FROM NONLINEAR SYSTEMS TO PORT CONTROLLED HAMILTONIAN SYSTEMS

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Abstract: This paper gives necessary and sufficient conditions under which an affine nonlinear system is feedback equivalent to a port controlled Hamiltonian system. In particular, we identify the minimum number of linear partial differential equations that need to be solved to achieve this transformation.

Keywords: Hamiltonian systems, pseudo–gradient systems, passivity-based control, feedback equivalence.

1. INTRODUCTION

In recent years the notion of passivity for analysis and control design has been studied widely, see e.g. (van der Schaft, 1999; Sepulchre, 1996; Ortega, et al, 1998). The central question of transforming a non–passive system into a passive system via state–feedback was elegantly settled in (Byrnes, et al, 1991) where succinct, necessary and sufficient, geometric conditions are given.

On the other hand, feedback equivalence to port controlled Hamiltonian (PCH) models, which are a class of passive systems, has attracted the attention of many researchers lately, in particular for stabilization objectives. A PCH system (with dissipation) is defined as (van der Schaft, 1999):

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\,\nabla H(x) + g(x)u, \quad x \in \mathbb{R}^{\mathrm{n}}, \\ y &= g^{\top}(x)\nabla H(x), \quad u \in \mathbb{R}^{\mathrm{m}},\ \mathrm{y} \in \mathbb{R}^{\mathrm{m}}, \end{aligned} \tag{1}$$

where $H : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ is the total stored energy, $J(x) = -J^{\top}(x)$ is known as the interconnection matrix, $R(x) = R^{\top}(x) \geq 0$ represents the dissipation and $\nabla = \frac{\partial}{\partial x}$. The vector signals u and y are the conjugated port variables and their product $u^{\top}y$ has units of power. It is easy to see that, if the total energy function is non-negative, then PCH systems are passive.

2. PROBLEM FORMULATION

Given an affine system

$$\Sigma_{f,G}: \quad \dot{x} = f(x) + G(x)u, \quad x \in \mathbb{R}^{\mathrm{n}},\ \mathrm{u} \in \mathbb{R}^{\mathrm{m}}, \tag{2}$$

and the matrix $J(x)-R(x)$, when does there exists a state feedback $\beta : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{m}}$ and an energy function $H : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ such that

$$f(x) + G(x)\beta(x) = [J(x) - R(x)]\,\nabla H(x). \tag{3}$$

In this case, we say that the system $\Sigma_{f,G}$ is feedback equivalent to a PCH system (with given interconnection and damping matrices.)

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Transforming a system to be controlled into a PCH system is the central idea of the Interconnection and Damping Assignment Passivity-based Control method firstly introduced in (Ortega, 2002). A summary of some recent developments may be found in (Ortega, 2002).

Remark 2.1 As will become clear in the sequel the particular structure of the matrix $J(x)-R(x)$ does not play any role in the characterization of the class of feedback equivalent systems, therefore we will address the slightly more general problem of feedback equivalence to a pseudo–gradient system. That is, instead of the PCH model (1) we will consider pseudo–gradient systems

$$\Sigma_F \;:\; \dot{x} = F(x)\nabla H(x)$$

where $F(x)$ is a fixed, but otherwise arbitrary, $n\times n$ matrix. This yields, instead of (3), the matching equation

$$f(x) + G(x)\beta(x) = F(x)\nabla H(x), \qquad (4)$$

If (4) holds, we will say that the system $\Sigma_{f,G}$ is feedback equivalent to Σ_F.

Before giving our solution to this problem a word on notation is in order.

- All vectors, including the gradient, are column vectors.
- For all vectors and matrices which are functions of x we will write explicitly this dependence only the first time they are defined.
- Throughout the paper we will assume that all functions are sufficiently smooth.
- Finally, no particular attention is given to the characterization of the domain of validity of our statements, to which the local qualifier should be attached. The global equivalence is discussed in Section 5. Where you can see that global rank condition is not enough to assure a global equivalence.

