SOLVABILITY CONDITIONS OF DISTURBANCE REJECTION BY MEASUREMENT FEEDBACK FOR MIMO NOLINEAR SYSTEMS

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Abstract: The problem of disturbance decoupling with measurement feedback (DDPMF) has been studied for single-input single-output systems and certain square-invertible nonlinear systems. In this paper, we deal with general multi-input multi-output nonlinear systems. Conditions and a computational algorithm for solving the DDPMF are presented in the differential vector space framework.

Keywords: Disturbance decoupling, Measurement feedback, Nonlinear systems.

1. INTRODUCTION

Disturbance decoupling has been studied for a long time. For linear systems, the problems of disturbance decoupling with state feedback (DDP), with state feedback and stability (DDPS) and with dynamic measurement feedback and internal stability (DDPMS) have been solved (refer to (Wonham, 1979)). The problem of disturbance decoupling with constant or static measurement feedback (DDPCM) for linear systems is a very difficult problem. There have been some results on this problem, for example, (Hamano and Furuta, 1975; Koumboulis and Tzierakis, 1998; Chen, 1997; Chen, 2000). The problem of disturbance decoupling with measurement feedback(DDPMF) for nonlinear systems is much more complicated than that of linear systems.

Consider the following nonlinear system with disturbance:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)w \\ y &= h(x), \quad z = h_m(x) \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n, y \in \mathcal{R}^{p_1}, u \in \mathcal{R}^m, z \in \mathcal{R}^{p_2}$ and $w \in \mathcal{R}^q$ are the state, controlled output, control input, measurement output and disturbance input of the system, respectively. In this paper, we consider that all functions, vector fields and maps in system (1) are analytic over an open and dense subset of its state space $\mathcal{R}^n$.

The problem of disturbance decoupling with measurement feedback (DDPMF) for nonlinear systems is stated as follows. *Find, if possible, a measurement feedback $u = \alpha(z) + \beta(z)v$ such that the disturbance w has no effect upon the controlled output y of the closed loop system.* For nonlinear systems there are only few results on the DDPMF. An early work was given by (Andiarti and Moog, 1996) and a complete solution of the DDPMF for single-input single-output

(SISO) nonlinear systems was presented by (Xia and Moog, 1999). (Pothin, 2001) studied the disturbance decoupling problem by static feedback of measured variables for some square-invertible nonlinear system.

In this paper, we study the solvability conditions of the DDPMF for more general MIMO nonlinear systems, which may be not square-invertible. Some algorithms are presented for checking the solvability conditions and finding the feedback laws. Using the algorithms one can check-up the insolvability of DDPMF in a finite number of steps. If the DDPMF is solvable, then one can find an exact solution or an approximation solution with sufficient accuracy.

2. SUBSPACES IN A DIFFERENTIAL VECTOR SPACE

In this work we use the notions of the differential field and differential vector space ((Di Benedetto and Moog, 1989); Conte, et al., 1999). Let $\mathcal{K}$ be the quotient field of analytic functions, i.e. the meromorphic function field (Conte, et al., 1999), of the variables x_i, w_j, and $u, \dot{u}, \ddot{u}, \cdots, u^{(k)}$ for $i \in \underline{n}$, $j \in \underline{m}$ and $k \geq 0$. Then $\mathcal{K}$ is defined by the nonlinear control system (1). Over the differential field $\mathcal{K}$, the system (1) defines two most fundamental linear vector spaces.

$$\mathcal{D} = span_{\mathcal{K}}\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial u_j^{(k)}}, \frac{\partial}{\partial w_r^{(s)}}; \quad i \in \underline{n},\ j \in \underline{m}, r \in \underline{q}\ k, s \geq 0\}$$

and its dual space of $\mathcal{D}$ *over* $\mathcal{K}$

$$\mathcal{D}^* = span_{\mathcal{K}}\{dx_i, du_j^{(k)}, dw_r^{(s)}; \quad i \in \underline{n},\ j \in \underline{m}, r \in \underline{q},\ k, s \geq 0, \}$$

