VARIABILITY METHOD FOR CYCLO-PERIOD ESTIMATION OF CYCLOSTATIONARY SIGNALS

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Abstract: This paper presents a new method, named as the variability method, to estimate the cyclo-period of a discrete-time cyclostationary signal. The method is essentially based on the time-varying correlation and/or mean, whose estimators are associated with some statistics of blocked signals; a plot of variability of these statistics as a function of the blocking operator index visually reveals a periodic pattern, from which the cyclo-period is obtained. The variability method is validated via simulation and real-life examples.

Keywords: Cyclostationary signals, cyclo-period estimation, blocking, variability

1. INTRODUCTION

A discrete-time cyclostationary signal is a signal whose mean and correlation are periodic sequences. Cyclostationary signals often arise due to the time-varying nature of physical phenomena, e.g., the weather (Martin, 1999), and certain man-made operations, e.g., the amplitude modulation, fractional sampling and multirate filtering (Gardner, 1994; Giannakis, 1999). The study of cyclostationary signals has applications in the area of signal processing and control, e.g., blind channel identification and equalization by fractional sampled signals (Tong *et al.*, 1994; Tong *et al.*, 1995), filter bank optimization by minimizing averaged variances of reconstruction errors (Sakai and Ohno, 1997; Ohno and Sakai, 1996), system identification by introducing cyclostationary external excitation (Gardner, 1990; Giannakis, 1995) and by fast sampling system outputs (Sun *et al.*, 2000; Wang *et al.*, 2004).

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The cyclo-period, defined as the least common multiple of periods of mean and correlation sequences, is the most fundamental parameter of a cyclostationary signal; hence, estimation of the cyclo-period should be regarded as the first step whenever the cyclo-period is required to be known *a priori*. The purpose of this paper is to present a new method to estimate the cyclo-period from a time series, i.e., given a realization of a cyclostationary signal x with unknown cyclo-period p,

$$\{x(t)\}_{t=0}^{N-1} := \{x(0), x(1), \cdots, x(N-1)\}, \quad (1)$$

how to estimate p?

In the literature, there were some methods aiming to or being applicable to cyclo-period estimation. Herbst (1965) tested the periodic fluctuation in the variance function of a cyclostationary signal via periodogram, which can be adapted to estimating the cyclo-period. Tian (1988) inferred the period from the mean function using the so-called cumulated autocovariances. Observing a phenomenon that the period of the mean function may be different from that of the variance function, Martin and Kedem (1993) formed a new periodic sequence having the period equal to the least common multiple of periods of the mean and variance functions of the original cyclostationary signal, and then detected the period via the periodogram associated with the new sequence. Hurd and Gerr (1991) obtained the cyclo-period from the bispectrum, which was estimated using the two-dimensional periodogram. Dandawaté and Giannakis (1994) aimed at the detection of cyclostationarity under a broader context, *almost* cyclostationary signals, through a statistical χ^2 test based on the cyclic covariance and cyclic spectrum, where the cyclo-period was actually estimated as well. Among these methods, Hurd-Gerr's, Martin-Kedem's, and Dandawaté-Giannakis's methods are more complete than the others and will be compared with our proposed method.

The new method to be proposed, referred to as the variability method, has many attractive features comparing to the other three methods mentioned earlier. First, it is not sensitive to stationary noises while Hurd-Gerr's and Martin-Kedem's methods are. Second, it can deal with a cyclo-period inconsistency problem, while Hurd-Gerr's and Dandawaté-Giannakis's methods cannot. Third, it is equally applicable to different types of ill-cyclostationary signals, while Martin-Kedem's method cannot handle a special ill-cyclostationarity.

The rest of the paper is organized as follows. Section 2 introduces the definition of cyclostationarity and the blocking operator. Theoretical foundation of the variability method is established in Section 3, while Section 4 illustrates the variability method through examples. Section 5 concludes the paper.

Some standard notation is used throughout the paper. $E[\cdot]$ denotes expectation. Symbols $\mathbb{Z}$ and $\mathbb{Z}_+$ stand for the set of integers and the set of nonnegative integers, respectively. Expressions $\lfloor \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the nearest integer function and the floor function, respectively. Function $\gcd(m,n)$ means the greatest common divisor of two integers m and n.