3. TWO EQUIVALENT CONDITIONS FOR FEEDBACK EQUIVALENCE

If the matrix F is full rank, Poincare's Lemma give us directly a necessary and sufficient condition for feedback equivalence. Indeed, the vector field $F^{-1}(f+G\beta)$ is a gradient vector field, that is, (4) is satisfied for some scalar function H, if and only if

$$\nabla[F^{-1}(f+G\beta)] = \left(\nabla[F^{-1}(f+G\beta)]\right)^\top. \qquad (5)$$

The latter condition—for fixed F, f and G—translates into $\frac{n}{2}(n-1)$ PDE's in terms of β. This was the method proposed in (Ortega et al, 2002). One of the objectives of this note is to show that we can significantly reduce the number of PDE's to be solved, therefore simplifying the associated computational problem. Actually, we will identify the minimal number of PDE's that needs to be solved to achieve the feedback equivalence.

Before presenting our characterization we recall a basic linear algebra lemma.

Lemma 3.1 *Consider two linear subspaces $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^n$. If, $\dim\mathcal{S}_1 = \dim\mathcal{S}_2$ and $\mathcal{S}_1 \subset \mathcal{S}_2$ (or $\mathcal{S}_2 \subset \mathcal{S}_1$), then $\mathcal{S}_1 = \mathcal{S}_2$.*

We need the following standard assumption.

Assumption A.1 For all points $p \in \mathbb{R}^n$ there exists an open and simply connected neighborhood $\mathcal{N}_p$ such that for all $x \in \mathcal{N}_p$ we have $rank G(x) = m$. Without loss of generality, we partition [2]

$$G(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix}, \quad G_2(x) \in \mathbb{R}^{m\times m}, \qquad (6)$$

where $rank G_2(x) = m$ for all $x \in \mathcal{N}_p$.

Remark 3.2 Note that under assumption A.1, the feedback control for PCH equivalence can be uniquely determined from (4) as

$$u(x) = [G^\top(x)G(x)]^{-1}G^\top(x)\,[F(x)\nabla H(x) - f(x)]\,. \qquad (7)$$

This control will be used through the paper.

Proposition 3.3 *Under Assumption A.1, the following statements are equivalent*

(1) $\Sigma_{f,G}$ *is feedback equivalent to* Σ_F.

(2) The n PDE's

$$[I_n - \Pi_G(x)][F(x)\nabla H(x) - f(x)] = 0 \qquad (8)$$

admit a solution, where

$$\Pi_G(x) = G(x)[G^\top(x)G(x)]^{-1}G^\top(x).$$

(3) The $n-m$ PDE's

$$\begin{aligned}[F_1(x) - G_1(x)G_2^{-1}(x)F_2(x)]\nabla H(x) \\ = f_1(x) - G_1(x)G_2^{-1}(x)f_2(x)\end{aligned} \qquad (9)$$

admit a solution, where

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}, \quad F_1(x) \in \mathbb{R}^{(n-m)\times n};$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \quad f_1(x) \in \mathbb{R}^{n-m}.$$

Proof. [i) $\Rightarrow$ iii)] Define the $(n-m)\times n$ matrix [3]

$$\xi(x) = \begin{bmatrix} I_{n-m} & -G_1(x)G_2^{-1}(x) \end{bmatrix}. \qquad (10)$$

We have the following chain of implications

$$\begin{aligned} i) &\Leftrightarrow f + G\beta = F\nabla H \\ &\Rightarrow \xi(f+G\beta) = \xi F\nabla H \\ &\Leftrightarrow \xi(F\nabla H - f) = 0 \\ &\Leftrightarrow iii) \end{aligned} \qquad (11)$$

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[2] This partition can always be locally achieved simply swapping and relabelling the state equations.

[3] Clearly, ξ is a left annihilator of G, that is $\xi G = 0$.

[iii) $\Rightarrow$ i)] This will be established by contradiction. Assume iii) does not hold. From (9), or (11), we see that this is equivalent to saying that

$$F\nabla H - f \ni \text{Ker } \xi, \tag{12}$$

We will prove now that Ker ξ = Im G. First, note that both spaces have the same dimension, m. Consider then the chain of implications:

$$\begin{aligned} a \in \text{Im } G &\Leftrightarrow \exists b \in \mathbb{R}^{\text{n}} \ : \ \text{a} = \text{Gb} \\ &\Rightarrow \xi a = \xi G b = 0 \\ &\Rightarrow a \in \text{Ker } \xi \\ &\Rightarrow \text{Im } G \subset \text{Ker } \xi \end{aligned}$$

Finally, we can invoke Lemma 3.1 to conclude that Ker ξ = Im G.

From the proof above we have that (12) is equivalent to

$$F\nabla H - f \ni \text{Im } G,$$

but the latter contradicts i), that states the existence of β such that $F\nabla H - f = G\beta$.