The dual space $\mathcal{D}^*$ is defined by the nonlinear control system (1) and is a differential linear vector space over $\mathcal{K}$.

$$\mathcal{X}^* := span_{\mathcal{K}}\{dx\}$$
$$\mathcal{U}^* := span_{\mathcal{K}}\{du, d\dot{u}, d\ddot{u}, \cdots, du^{(k)}, \cdots\}$$
$$\mathcal{W}^* := span_{\mathcal{K}}\{dw, d\dot{w}, \cdots\}$$
$$\mathcal{Y}^* := span_{\mathcal{K}}\{dy, d\dot{y}, d\ddot{y}, \cdots, dy^{(k)}, \cdots\}$$
$$\mathcal{Z}^* := span_{\mathcal{K}}\{dz, d\dot{z}, d\ddot{z}, \cdots, dz^{(k)}, \cdots\}$$

where dx stands for $\{dx_1, dx_2, \cdots, dx_n\}$, $du^{(k)}$ stands for $\{du_1^{(k)}, du_2^{(k)}, \cdots, du_m^{(k)}\}$; $k = 0, 1, 2, \cdots$, and so on. $\mathcal{X}^*$, $\mathcal{U}^*$, $\mathcal{Y}^*$ and $\mathcal{Z}^*$ are called the *dual* state space, *dual* input space, *dual* controlled output space and *dual* measurable output space, respectively, where the superscript * is to emphasize that these dual spaces are vector spaces in the differential form.

Let $\mathcal{C}^*$ be a s-dimensional integrable subspace of $\mathcal{X}^*$, i.e. we can write $\mathcal{C}^* = span_{\mathcal{K}}\{d\phi(x)\}$, where $\phi(x) := (\phi_1(x), \phi_2(x), \cdots, \phi_s(x))^\tau$, $\phi_i(x) \in \mathcal{K}, i \in \underline{s}$. The derivative subspace of $\mathcal{C}^*$, denoted by $\dot{\mathcal{C}}^*$, is defined as $\dot{\mathcal{C}}^* = span_{\mathcal{K}}\{d\dot{\phi}(x)\}$.

Without disturbance, the nonlinear system (1) is written as

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{2}$$

Definition 2.1 Given system (2), an integrable subspace $\mathcal{C}^*$ of $\mathcal{X}^*$ is called controlled invariant if there are $m' \leq m$ functions affine in u and written as $\phi_i(x, u) = a_i(x) + \sum_{j=1}^{m} b_{ij}(x)u_j; i \in \underline{m'}$ with $\{d\phi_i(x, u); i \in \underline{m'}\}$ (mod $\mathcal{X}^*$) being a set of independent vectors, such that

$$\mathcal{C}^* + \dot{\mathcal{C}}^* = \mathcal{C}^* \oplus span_{\mathcal{K}}\{d\phi_i;\ i \in \underline{m'}\}$$

The controlled invariant subspace of $\mathcal{X}^*$ was defined by (Huijberts and Andiarti, 1997) with non-exact one-forms and it is a dual-notion of the *controlled invariant distribution* in the differential geometry approach ((Isidori, 11995).

Definition 2.2 (Zheng and Evans, 2000) A function of the system state $\varphi(x) \in \mathcal{K}$ is observable for system (2) if under the condition $\varphi(x_a(0)) \neq \varphi(x_b(0))$, for two initial states $x_a(0)$ and $x_b(0)$, there exists a control $u(t)$ such that the system output satisfies

$$y(t, x_a(0), u(t)) \not\equiv y(t, x_b(0), u(t))$$

The observable dual state space, denoted by $\mathcal{O}^*$, of the nonlinear system (2) is defined as

$$\mathcal{O}^* = span_{\mathcal{K}}\{d\varphi(x); \varphi(x) \text{ is observable}\}$$

Definition 2.3 (Zheng and Evans, 2000) A function of the system state $\varphi_s(x) \in \mathcal{K}$ is strongly observable for system (2) if for any state feedback control $u = \alpha(x) + u_s$ it is an observable function for its closed loop system.