2. PRELIMINARY

A discrete-time signal x, real-valued and univariate [2], is said to be cyclostationary or cyclo-wide-sense-stationary (CWSS), if its mean and correlation are periodic sequences (Gladyshev, 1961; Gardner, 1994). In particular, x is called first-order cyclostationary (Giannakis, 1999) if its mean $m_x(t) := E[x(t)]$ is periodic:

$$m_x(t + lp_1) = m_x(t), \quad \forall t, l \in \mathbb{Z}; \qquad (2)$$

x is second-order cyclostationary (Giannakis, 1999) if its correlation

$$R_{xx}(t;\tau) := R_{xx}(t, t+\tau) := E[x(t)x(t+\tau)] \quad (3)$$

is periodic in t for a fixed τ such that

$$R_{xx}(t + lp_2; \tau) = R_{xx}(t;\tau), \forall t, l \in \mathbb{Z}. \quad (4)$$

Note that p_1 and p_2 are the smallest positive integers such that (2) and (4) hold, respectively. Due to the time dependence, $m_x(t)$ and $R_{xx}(t;\tau)$ are usually named the time-varying mean and correlation, respectively. A cyclo-period inconsistency problem occurs frequently, i.e., $p1 \neq p2$; thus, the cyclo-period p is defined as the least common multiple of p_1 and p_2, and x is said to be cyclostationary with period p, abbreviated as $(\text{CWSS})_p$. If $p_1 = p_2 = 1$, (2) and (4) say that cyclostationary signals with period 1 reduce to stationary or wide-sense stationary (Papoulis, 1965).

The n-fold discrete blocking operator L_n is defined as the mapping from a scalar sequence x to a n-dimensional vector sequence $\underline{x}_n$, where underlining denotes blocking (Meyer and Burrus, 1975; Vaidyanathan, 1993):

$$x \mapsto \underline{x}_n = \begin{bmatrix} x_n^{(0)}(t) \\ x_n^{(1)}(t) \\ \vdots \\ x_n^{(n-1)}(t) \end{bmatrix} = \left\{ \begin{bmatrix} x(nt) \\ x(nt+1) \\ \vdots \\ x(nt+n-1) \end{bmatrix} \right\}. \quad (5)$$

3. VARIABILITY ANALYSIS

The purpose of this section is to build a theoretical foundation for the variability method.

3.1 *Relationship between Two Estimators*

We present an estimator of $R_{xx}(t;\tau)$ defined in (3) and an estimator of "correlation" of $x_n^{(k)}$ defined in (5), and explore the relationship between the two estimators.

Given $\{x(t)\}_{t=0}^{N-1}$ in (1), an asymptotically unbiased and consistent estimator of $R_{xx}(t;\tau)$ can be shown to be

$$\hat{R}_{xx}(t;\tau) = \frac{1}{\left\lfloor \frac{N}{p} \right\rfloor} \sum_{k=0}^{\left\lfloor \frac{N-\tau}{p} \right\rfloor - 1} x(kp+t) \cdot x(kp+t+\tau), \qquad (6)$$

[2] The two restrictions are not necessary for the definition of cyclostationarity; they are introduced only for the sake of an easier presentation.

where $0 \leq t \leq p-1$ and $0 \leq \tau \leq N-p$[3]. Even though $x_n^{(k)}$'s are jointly stationary only at $n = lp$ (Sathe and Vaidyanathan, 1993; Sakai and Ohno, 1997), the natural correlation estimator defined for stationary signals can still be applied to obtain the following statistics,

$$\hat{R}_{x_n^{(k_1)} x_n^{(k_2)}}(\tau) = \frac{1}{\left\lfloor \frac{N}{n} \right\rfloor} \sum_{r=0}^{\left\lfloor \frac{N}{n} \right\rfloor - 1 - \tau} x(rn + k_1) \cdot x(rn + k_2 + \tau n), \quad (7)$$

where $0 \leq \tau \leq \lfloor N/n \rfloor - 1$, $0 \leq k_1 \leq n-1$, $k_1 \leq k_2$, and $x_n^{(k_2)} = q^{\lfloor k_2/n \rfloor} x_n^{(k_2 - n\lfloor k_2/n \rfloor)}$. Here q is a forward shift operator, i.e., $q\,x(t) = x(t+1)$.