[iii) $\Leftrightarrow$ ii)] To prove this equivalence we will establish that

$$\text{Ker } \xi = \text{Ker } (I_n - \Pi_G),$$

which, together with (8) and (11), shows that the set of solutions of both PDE's are the same—completing the proof. Towards this end, define two $n \times n$ matrices

$$C(x) = \left[\xi^\top(x)[\xi(x)\xi^\top(x)]^{-1} \ G(x)\right],$$

$$D(x) = \begin{bmatrix} \xi(x) \\ (G^\top(x)G(x))^{-1}G^\top(x) \end{bmatrix}.$$

Now, using $\xi G = 0$ we have that $DC = I_n$, and consequently $D = C^{-1}$. This also implies that $CD = I_n$, which doing the computations is equivalent to

$$\Pi_{\xi^\top}(x) = I_n - \Pi_G(x). \tag{13}$$

where we have defined the projector matrix

$$\Pi_{\xi^\top}(x) = \xi^\top(x)[\xi(x)\xi^\top(x)]^{-1}\xi(x).$$

We will prove now, by contradiction, that

$rank\Pi_{\xi^\top} = n - m$. Assume $rank\Pi_{\xi^\top} < n - m$. We have the following set of equations, that lead to a contradiction,

$$\begin{aligned} n - m &= rank\xi^\top \\ &= rank\Pi_{\xi^\top}\xi^\top \\ &\leq \min\{rank\Pi_{\xi^\top}, rank\xi^\top\} \\ &< n - m, \end{aligned}$$

where we used $\Pi_{\xi^\top}\xi^\top = \xi^\top$ for the second identity, and to obtain the third line we invoked the fact that, for any pair of conformal matrices A, B,

$$rankAB \leq \min\{rankA, rankB\}.$$

Therefore, $rank\Pi_{\xi^\top} = n - m$ and consequently $\dim \text{Ker } \Pi_{\xi^\top} = m$.

To conclude the proof we recall that $\dim \text{Ker } \xi = m$, and given that Ker $\xi \subset$ Ker $\Pi_{\xi^\top}$, we have that

$$\begin{aligned} \text{Ker } \xi &= \text{Ker } \Pi_{\xi^\top} \\ &= \text{Ker } [I_n - \Pi_G] \end{aligned}$$

where we have invoked Lemma 3.1 for the first identity and (13) for the second. □

Remark 3.4 The proposition establishes the interesting fact that the set of solutions of the n PDE's (8) exactly coincides with the set of solutions of the $n - m$ PDE's (9)—equivalently, (11). (The lack of such a formal statement was a source of some confusion on the literature, see e.g., (Ortega, 2002).) The proposition also gives alternative parameterizations of the matching equation (4) that complements the original proposal of (Ortega, 2002) to solve the $\frac{n}{2}(n-1)$ PDE's (5).

Remark 3.5 Although the left annihilator matrix of G is not uniquely defined the proposition remains unaffected if we choose a matrix different from (10).[4] Indeed, let us take any left annihilator, say $\bar{\xi}(x)$, of G, and partition it as

$$\bar{\xi}(x) = \left[\bar{\xi}_1(x) \ \bar{\xi}_2(x)\right], \quad \bar{\xi}_1(x) \in \mathbb{R}^{(\text{n}-\text{m})\times(\text{n}-\text{m})}.$$

The left annihilator condition $\bar{\xi}G = 0$ imposes the relationship $\bar{\xi}_2 = -\bar{\xi}_1 G_1 G_2^{-1}$ that leads to the factorization

$$\bar{\xi} = \bar{\xi}_1 \left[I_{n-m} \ -G_1 G_2^{-1}\right].$$

For all full–rank matrices $\bar{\xi}_1$, $\bar{\xi}$ has the same kernel as ξ, whence the proposition remains unaltered by the use of this "new" left annihilator.

Remark 3.6 The necessity of (8) for feedback equivalence can be easily established as follows. (4) implies that

$$G^\top G\beta = G^\top(F\nabla H - f),$$

which together with invertibility of[5] $G^\top G$ defines, uniquely, the control β. Equation (8) is then obtained plugging the expression of β in (4).

Remark 3.7 It is easy to prove that Im $\xi^\top$ is the orthogonal complement of Im G, that is

$$\text{Im } G \oplus \text{Im } \xi^\top = \mathbb{R}^{\text{n}},$$

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[4] Of course, as a space, the orthogonal complement of Im G is uniquely defined. See Remark 11.