The strongly observable dual state space, denoted by $\mathcal{O}_s^*$, of the nonlinear system (2) is defined as $\mathcal{O}_s^* = span_{\mathcal{K}}\{d\varphi_s(x); \varphi_s(x) \text{ is strongly observable}\}$

Proposition 2.1 (i) The subspace $\mathcal{O}_s^* (\subset \mathcal{O}^*)$ of $\mathcal{X}^*$ is invariant under state feedbacks. (ii) $\mathcal{O}_s^*$ is a controlled invariant subspace of $\mathcal{X}^*$, i.e. there exists a $\phi_u^\tau = (\phi_1(x, u), \cdots, \phi_{m_1}(x, u))$ with dim $span_{\mathcal{K}}\{d\phi_u\} = m_1 \leq m$ (mod $\mathcal{X}^*$) such that

$$\mathcal{O}_s^* + \dot{\mathcal{O}}_s^* = \mathcal{O}_s^* \oplus span_{\mathcal{K}}\{d\phi_u\} \tag{3}$$

Remark 2.1 In Proposition 2.1, ϕ_u may not be uniquely defined, but $\mathcal{O}_s^* + \dot{\mathcal{O}}_s^*$ is invariant.

As $p(x)$ in system (1) can be written as $p(x) = (p_1(x)\ \ p_2(x)\ \ \cdots\ \ p_q(x))$, where $p_i; i \in \underline{q}$ are vector fields, the distribution spanned by $p(x)$ is denoted by $\mathcal{P} = span_{\mathcal{K}}\{p_1, p_2, \cdots, p_q\}(\subset \mathcal{D})$. The observable subspace $\mathcal{O}_s^*$ of $\mathcal{X}^*$ can be constructed by $\mathcal{O}_s^*$-*Algorithm* (see (Zheng, 1993)) and the DDP of system (1) is solvable if and only if

$$\mathcal{O}_s^* \subset \mathcal{P}^{\perp} \tag{4}$$

The (4) is not a sufficient condition for the DDPMF (see (Xia and Moog, 1999)). In the remainder of this paper, it is assumed that the system (1) is observable from the measurement of y and its DDP is solvable. Furthermore, we introduce a controlled invariant subspace Ω^* of $\mathcal{X}^*$, which is equivalent to that of **4.3** in (Huijberts and Andiarti, 1997) or (Xia and Moog, 1999) and constructed by applying an Ω^*-Algorithm, such that

$$\mathcal{O}_s^* \subseteq \Omega^* := \{d\psi \in \mathcal{X}^*;\ d\dot{\psi} \in \Omega^* + \dot{\mathcal{O}}_s^*\}$$

$$\begin{aligned} \Omega^* + \dot{\Omega}^* &= \Omega^* \oplus span_{\mathcal{K}}\{d\phi_u\} \\ &= \Omega^* + \dot{\mathcal{O}}_s^* \subset \mathcal{P}^{\perp} \end{aligned} \tag{5}$$

Lemma 2.1 Given system (2) there exists an controllability decomposition in dual-state space $\mathcal{X}^*$ such that

$$\mathcal{X}^* = \mathcal{X}_c^* \oplus \mathcal{X}_{\bar{c}}^*$$

where $\mathcal{X}_c^*$ is controllable subspace and the uncontrollable subspace $\mathcal{X}_{\bar{c}}^*$ is uniquely defined, integrable and invariant under coordinate transformation.

The algorithm for constructing $\mathcal{X}_{\bar{c}}^*$ named $\mathcal{X}_{\bar{c}}^*$-Algorithm is given by (Zheng and Zhang, 1999).