Theorem 1. Given a fixed τ and $\{x(t)\}_{t=0}^{N-1}$ in (1), if $N \gg n$, $N \gg p$ and $N \gg \tau$, the two estimators in (6) and (7) are connected as

$$\hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0) \simeq \frac{1}{\bar{p}} \sum_{r=0}^{\bar{p}-1} \hat{R}_{xx}(rn + k; \tau), \quad (8)$$

where $\bar{p} = p/\gcd(p, n)$ and $0 \leq k \leq n-1$. The difference is ignorable for a large N and reduces to zero under some configurations, e.g., $\tau = 0$ and N is a common multiple of n and p.

Proof: It follows from definitions of the two estimators and the periodicity implied in (4). Details are omitted due to space limitation.

3.2 *Largest Variability at $n = lp$*

For a fixed τ, the sequence $\left\{\hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0)\right\}_{k=0}^{n-1}$ has the largest variability at $n = lp$ for $l \in \mathbb{Z}_+$, which is intuitively true:

- If $n = lp$ for $l \in \mathbb{Z}_+$, (8) reduces to

$$\begin{aligned} \hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0) &= \frac{1}{\bar{p}} \sum_{r=0}^{\bar{p}-1} \hat{R}_{xx}(rlp + k; \tau) \\ &= \frac{1}{\bar{p}} \sum_{r=0}^{\bar{p}-1} \hat{R}_{xx}(k; \tau) \\ &= \hat{R}_{xx}(k; \tau), \end{aligned}$$

  where the second equality is from (4). Cyclostationarity implies that $\hat{R}_{xx}(k;\tau)$'s are generally not the same for all integers k, i.e., some variability exists among the sequence.
- If p and n are coprime or relatively prime, it can be shown that (8) becomes

$$\hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0) = \frac{1}{p} \sum_{t=0}^{p-1} \hat{R}_{xx}(t; \tau).$$

  Thus, $\hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0)$ is invariant to k and the variability of the sequence is zero.
- Besides the above two cases, $\hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0)$ is the average of some subset of $\left\{\hat{R}_{xx}(k;\tau)\right\}_{k=0}^{p-1}$; hence, there may exist some variability, but the variability is generally smaller than that at $n = lp$ because of the averaging effect.

This heuristic argument is formally stated in Theorem 2, which needs a new definition for some special cyclostationary signals.

Definition 1. A discrete-time $(\text{CWSS})_p$ signal x is called **ill-cyclostationary in correlation at lag** τ if $R_{xx}(t;\tau)$'s are the same for all t, i.e.,

$$R_{xx}(0;\tau) = R_{xx}(1;\tau) = \cdots = R_{xx}(p-1;\tau).$$

Theorem 2. If x is not ill-cyclostationary in correlation at lag τ, the sequence $\left\{\hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0)\right\}_{k=0}^{n-1}$ has the largest variability at $n = lp$ for $l \in \mathbb{Z}_+$, in terms of the sample variance

$$\frac{1}{n} \sum_{k=0}^{n-1} \left(\hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0) - \frac{1}{n} \sum_{k=0}^{n-1} \hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0) \right)^2. \quad (9)$$

Proof: Omitted due to space limitation.

3.3 *Generalization Based on Time-Varying Mean*

Given $\{x(t)\}_{t=0}^{N-1}$ in (1), an asymptotically unbiased and consistent estimator of $m_x(t)$ is

$$\hat{m}_x(t) = \frac{1}{\lfloor N/p \rfloor} \sum_{k=0}^{\lfloor N/p \rfloor - 1} x(kp + t), \quad (10)$$

where $0 \leq t \leq p-1$. Analogously to (7), the "mean" estimator for $x_n^{(k)}$ is,

$$\hat{m}_{x_n^{(k)}} = \frac{1}{\lfloor N/n \rfloor} \sum_{r=0}^{\lfloor N/n \rfloor - 1} x(rn + k), \quad (11)$$

where $0 \leq k \leq n-1$. Then, the following are the counterparts of Theorem 1, Definition 1 and Theorem 2.

Theorem 3. Given $\{x(t)\}_{t=0}^{N-1}$ in (1), if $N \gg n$ and $N \gg p$, the two estimators in (10) and (11) are connected as

$$\hat{m}_{x_n^{(k)}} \simeq \frac{1}{\bar{p}} \sum_{r=0}^{\bar{p}-1} \hat{m}_x(rn + k),$$

where $\bar{p} = p/\gcd(p, n)$ and $0 \leq k \leq n-1$. The difference is ignorable for a large N and reduces to zero if N is a common multiple of n and p.

[3] Eq. (3) implies that $R_{xx}(t; -\tau) = R_{xx}(t; \tau)$.

