamma}_k\boldsymbol{\alpha}_k^H), \tag{16}$$

where $\boldsymbol{\gamma}_k$ and $\boldsymbol{\alpha}_k^H$ are defined as

$$\begin{aligned}\boldsymbol{\gamma}_k &\triangleq \mathbf{C}^T\boldsymbol{\rho}_k,\\ \boldsymbol{\alpha}_k^H &\triangleq \boldsymbol{\xi}_k^H\mathbf{P}\mathbf{C},\end{aligned}$$

and $\mathbf{\Gamma}(s)$ is the matrix whose (k,l)th element is equal to

$$[\mathbf{\Gamma}(s)]_{k,l} = \begin{cases}e^{(t_k-t_l)s}, & k \geq l,\\ 0, & \text{otherwise.}\end{cases}$$

To obtain convenient formulas for $\mathbf{R}'_k$, it is first noted that $\mathbf{R}'_k$ possesses the same structure as $\mathbf{R}$, i.e.,

$$\mathbf{R}'_k = \tilde{\mathbf{R}}'_k + \mathbf{U}_k^T, \tag{17}$$

where

$$\tilde{\mathbf{R}}'_k \triangleq \mathbf{I} \otimes \mathbf{r}_k(0) + \mathbf{U}_k,$$

with $\mathbf{U}_k = \partial\mathbf{U}/\partial\psi_k$ and $\mathbf{r}_k(0) = \partial\mathbf{r}(0)/\partial\psi_k$. Next, we will show how $\tilde{\mathbf{R}}'_k$ can be computed. Once $\tilde{\mathbf{R}}'_k$ is found, $\mathbf{R}'_k$ is easily constructed from (17).

First, notice that differentiation of (14) with respect to ψ_i yields

$$\begin{aligned}&\mathbf{A}\mathbf{P}_i + \mathbf{P}_i\mathbf{A}^T + \mathbf{A}_i\mathbf{P} + \mathbf{P}\mathbf{A}_i^T \\ &+ \mathbf{B}_i\mathbf{\Sigma}\mathbf{B}^T + \mathbf{B}\mathbf{\Sigma}_i\mathbf{B}^T + \mathbf{B}\mathbf{\Sigma}\mathbf{B}_i^T = 0,\end{aligned} \tag{18}$$

where the derivatives $\mathbf{P}_i = \partial\mathbf{P}/\partial\psi_i$, $\mathbf{A}_i = \partial\mathbf{A}/\partial\psi_i$, $\mathbf{B}_i = \partial\mathbf{B}/\partial\psi_i$ and $\mathbf{\Sigma}_i = \partial\mathbf{\Sigma}/\partial\psi_i$. The derivatives $\mathbf{A}_i$, $\mathbf{B}_i$ and $\mathbf{\Sigma}_i$, for all elements ψ_i of $\boldsymbol{\psi}$ in (9), i.e., with respect to $\{a_i, b_i, \sigma_e^2, c_i, d_i, \sigma_v^2\}$, are given in Appendix A. The Lyapunov equation (18) is straightforward to solve with respect to $\mathbf{P}_i$. Next, differentiation of $\mathbf{r}(\tau)$ in (12) with respect to ψ_i, yields

$$\begin{aligned}\mathbf{r}_i(\tau) &= \frac{\partial}{\partial\psi_i}\mathbf{r}(\tau) \\ &= \mathbf{C}^T\left(\frac{\partial}{\partial\psi_i}e^{\mathbf{A}\tau}\right)\mathbf{P}\mathbf{C} + \mathbf{C}^T e^{\mathbf{A}\tau}\mathbf{P}_i\mathbf{C},\end{aligned}$$

where (Brewer, 1977)

$$\frac{\partial}{\partial\psi_i}e^{\mathbf{A}\tau} = \sum_{k=1}^{\eta}\sum_{l=1}^{\eta}\boldsymbol{\rho}_k\boldsymbol{\xi}_k^H\mathbf{A}_i\boldsymbol{\rho}_l\boldsymbol{\xi}_l^H f_{kl}(\tau),$$

and

$$f_{kl}(\tau) = \begin{cases}\tau e^{\lambda_k\tau}, & \lambda_k = \lambda_l, \\ \dfrac{e^{\lambda_l\tau} - e^{\lambda_k\tau}}{\lambda_l - \lambda_k}, & \text{otherwise.}\end{cases}$$