[5] This follows from the fact that $G^\top G$ is the Gram matrix of a set of linearly independent vectors.

with $\oplus$ denoting direct sum. This stems from the fact that $\Pi_{\xi^\top}$ is an orthogonal projector (over the rows of ξ), whence

$$\begin{aligned} \text{Im } \Pi_{\xi^\top} \oplus \text{Ker } \Pi_{\xi^\top} &= \mathbb{R}^n, \\ \text{Im } \Pi_{\xi^\top} &= \text{Im } \xi^\top, \\ \text{Ker } \Pi_{\xi^\top} &= \text{Ker } \xi \end{aligned}$$

and that, as shown in the proof above, Ker $\xi =$ Im G.

4. REDUCING THE NUMBER OF PDE'S

This section provides the minimum number of linear PDE's to be solved for feedback equivalence. Rewrite (9) in the form

$$W(x)\nabla H(x) = s(x) \tag{14}$$

where we have defined

$$\begin{aligned} W(x) &= F_1(x) - G_1(x)G_2^{-1}(x)F_2(x), \\ s(x) &= f_1(x) - G_1(x)G_2^{-1}(x)f_2(x). \end{aligned}$$

Assumption A.2 For all points $p \in \mathbb{R}^n$ there exists an open and simply connected neighborhood $\mathcal{N}_p$ such that, for all $x \in \mathcal{N}_p$, $rank W(x) = \ell$ and assume

A.2.1 $\ell < n - m$, **A.2.2** $s \in \text{Im } W$

Clearly, if $\ell = n - m$ we are in the situation of iii) of Proposition 1. Hence, to reduce the number of PDE's we need Assumption A.2.1. Furthermore, if the latter holds, Assumption A.2.2 is necessary for solvability of (14).

It is well known that there exists an $n \times n$ nonsingular matrix $Q(x)$ that extracts the full–rank part of W. That is, such that

$$W(x)Q(x) = \left[W_0(x) \ 0_{(n-m)\times(n-\ell)}\right], \tag{15}$$

where $W_0 \in \mathbb{R}^{(n-m)\times \ell}$ verifies $rank W_0 = \ell$. Let us partition

$$Q^{-1}(x) = \begin{bmatrix} M(x) \\ N(x) \end{bmatrix}, \quad M(x) \in \mathbb{R}^{\ell \times n}. \tag{16}$$

Proposion 4.1 *Under Assumptions A.1 and A.2, the following statements are equivalent*

(1) $\Sigma_{f,G}$ *is feedback equivalent to* Σ_F.
(2) The ℓ *PDE's*

$$M(x)\nabla H(x) = [W_0^\top(x)W_0(x)]^{-1}W_0^\top(x)s(x) \tag{17}$$

admit a solution.

Furthermore, the equation (17) is independent of the choice of the matrix Q *in the construction of (15).*

Proof. First, notice that since Im $W =$ Im W_0 we have that Assumption A.2.2 can be equivalently stated as $s \in$ Im W_0. Now, this condition implies that

$$\Pi_{W_0}(x)s(x) = s(x) \tag{18}$$

where

$$\Pi_{W_0}(x) = W_0(x)[W_0^\top(x)W_0(x)]^{-1}W_0^\top(x)$$

is an orthogonal projector (over the columns of W_0.)

We have established in Proposition 1 that $\Sigma_{f,G}$ is feedback equivalent to Σ_F if and only if (9), or equivalently (14), hold. We then have the following set of equivalences

$$\begin{aligned} W\nabla H = s &\Leftrightarrow WQQ^{-1}\nabla H = s \\ &\Leftrightarrow \left[W_0 \ 0_{(n-m)\times(n-\ell)}\right] \begin{bmatrix} M \\ N \end{bmatrix} \nabla H = s \\ &\Leftrightarrow W_0 M \nabla H = \Pi_{W_0} s \\ &\Leftrightarrow (17), \end{aligned}$$

where we have used (15) and (16) to get the second equivalence, (18) in the third, and the full rank condition of W_0 for the latter. This establishes the equivalence between i) and ii).