We now recall the notion of *covering space* from (Cao and Zheng, 1992). Let $d\psi(x) \in \mathcal{O}_s^*$ and $d\dot{\psi} \equiv \omega$ (mod $\mathcal{U}$, then ω can be represented by $\omega = d\xi_0 + u_1 d\xi_1 + u_2 d\xi_2 + \cdots + u_m d\xi_m$ where $d\xi_i \in \mathcal{X}^*,\ i = 0, 1, 2, \cdots, m$ as system (2) is affine in u. We define $[\omega] := span_{\mathcal{K}}\{d\xi_0, d\xi_1, \cdots, d\xi_m\}$ to be the covering space of the vector ω.

Using the definition of the covering space, we have

$$[\dot{\mathcal{O}}_s^*] := span_{\mathcal{K}}\{[\omega];$$
$$\omega \equiv d\dot{\psi} \text{ (mod } \mathcal{U}^*) \ \forall\ d\psi \in \mathcal{O}_s^*\}$$

Given system (1) and a function $\psi(x)$, the relative degree of $d\psi(x)$ with respect to input u, denoted by $deg_u(d\psi)$, is $deg_u(d\psi) = r\ (\geq 1)$ if $d\psi^{(r-1)} \in \mathcal{X}^*$ (mod $\mathcal{W}^*$) and $d\psi^{(r)} \notin \mathcal{X}^*$ (mod $\mathcal{W}^*$) are satisfied. Similarly, the relative degree of $d\psi(x)$ with respect to input w, denoted by $deg_w(d\psi)$, is $deg_w(d\psi) = r\ (\geq 1)$ if $d\psi^{(r-1)} \in \mathcal{X}^*$ (mod $\mathcal{U}^*$) and $d\psi^{(r)} \notin \mathcal{X}^*$ (mod $\mathcal{U}^*$) are satisfied.

Now we are in the position to define a new controlled invariant subspaces $\mathcal{Q}^*$ of $\mathcal{X}^*$ based on the controlled subspaces $\mathcal{O}_s^*$ and introduce $\mathcal{Q}^*$-*Algorithm*based on the three algorithms, $\mathcal{O}_s^*$-, Ω^*- and $\mathcal{X}_{\bar{c}}^*$-*Algorithms*.

$\mathcal{Q}^*$ **Algorithm** *(1) Define a set of functions denoted by $y_e = h_e(x)$ such that $span_{\mathcal{K}}\{dy_e\} := span_{\mathcal{K}}\{dy\} + [\dot{\mathcal{O}}_s^*]$, where y_e is considered as an extended controlled output.*

(2) Apply $\mathcal{O}_s^*$-Algorithm to system (1) with the extended output y_e to obtain an extended strongly observable subspace $\mathcal{O}_{es}^*$ of $\mathcal{X}^*$. $\mathcal{O}_{es}^*$ is a controlled invariant subspace of $\mathcal{X}^*$ and satisfies $\mathcal{O}_{es}^* + \dot{\mathcal{O}}_{es}^* = \mathcal{O}_{es}^* + span_{\mathcal{K}}\{d\phi_{u1}, d\tilde{\phi}_{u2}\}$ (mod $\mathcal{W}^*$), where $\phi_{u1}, \tilde{\phi}_{u2}$ are independent (mod $\mathcal{X}^*$). (Notice that the ϕ_u in (3) is replaced by ϕ_{u1} here.)