Hence,

$$\begin{aligned}\tilde{\mathbf{R}}'_i = &\sum_{k=1}^{\eta}\mathbf{\Gamma}(\lambda_k)\otimes(\boldsymbol{\gamma}_k\boldsymbol{\alpha}_{k,\psi_i}^H) \\ &+ \sum_{k=1}^{\eta}\sum_{l=1}^{\eta}\mathbf{G}_{k,l}\otimes(\boldsymbol{\gamma}_k\boldsymbol{\alpha}_l^H\beta_{k,l}^{(i)}),\end{aligned} \tag{19}$$

where

$$\boldsymbol{\alpha}_{k,\psi_i}^H \triangleq \boldsymbol{\xi}_k^H\mathbf{P}_i\mathbf{C},$$
$$\beta_{k,l}^{(i)} \triangleq \boldsymbol{\xi}_k^H\mathbf{A}_i\boldsymbol{\rho}_l,$$
$$\mathbf{G}_{k,l} = \begin{cases}\mathbf{\Omega}\odot\mathbf{\Gamma}(\lambda_k), & \lambda_k = \lambda_l, \\ \dfrac{1}{\lambda_l - \lambda_k}(\mathbf{\Gamma}(\lambda_l) - \mathbf{\Gamma}(\lambda_k)), & \text{otherwise,}\end{cases}$$

and where $\mathbf{\Omega}$ is a matrix whose (k,l)th element is equal to

$$[\mathbf{\Omega}]_{k,l} = |t_k - t_l|.$$

This completes the derivation of the expressions for $\mathbf{R}$ and $\mathbf{R}'_k$. We summarize our algorithm for computing the CRB as follows:

Step 1. Compute $\mathbf{P}$, and $\mathbf{P}_i$ by solving the Lyapunov equations (14), and (18).

Step 2. Compute $\tilde{\mathbf{R}}$ via (16). Thereafter construct $\mathbf{R}$ by using (15).

Step 3. Compute $\tilde{\mathbf{R}}'_k$ by using (19), and form $\mathbf{R}'_k$ via (17).

Step 4. Compute $\mathbf{J}$ via (10) and obtain the CRB for $\boldsymbol{\theta}$ by extracting the upper left block of $\mathbf{J}^{-1}$.

Let CRB_N denote the CRB given N samples of $y(t)$ and $u(t)$. The evaluation of CRB_N can be impractical if N is very large. Theorem 2 in (Larsson and Larsson, 2002) provides some remedy to this problem by showing that under certain circumstances, the CRB becomes inversely proportional to N, and hence it can be extrapolated as

$$\mathrm{CRB}_N \overset{as.}{=} \frac{N_0}{N}\mathrm{CRB}_{N_0},$$

where $N_0 < N$.

4. NUMERICAL STUDIES

We consider $N = 50$ samples of the CARX process

$$A(p)y(t) = B(p)u(t) + e(t) \tag{20}$$

where $e(t)$ is continuous-time white noise of intensity σ_e^2, and

$$\begin{aligned}A(p) &= p^2 + a_1 p + a_2 = (p - \tilde{p}_1)(p - \tilde{p}_1^c), \\ B(p) &= b_1 p + b_2.\end{aligned}$$

The input $u(t)$ is given by the CAR process

$$C(p)u(t) = v(t) \tag{21}$$

where $v(t)$ is continuous-time white noise, independent of $e(t)$, with intensity σ_v^2, and

$$C(p) = p^2 + c_1 p + c_2 = (p - n_1)(p - n_1^c).$$

In the example to follow, we will consider the following four sampling strategies:

(a) *(Deterministic) uniform*: Here $t_n = nT$, $n = 1, \ldots, N$.

(b) *Uniformly distributed*: Here $t_n = nT + \sum_{k=1}^{n}\delta_k$, $n = 1, \ldots, N$, where δ_k is uniformly distributed between $-\tilde{\delta}$ and $\tilde{\delta}$; δ_k are independent of $e(t)$ for all t and k, and δ_k are independent of δ_j for all $j \neq k$. We choose $\tilde{\delta} = T/5$ in our example.

(c) *Two-point distributed*: $t_n = \sum_{k=1}^{n}\Delta t_k$, $n = 1, \ldots, N$, where

$$\Delta t_k = \begin{cases}\Delta_0 & \text{with probability } p_0 \\ \Delta_1 & \text{with probability } 1 - p_0.\end{cases} \tag{22}$$

Here Δt_k are independent of each other and of $e(t)$. In this case the average sample interval T is

$$T = \mathrm{E}\{\Delta t_k\} = p_0\Delta_0 + (1 - p_0)\Delta_1. \tag{23}$$

In Example 1 below, we choose $p_0 = 1/2$, $\Delta_0 = 2T/3$ and $\Delta_1 = 4T/3$. Note that this particular choice of parameters effectively models uniform sampling with randomly missing observations, which is important in some system identification applications (Isaksson, 1993).