Assume now that the extraction of the full–rank part of W is done with another matrix $\bar{Q}(x)$, that is, instead of (15) we have

$$W(x)\bar{Q}(x) = \left[\bar{W}_0(x) \ 0_{(n-m)\times(n-\ell)}\right], \tag{19}$$

where $\bar{W}_0 \in \mathbb{R}^{(n-m)\times \ell}$ verifies $rank \bar{W}_0 = \ell$. We note that Im $W_0 =$ Im $\bar{W}_0$, therefore we can also construct a projector $\Pi_{\bar{W}_0}$ that leaves s invariant as in (18). It can actually be shown that the two projectors are the same, that is $\Pi_{W_0} = \Pi_{\bar{W}_0}$. Mimicking the steps of the proof with the new matrix $\bar{Q}$ we obtain in the one before the last step

$$\bar{W}_0\bar{M}\nabla H = \Pi_{W_0}s,$$

where we have partitioned

$$\bar{Q}^{-1} = \begin{bmatrix} \bar{M} \\ \bar{N} \end{bmatrix}, \quad \bar{M} \in \mathbb{R}^{\ell \times n}.$$

Now, from (19) and (15) one obtains

$$\left[\bar{W}_0 \ 0_{(n-m)\times(n-\ell)}\right]\bar{Q}^{-1} = \left[W_0 \ 0_{(n-m)\times(n-\ell)}\right]Q^{-1}$$

which proves that $\bar{W}_0\bar{M} = W_0 M$, completing the proof. □

Remark 4.2 Although the construction of the matrix Q is standard, for the sake of completeness, we outline here the procedure. Referring to (16), select the $\ell \times n$ matrix M such that its ℓ rows span Im W, this can be done choosing elements proportional to the euclidian orthonormal basis of $\mathbb{R}^\ell$. Then, select and $(n-\ell)\times n$ matrix N, also from the orthonormal basis, to complete the rank. It is easy to see that this construction yields the desired decomposition (15).

5. GLOBAL EQUIVALENCE

This section considers global equivalence to PCH. We assume the following, which is a global version of A.1:

Assumption A.3 There exists an $n \times (n-m)$ matrix $\Psi(x)$ such that

$$B(x) := \left[G(x)\ \ \Psi(x)\right], \quad x \in \mathbb{R}^n$$

is nonsingular.

Next, we define

$$\begin{aligned} E(x) &:= \left[I_n - \prod_G(x)G(x)\right] \\ &= \left[I - G(x)(G(x)^\top G(x))^{-1}G(x)^\top\right]. \end{aligned}$$

We first claim that $E(x)$ has constant rank.

Lemma 5.1 *Assume A1 holds. Then*

$$rank(E(x)) = n - m. \tag{20}$$

Proof. From A1 one sees that $g_1(x), \cdots, g_m(x)$ are linearly independent. It is easy to see that

$$E(x)\left(g_1(x)\ \cdots\ g_m(x)\right) = 0.$$

So $rank(E(x)) \le n - m$.

Denote by $\mathcal{G}(x) = Span\{g_1(x), \cdots, g_m(x)\}$. Then for any vector field $Z(x) \in \mathcal{G}^\perp(x)$

$$E(x)Z(x) = Z(x), \quad \forall Z(x) \in \mathcal{G}^\perp(x).$$

That is, $Span\,\mathrm{col}\{E(x)\} \supset \mathcal{G}^\perp(x)$ Hence

$rank(E(x)) \ge n - m$. Therefore, (20) follows. □

Lemma 5.2 *Assume A3 holds. Then there exists an orthogonal matrix $P(x)$, $x \in \mathbb{R}^n$, such that*

$$P(x)^{-1}E(x)P(x) = \begin{bmatrix} I_{n-m} & 0 \\ 0 & 0 \end{bmatrix}, \quad x \in \mathbb{R}^n. \tag{21}$$

Proof. Using A3, we denote

$$B^{-1}(x) = \begin{bmatrix} * \\ \xi_1^T(x) \\ \vdots \\ \xi_{n-m}^T(x) \end{bmatrix}, \quad x \in \mathbb{R}^n,$$

where $*$ is the first m less important rows. Now it is clear that

$$\mathcal{G}^\perp(x) = Span\{\xi_1(x), \cdots, \xi_{n-m}(x)\}, \quad x \in \mathbb{R}^n.$$