Definition 2. A discrete-time $(\mathrm{CWSS})_p$ signal x is called **ill-cyclostationary in mean** if $m_x(t)$'s are the same for all integers t, i.e.,

$$m_x(0) = m_x(1) = \cdots = m_x(p-1).$$

Theorem 4. If x is not ill-cyclostationary in mean, the sequence $\left\{\hat{m}_{x_n^{(k)}}\right\}_{k=0}^{n-1}$ has the largest variability at $n = lp$ for $l \in \mathbb{Z}_+$, in terms of the sample variance

$$\frac{1}{n}\sum_{k=0}^{n-1}\left(\hat{m}_{x_n^{(k)}} - \frac{1}{n}\sum_{k=0}^{n-1}\hat{m}_{x_n^{(k)}}\right)^2. \qquad (12)$$

4. EXAMPLES

This section presents examples in a comparative manner to confirm the performance of the variability method.

Algorithms: Given $\{x(t)\}_{t=0}^{N-1}$ in (1), the cyclo-period p can be estimated by the following steps:

(1) Start with $n = 2$, i.e., block x by L_2 as defined in (5). Select an integer lag τ (usually 0 or 1) if the variability method is based on the time-varying correlation.
(2) Compute $\hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0)$ in (7) and/or $\hat{m}_{x_n^{(k)}}$ in (11), and the sample variance in (9) and/or that in (12).
(3) Repeat Steps 1 and 2 by increasing n until a reasonable number n_{max}, where $n_{max} \geq lp$ for some positive integer l. A good rule of thumb is $50 n_{max} \leq N$ (Box *et al.*, 1994).
(4) Plot the sample variance as a function of the blocking operator index n, where the largest peaks display a periodic pattern at $n = lp$, i.e., p is the smallest integer among the cluster of the largest peaks.

Example 1 The example is from (Hurd and Gerr, 1991) (see Eq. (14) therein):

$$x(t) = \left[1 + \cos\left(\frac{2\pi t}{16}\right)\right] \cdot w(t) + v(t).$$

Here w is a white noise with zero-mean and unit variance, abbreviated as $\mathrm{WN}(0,1)$, v is $\mathrm{WN}(0,\sigma_v^2)$, and w and v are mutually independent. x is second-order cyclostationary with period $p = 16$. First, to study the noise-free performance, i.e., $\sigma_v^2 = 0$, 100 independent trials in a Monte Carlo simulation are implemented and shown in Figure 1. All four methods estimate the cyclo-period $p = 16$ correctly. Second, the effect of a stationary noise is investigated by increasing σ_v^2 to 5. Figure 2 takes one typical sample of 100 trials. The variability method and Dandawaté-Giannakis's method correctly estimate the cyclo-period, while Hurd-Gerr's and Martin-Kedem's methods fail. In the variability method, a stationary noise equally contributes to $\hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0)$ or $\hat{m}_{x_n^{(k)}}$ for a fixed n; its effect is canceled off in the step of computing the variability of these statistics.

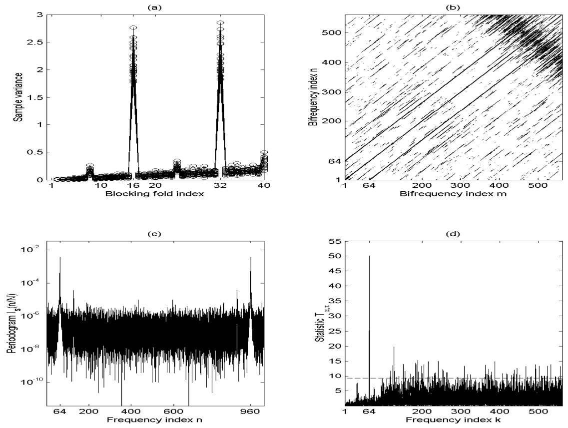

Fig. 1. $\sigma_v^2 = 0$, 100 independent trials (a) The variability method ($n = 16l$), (b) Hurd-Gerr's method ($\lfloor 1024/64 \rceil = 16$), (c) Martin-Kedem's method ($\lfloor 1024/64 \rceil = 16$), (d) Dandawaté-Giannakis's method ($\lfloor 1024/64 \rceil = 16$), 1024 is the Fast Fourier Transformation (FFT) length.