(d) *Gapped-data sampling*:

$$\begin{aligned}\{t_n\} = \{&0, \tilde{\Delta}, \dots, (\tilde{N}-1)\tilde{\Delta}, \Delta, \Delta+\tilde{\Delta}, \dots,\\ &\Delta+(\tilde{N}-1)\tilde{\Delta}, \dots,\\ &(N_\#-1)\Delta, (N_\#-1)\Delta+\tilde{\Delta}, \dots,\\ &(N_\#-1)\Delta+(\tilde{N}-1)\tilde{\Delta}\}.\end{aligned} \tag{24}$$

Here the average sample interval is

$$T = \mathrm{E}\{t_n - t_{n-1}\} = \frac{(N_\#-1)\Delta + (\tilde{N}-1)\tilde{\Delta}}{N_\#\tilde{N}-1}. \tag{25}$$

Gapped-data sampling is of particular relevance in time-series analysis applications (Adorf, 1995; Stoica *et al.*, 2000) and radar imaging (Larsson *et al.*, 2002) but its use in system identification can also be envisioned. In the example below, the values of the parameters are $\tilde{N} = 5$, $\tilde{\Delta} = 2T/3$, $N_\# = N/\tilde{N}$ and Δ is chosen such that (25) is fulfilled.

Example 1. The aim with this example is to show how the spectrum of the input may influence the accuracy of the parameter estimates. When dealing with evenly sampled models it is well-known that the input should have its major energy at the frequencies that dominate the system. The process is described by (20) and (21), and we study the following situations:

$$\begin{aligned}\times &: T = 0.125; \tilde{p}_1 = -0.05 + i; n_1 = -0.05 + i\gamma\\ \circ &: T = 0.5; \tilde{p}_1 = -0.05 + i; n_1 = -0.05 + i\gamma\\ \square &: T = 0.5; \tilde{p}_1 = -0.02 + i; n_1 = -0.02 + i\gamma\end{aligned} \tag{26}$$

where γ varies between 0.1 and 3 and $T = \mathrm{E}[t_k - t_{k-1}]$ is the average sampling interval. In all situations we choose $b_1 = 1$, $b_2 = 1$, $\sigma_e^2 = 0.1$ and σ_v^2 such that $\operatorname{var} u(t) = 1$. All situations in (26) yield a process (20) with a transfer function $B(s)/A(s)$ with a narrow-band characteristic; see Figure 1. The spectral peak is situated at $\omega = 1$ in all situations, while the spectral width (which is associated with the real part of $\tilde{p}_1$) is different in some of the situations. Note that the spectrum of the input is affected by γ; the value of γ determines the location of the spectral peak for the spectrum of the input signal.

The expected CRB is computed by using the results in Section 3 and averaging over 50 realizations by means of Monte-Carlo simulation. The value of $\det \mathrm{CRB}(\boldsymbol{\theta})$ as a function of γ is shown in Figure 2. It is clear from the figure that $\det \mathrm{CRB}(\boldsymbol{\theta})$ has a minimum for $\gamma = 1$. This supports the claim that the input should have its energy in the frequency region that dominates the system. Furthermore, we see that there is, at least for this example, no big difference between the CRBs for different sampling strategies. ■

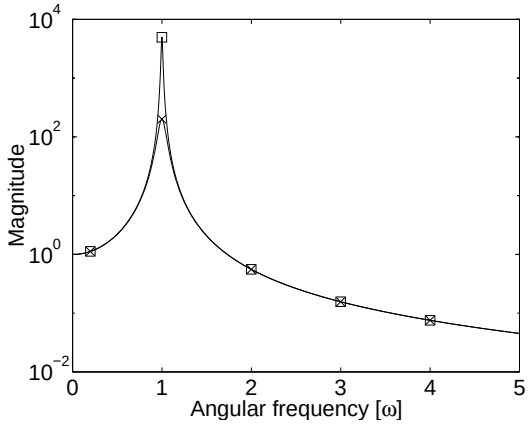

Fig. 1. Bode diagram for the transfer function $B(s)/A(s)$ of the system (20). The characteristics of the two different curves are explained in (26).