Moreover, from the proof of Lemma 5.1, we have

$$E(x)\xi_i(x) = \xi_i(x), \quad i = 1, \cdots, n-m, \quad x \in \mathbb{R}^n; \tag{22}$$

and

$$E(x)g_i(x) = 0, \quad i = 1, \cdots, m, \quad x \in \mathbb{R}^n. \tag{23}$$

Normalizing $\{\xi_1(x), \cdots, \xi_{n-m}(x)\}$ by Gram-Schmidt orthogonalization algorithm as

$$\bar{\xi}_1(x) = \xi_1(x)/\|\xi_1(x)\|;$$

$$\begin{cases} \tilde{\xi}_k(x) = \xi_k(x) - \left\langle \xi_k(x), \bar{\xi}_1(x)\right\rangle \bar{\xi}_1(x) - \cdots \\ \qquad\quad - \left\langle \xi_k(x), \bar{\xi}_{k-1}(x)\right\rangle \bar{\xi}_{k-1}(x), \\ \bar{\xi}_k(x) = \tilde{\xi}_k(x)/\|\tilde{\xi}_k(x)\|, \quad k = 2, \cdots, n-m. \end{cases}$$

Similarly, $g_1(x), \cdots, g_m(x)$ can be normalized as $\bar{g}_1(x), \cdots, \bar{g}_m(x)$.

Define

$$P(x) = \left(\bar{\xi}_1(x), \cdots, \bar{\xi}_{n-m}(x)\ \bar{g}_1(x)\ \cdots\ \bar{g}_m(x)\right), \quad x \in \mathbb{R}^n.$$

Then it is ready to check that $P(x)$ is an orthogonal matrix and (21) holds. □

Now left multiplying (8) by $P^{-1}(x)$ yields

$$\begin{bmatrix} I_{n-m} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}(x) \left[F(x)\frac{\partial H}{\partial x} - f(x)\right] = 0. \tag{24}$$

Denote

$$P^{-1}(x)F(x) := \begin{pmatrix} W_{11}(x) & W_{12}(x) \\ W_{21}(x) & W_{22}(x) \end{pmatrix}, \tag{25}$$

where $W_{11}(x)$ is an $(n-m) \times (n-m)$ matrix, the other blocks have corresponding dimensions. Similarly, decompose

$$P^{-1}(x)f(x) := \begin{pmatrix} s(x) \\ s^c(x) \end{pmatrix}, \tag{26}$$

$s(x)$ is the first $n-m$ components.

Now we need the global version of assumption A2 as

Assumption A.4 Let

$$W := \left[W_{11}(x)\ \ W_{12}(x)\right].$$

Then $rank W(x) = \ell$, $x \in \mathbb{R}^n$ and assume

A.4.1 $\ell < n - m$, **A.4.2** $s(x) \in \operatorname{Im} W$;

A.4.3 there exists a nonsingular matrix $Q(x)$, such that

$$\left(W(x)\right)Q(x) = \left(W_0(x)\ \ 0\right), \quad x \in \mathbb{R}^n, \tag{27}$$

where $W_0(x)$ is an $(n-m) \times \ell$ matrix of rank ℓ.

Then Proposition 4.1 becomes a global result:

Corollary 5.3 *Assume A3 and A4. Then Proposition 4.1 is globally true on $\mathbb{R}^n$*

6. AN ILLUSTRATIVE EXAMPLE

Consider global equivalence to PCH of the following system

$$\begin{aligned} \dot{x} = &\begin{bmatrix} \cos x_2(\cos x_2 - \sin x_2) + 1 \\ \sin x_2(\cos x_2 - \sin x_2) \\ -\cos x_2 - \sin^2 x_2 \end{bmatrix} \phi(x) \\ &+ \begin{bmatrix} \cos x_2 \\ \sin x_2 \\ \cos x_2 - \sin x_2 \end{bmatrix} u, \end{aligned} \tag{28}$$

where $x \in \mathbb{R}^3$ and $\phi(x) = x_3^3 - x_1$. For notational compactness, we denote

$$S = \sin x_2, \quad C = \cos x_2, \quad \mu = \sqrt{2 - 2SC}.$$

Choosing

$$\Psi(x) = \begin{bmatrix} 1 & S \\ -1 & -C \\ -1 & 0 \end{bmatrix},$$

it is easy to check that $B(x) = [g(x) \quad \Psi(x)]$ is nonsingular and

$$B^{-1}(x) = \frac{1}{\mu^2} \begin{bmatrix} C & S & -S + C \\ C(C-S) & S(C-S) & -1 \\ 2S - C & S - 2C & C + S \end{bmatrix}.$$