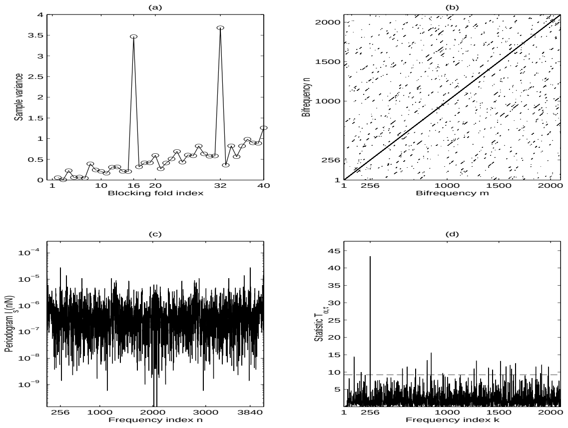

Fig. 2. $\sigma_v^2 = 5$ (a) The variability method ($n = 16l$), (b) Hurd-Gerr's method, (c) Martin-Kedem's method, (d) Dandawaté-Giannakis's method ($\lfloor 4096/256 \rceil = 16$).

The variability method based on time-varying mean is useful, especially in dealing with the cyclo-period inconsistency problem (see Section 2).

Example 2 This is an example in (Martin and Kedem, 1993) with

$$x(t) = \cos\left(\frac{2\pi t}{10}\right) + w(t),$$

where w is generated by filtering a $\mathrm{WN}(0,1)$ signal through an AR filter with parameter 0.3. x is both first- and second-order cyclostationary with periods 10 and 5, respectively. Figures 3-(a) and

(b) are obtained via the variability methods based on $m_x(t)$ and $R_x(t;0)$, respectively; the former estimates $\hat{p}_1 = 10$ and the latter gives $\hat{p}_2 = 5$; thus, their least common multiple is the correct cyclo-period $p = 10$. Martin-Kedem's method uses both the mean and variance information (see Section 1) and thus gives the correct estimate $p = 10$, shown in Figure 3-(c). Hurd-Gerr's and Dandawaté-Giannakis's methods are exclusively based on second-order statistics; therefore, they give an incorrect estimate, $\hat{p} = 5$. Figure 3-(d) shows the result of Dandawaté-Giannakis's method.

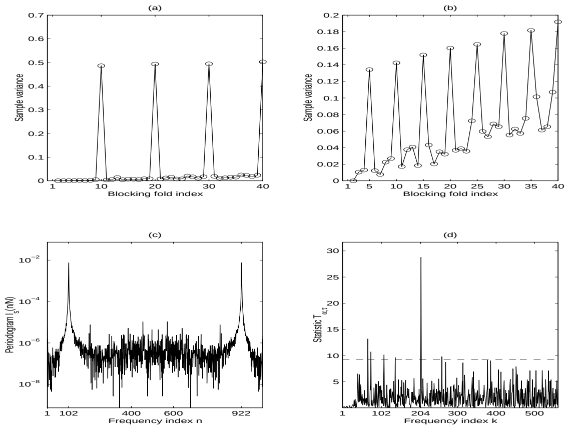

Fig. 3. Cyclo-period inconsistency (a) The variability method based on $m_x(t)$ $(n = 10l)$, (b) The variability method based on $R_{xx}(t;0)$ $(n = 5l)$, (c) Martin-Kedem's method $(\lfloor 1024/102 \rceil = 10)$, (d) Dandawaté-Giannakis's method $(\lfloor 1024/204 \rceil = 5)$.

It seems that for most second-order cyclostationary signals, $\hat{R}_{x_n^{(k)} x_n^{(k)}}(0)$ provides enough information for cyclo-period estimation. However, if the ill-cyclostationarity in correlation at lag $\tau = 0$ arises (see Definition 1), the variability method needs to rely upon $\hat{R}_{x_n^{(k)} x_n^{(k+\tau)}}(0)$ for $\tau \neq 0$.