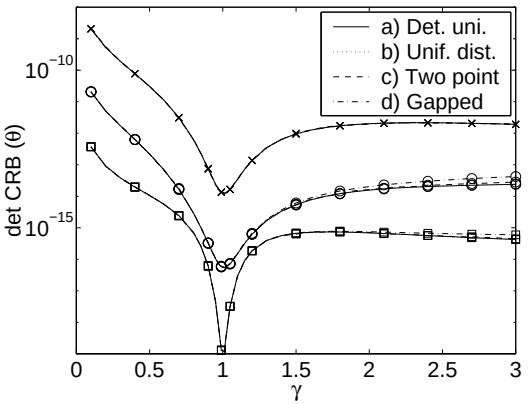

Fig. 2. Illustration of $\det \mathrm{CRB}(\boldsymbol{\theta})$ as a function of γ, for the CARX processes described in Example 1. The legend is further explained in (26). Note that many curves overlap.

5. CONCLUSIONS

A simple algorithm for computing the Cramér-Rao bound for estimation of parameters in continuous-time ARX models from irregularly sampled data was presented. The derivation was based on the Slepian-Bangs formula together with a state space framework, resulting in closed form expressions for the Cramér-Rao bound. A numerical example was included to illustrate the behavior of the CRB for different scenarios. The framework presented here for calculating the CRB for CARX

models can easily be modified to include other model structures. The results also provide us with useful tools for answering questions regarding, for example, optimal input design for irregularly sampled CARX models.

Appendix A. MATRIX DERIVATIVES

The derivatives $\mathbf{A}_i = \partial\mathbf{A}/\partial\psi_i$, $\mathbf{B}_i = \partial\mathbf{B}/\partial\psi_i$ and $\mathbf{\Sigma}_i = \partial\mathbf{\Sigma}/\partial\psi_i$, with respect to any component ψ_i of $\boldsymbol{\psi}$ in (9), i.e., with respect to $\{a_i, b_i, \sigma_e^2, c_i, d_i, \sigma_v^2\}$, are given in this appendix. The derivatives are needed in the Lyapunov equation (18), and in (19). Let $\mathbf{e}_k(j)$ be the kth column of the identity matrix of dimension j. Straightforward calculations then give

$$\frac{\partial\mathbf{A}}{\partial a_i} = \begin{bmatrix} -\mathbf{e}_i(n)\mathbf{e}_1^T(n) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \frac{\partial\mathbf{B}}{\partial a_i} = \mathbf{0}, \quad \frac{\partial\mathbf{\Sigma}}{\partial a_i} = \mathbf{0},$$

$i = 1, \ldots, n,$

$$\frac{\partial\mathbf{A}}{\partial b_i} = \begin{bmatrix} \mathbf{0} & \mathbf{e}_i(n)\mathbf{c}_u^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \frac{\partial\mathbf{B}}{\partial b_i} = \mathbf{0}, \quad \frac{\partial\mathbf{\Sigma}}{\partial b_i} = \mathbf{0},$$

$i = 1, \ldots, n,$

$$\frac{\partial\mathbf{A}}{\partial\sigma_e^2} = \mathbf{0}, \qquad \frac{\partial\mathbf{B}}{\partial\sigma_e^2} = \mathbf{0}, \qquad \frac{\partial\mathbf{\Sigma}}{\partial\sigma_e^2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\frac{\partial\mathbf{A}}{\partial c_i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{e}_i(m_1)\mathbf{e}_1^T(m_1) \end{bmatrix}, \quad \frac{\partial\mathbf{B}}{\partial c_i} = \mathbf{0}, \quad \frac{\partial\mathbf{\Sigma}}{\partial c_i} = \mathbf{0},$$

$i = 1, \ldots, m_1,$

$$\frac{\partial\mathbf{A}}{\partial d_i} = \mathbf{0}, \quad \frac{\partial\mathbf{B}}{\partial d_i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}_i(m_2) \end{bmatrix}, \quad \frac{\partial\mathbf{\Sigma}}{\partial d_i} = \mathbf{0},$$

$i = 1, \ldots, m_2,$

$$\frac{\partial\mathbf{A}}{\partial\sigma_v^2} = \mathbf{0}, \qquad \frac{\partial\mathbf{B}}{\partial\sigma_v^2} = \mathbf{0}, \qquad \frac{\partial\mathbf{\Sigma}}{\partial\sigma_v^2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

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