Hence

$$\xi_1 = \frac{1}{\mu^2} \begin{bmatrix} C(C-S) \\ S(C-S) \\ -1 \end{bmatrix}, \quad \xi_2 = \frac{1}{\mu^2} \begin{bmatrix} 2S - C \\ S - 2C \\ C + S \end{bmatrix}$$

Normalizing them, we have

$$\bar{\xi}_1 = \frac{1}{\mu} \begin{bmatrix} C(C-S) \\ S(C-S) \\ -1 \end{bmatrix}, \quad \bar{\xi}_2 = \begin{bmatrix} S \\ -C \\ 0 \end{bmatrix}.$$

Then

$$P = [\bar{g} \ \bar{\xi}_1 \ \bar{\xi}_2] = \begin{bmatrix} \frac{C(C-S)}{\mu} & S & -\frac{C}{\mu} \\ \frac{S(C-S)}{\mu} & -C & \frac{S}{\mu} \\ -\frac{1}{\mu} & 0 & \frac{C-S}{\mu} \end{bmatrix},$$

and $P^{-1} = P^T$.

Now assume the structure required is

$$J = \begin{bmatrix} 0 & (-SC - S^2)/2 & S^2 - 0.5 \\ (SC + S^2)/2 & 0 & -S^2 - SC \\ 0.5 - S^2 & S^2 + SC & 0 \end{bmatrix}, \tag{29}$$

$$R = \begin{bmatrix} C^2 + S^2 - CS & S(C-S)/2 & -S^2 - 0.5 \\ S(C-S)/2 & S^2 & (3SC - S^2)/2 \\ -S^2 - 0.5 & (3SC - S^2)/2 & 2 + 2(C-S)^2 \end{bmatrix}. \tag{30}$$

Then

$$P^{-1}F(x) = P^{-1}(J-R) = \begin{bmatrix} -\mu & 0 & 2\mu \\ -S & 0 & 2S \\ 0 & -S\mu & 2(S-C)\mu \end{bmatrix}.$$

It follows that

$$W = \begin{bmatrix} -\mu & 0 & 2\mu \\ -S & 0 & 2S \end{bmatrix}.$$

$$\ell = rank(W) = 1.$$

It is easy to calculate that

$$P^{-1}f(x) = \begin{bmatrix} \mu \\ S \\ 2C - 2C^2S \end{bmatrix} \phi(x)$$

So

$$s(x) = \begin{pmatrix} \mu \\ S \end{pmatrix} \phi(x) \in Span\{W\}.$$

Now all the conditions of A3 and A4 are satisfied. So the only thing we have to check is the solvability of (17). Now

$$Q(x) = Q = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -\mu \\ -S \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M(x) = M = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}.$$

Equation (17) can be easily obtained as

$$\frac{\partial H}{\partial x_1} - 2\frac{\partial H}{\partial x_3} = x_1 - x_3^3. \tag{31}$$

An obvious solution is:

$$H(x) = \frac{1}{2}x_1^2 + x_2^2 + \frac{1}{8}x_3^4. \tag{32}$$

So system (28) has a feedback Hamiltonian equivalent form with Structure matrices J, R and Hamiltonian function H as in (29), (30) and (32) respectively.

7. CONCLUSION

The problem of feedback equivalence—via state–feedback control—of affine systems to PCH systems is investigated. Necessary and sufficient conditions, expressed in terms of solvability of sets of linear PDE's, are given. The minimum number of linear PDE's necessary for the solvability is presented.

REFERENCES

C. Byrnes, A. Isidori and J.C. Willems, Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems, *IEEE Trans Aut. Cont.*, Vol.36, no.11, pp. 1228–1240, 1991.

R. Ortega, A. Loria, P.J. Nicklasson and H. Sira-Ramirez, *Passivity–based control of Euler–Lagrange systems*, Springer-Verlag, Berlin, Communications and Control Engineering, Sept. 1998.

R. Ortega, A. van der Schaft, B. Maschke and G. Escobar, Interconnection and damping assignment passivity–based control of port– controlled hamiltonian systems. *Automatica*, vol. 38, no. 4, April 2002.

R. Sepulchre, M. Jankovic and P.V. Kokotović, *Constructive nonlinear control*, Springer-Verlag, Berlin, 1996.

A. J. van der Schaft, *L_2-gain and passivity techniques in nonlinear control*, Berlin, Springer, 1999.