Example 3 A discrete-time Zero-Order-Holder (ZOH) with an integer factor p, H_p, is an useful operator, e.g., in the closed-loop output fast sampling (Wang *et al.*, 2004). It takes an input sequence x and produces an output sequence $y(t) = x(\lfloor t/p \rfloor)$. The output of H_7 that is driven by a zero-mean colored noise x can be shown to be $(\text{CWSS})_7$ and ill-cyclostationary in correlation at lag $\tau = 0$. Figure 4-(a) displays the variability of $\hat{R}_{y_n^{(k)} y_n^{(k)}}(0)$ as a function of n, where an aperiodic pattern occurs, as expected. Figure 4-(b) is based on $\hat{R}_{y_n^{(k)} y_n^{(k+1)}}(0)$, i.e., $\tau = 1$, where the cyclo-period $p = 7$ is correctly estimated. Since Martin-Kedem's method is based on the mean and variance functions, it cannot deal with such an ill-cyclostationary signal, which is confirmed by Figure 4-(c). Both Hurd-Gerr's and Dandawaté-Giannakis's methods succeed in the estimation, for they are capable of using all the second-order statistical information; Figure 4-(d) shows the result obtained by Hurd-Gerr's method.

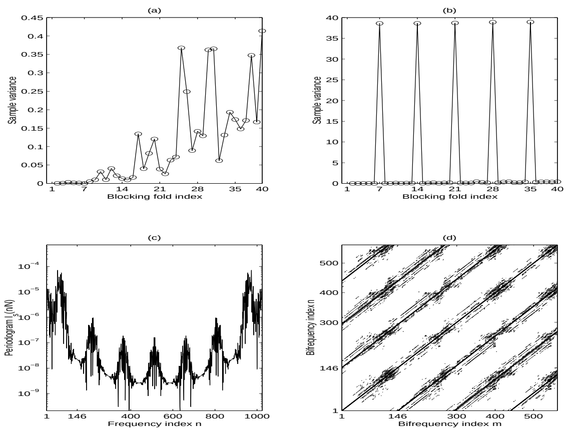

Fig. 4. Ill-cyclostationarity in correlation at lag 0 (a) The variability method based on $\hat{R}_{y_n^{(k)} y_n^{(k)}}(0)$, (b) The variability method based on $\hat{R}_{y_n^{(k)} y_n^{(k+1)}}(0)$ $(n = 7l)$, (c) Martin-Kedem's method, (d) Hurd-Gerr's method $(\lfloor 1024/146 \rceil = 7)$.

Finally, we have a real-life example to further capture the performance of the four cyclo-period estimation methods.

Example 4 The global irradiance was measured hourly for three years (1990-1992) at a meteorological station DELTA on Ellsmere Island, N.W.T., Canada. Data and more information are available at the Taconite Inlet Project website: http://www.geo.umass.edu/climate/TILPHTML /TILPhome.html. As the global irradiance is one of the measurements of solar radiation, it is plausible to conjecture that some statistics of the global irradiance have a 24 hour period rhythm; in other words, the global irradiance can be modeled as a cyclostationary signal with period 24. The four cyclo-period estimation methods are applied with results shown in Figure 5. All give $\hat{p} = 24$ that is consistent with the conjecture; however, their performances are quite different: The variability method displays a very clear periodic pattern and the other three existing methods more or less suffer from some disturbing lines/peaks. Here the data length is 4196 and the FFT lengths are given in the caption of Figure 5.

5. CONCLUSION

A new method, named as the variability method, is proposed to estimate the cyclo-period p of a discrete-time cyclostationary signal x. If x is blocked by the blocking operator L_n, $\underline{x}_n = L_n x$ defined in (5), estimators of the time-varying correlation and mean are associated with some

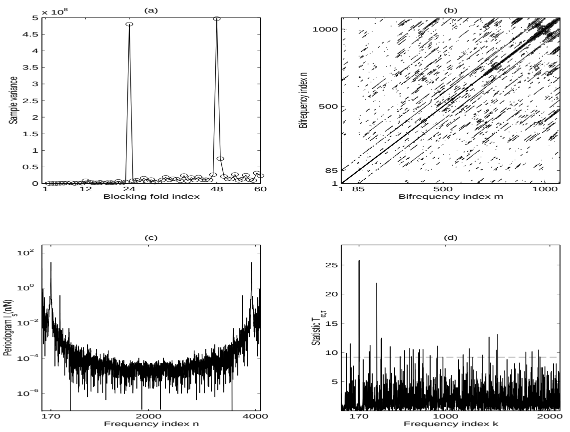

Fig. 5. Global irradiance (a) The variability method ($n = 24l$), (b) Hurd-Gerr's method ($\lfloor 2048/85 \rceil = 24$), (c) Martin-Kedem's method ($\lfloor 4096/170 \rceil = 24$), (d) Dandawaté-Giannakis's method ($\lfloor 4096/170 \rceil = 24$).

statistics of the blocked signal $\underline{x}_n$ in Theorems 1 and 3, respectively. When n is an integer multiple of p, Theorems 2 and 4 show that variability of these statistics of $\underline{x}_n$ achieves the maximum, from which the cyclo-period p is obtained. Simulation and real-life examples confirm the effectiveness of the variablity method. Comparing with the other three existing methods, the variability method shows many attractive features.

6. REFERENCES

Box, G. E. P., G. M. Jenkins and G. C. Reinsel (1994). *Time Series Analysis: Forecasting and Control*, Englewood Cliffs, NJ: Prentice-Hill.

Dandawaté, A. V. and G. B. Giannakis (1994). Statistical tests for presence of cyclostationarity, *IEEE Trans. Signal Processing*, **42(9)**, pp. 2355-2369.

Gardner, W. A. (1990). Identification of systems with cyclostationary input and correlated input/output measurement noise, *IEEE Trans. Automat. Control*, **35(4)**, pp. 449-452.

Gardner, W. A. (1994). An introduction to cyclostationary signals, in *Cyclostationarity in Communications and Signal Processing*, W. A. Gardner, Ed., Piscataway, NJ: IEEE Press.

Giannakis, G. B. (1995). Polyspectral and cyclostationary approaches for identification of closed-loop systems, *IEEE Trans. Automat. Control*, **40(5)**, pp. 882-885.

Giannakis, G. B. (1999). Cyclostationary signal analysis, in *Digital Signal Processing Handbook*, Boca Raton: CRC Press LLC.

Gladyshev, E. G. (1961). Periodically correlated random sequence, *Soviet Math.*, **2**, pp. 385-388.

Herbst, L. J. (1965). The statistical Fourier analysis of variances, *J. R. Statist. Soc.*, **B 27**, pp. 159-165.

Hurd, H. L. and N. L. Gerr (1991). Graphical methods for determining the presence of periodic correlation, *J. Times Ser. Anal.*, **12(4)**, pp. 337-350.

Martin, D. E. K. and B. Kedem (1993). Estimation of the period of periodically correlated sequences, *J. Times Ser. Anal.*, **14(2)**, pp. 193-205.

Martin, D. E. K. (1999). Detection of periodic autocorrelation in time series data via zero-crossing, *J. Times Ser. Anal.*, **20(4)**, pp. 435-452.

Meyer, R. A. and C. S. Burrus (1975). A unified analysis of multirate and periodically time-varying digital filters, *IEEE Trans. Circuits Syst.*, **22(3)**, pp. 162-168.

Ohno, S. and H. Sakai (1996). Optimization of filter banks using cyclostationary spectral analysis, *IEEE Trans. Signal Processing*, **44(11)**, pp. 2718-2725.

Papoulis, A. (1965), *Probability, Random Variables and Stochastic Processes*, New York: McGraw-Hill.

Sakai, H. and S. Ohno (1997). Theory of cyclostationary processes and its application, in *Statistical Methods in Control and Signal Processing*, pp. 327-354, New York: Marcel Dekker.

Sathe, V. P. and P. P. Vaidyanathan (1993). Effects of multirate systems on the statistical properties of random signals, *IEEE Trans. Signal Processing*, **41(1)**, pp. 131-146.

Sun, L., H. Ohmori and A. Sano (2000). Frequency domain approach to closed-loop identification based on output inter-sampling scheme, *Proc. of the American Control Conference*, **3**, pp. 1802-1806.

Tian, C. J. (1988). A limiting property of sample autocovariances of periodically correlated processes with application to period determination, *J. Times Ser. Anal.*, **9(4)**, pp. 411-417.

Tong, L., G. Xu, and T. Kailath (1994). Blind identification and equalization based on second-order statistics: a time domain approach, *IEEE Trans. Inform. Theory*, **40(2)**, pp. 340-349.

Tong, L., G. Xu, B. Hassibi and T. Kailath (1995). Blind identification and equalization based on second-order statistics: a frequency-domain approach, *IEEE Trans. Inform. Theory*, **41(1)**, pp. 329-334.

Wang, J., T. Chen and B. Huang (2004). Closed-loop identification via output fast sampling, *J. Process Control*, **14(5)**, pp. 555-570.

Vaidyanathan, P. P. (1993). *Multirate Systems and Filter Banks*, Englewood Cliffs, NJ: Prentice-Hall.