(3) Apply $\mathcal{X}_{\bar{c}}^*$-Algorithm to $\mathcal{O}_{es}^*$ to obtain the uncontrollable subspace $\mathcal{O}_{es\bar{c}}^*$ of $\mathcal{O}_{es}^*$ with respect to the control input w, i.e. $\mathcal{O}_{es\bar{c}}^* + \dot{\mathcal{O}}_{es\bar{c}}^* = \mathcal{O}_{es\bar{c}}^*$ (mod $span_{\mathcal{K}}\{d\phi_{u1}, d\tilde{\phi}_{u2}\}$). We write

$$\mathcal{O}_{es\bar{c}}^* + \dot{\mathcal{O}}_{es\bar{c}}^* = \mathcal{O}_{es\bar{c}}^* + span_{\mathcal{K}}\{d\phi_{u1}, d\phi_{u2}\}$$

(4) Apply Ω^*-Algorithm to $\mathcal{O}_{es\bar{c}}^*$ to construct controlled invariant subspace $\mathcal{Q}^*$ of $\mathcal{X}^*$, which contains $\mathcal{O}_{es\bar{c}}^*$ and is contained in $\mathcal{P}^{\perp}$.

Thus, we have constructed three controlled invariant subspaces in $\mathcal{P}^{\perp}$ with respect to the controlled output y, i.e.

$$\begin{aligned} \mathcal{O}_s^* + \dot{\mathcal{O}}_s^* &= \mathcal{O}_s^* \oplus span_{\mathcal{K}}\{d\phi_{u1}\} \\ \Omega^* + \dot{\Omega}^* &= \Omega^* \oplus span_{\mathcal{K}}\{d\phi_{u1}\} \\ \mathcal{Q}^* + \dot{\mathcal{Q}}^* &= \mathcal{Q}^* \oplus span_{\mathcal{K}}\{d\phi_{u1}, d\phi_{u2}\} \end{aligned} \tag{6}$$

where dim $span_{\mathcal{K}}\{d\phi_{u1}, d\phi_{u2}\} = m_1 + m_2 \leq m$ (mod $\mathcal{X}^*$) and $\mathcal{O}_s^* \subseteq \Omega^* \subseteq \mathcal{O}_{es\bar{c}}^* \subseteq \mathcal{Q}^* \subset \mathcal{P}^{\perp}$.

Let $\bar{y} = \begin{pmatrix} y \\ z \end{pmatrix}$ be an augmented output of the system (1). With respect to the output $\bar{y}$ we can construct the strongly observable subspace $\bar{\mathcal{O}}_s^*$. It is controlled invariant and that $\mathcal{O}_s^* \subset \bar{\mathcal{O}}_s^*$, but "$\bar{\mathcal{O}}_s^* \subset \mathcal{P}^{\perp}$" may not not satisfied. Thus, we construct the uncontrollable subspace $\bar{\mathcal{O}}_{s\bar{c}}^*$ of $\bar{\mathcal{O}}_s^*$ with respect to the input w, which satisfies that $\bar{\mathcal{O}}_{s\bar{c}}^* + \dot{\bar{\mathcal{O}}}_{s\bar{c}}^* = \bar{\mathcal{O}}_{s\bar{c}}^* \oplus span_{\mathcal{K}}\{d\bar{\phi}_u\}$. Apply Ω^*-Algorithm to $\bar{\mathcal{O}}_{s\bar{c}}^*$, we obtain the controlled invariant subspace $\bar{\Omega}^*$ satisfying

$$\begin{aligned} \Omega^* \subseteq \bar{\Omega}^* \subset \mathcal{P}^{\perp} \\ \bar{\Omega}^* + \dot{\bar{\Omega}}^* = \bar{\Omega}^* \oplus span_{\mathcal{K}}\{d\bar{\phi}_{u1}\} \end{aligned} \tag{7}$$

where $d\bar{\phi}_{u1}$ is a set of independent vectors (mod $\mathcal{X}^*$).

3. SOLVABILITY CONDITIONS FOR DDPMF AND ALGORITHM

Let the dual controlled output subspace of the closed loop system of (1) be $\tilde{\mathcal{Y}}^*$, where $\tilde{\mathcal{Y}}^* :=$

$span_{\mathcal{K}}\{d\tilde{y}, d\dot{\tilde{y}}, d\ddot{\tilde{y}}, \cdots\}$. By definition, the disturbance decoupling for the closed loop system of (1) implies

$$\tilde{\mathcal{Y}}^* \subset \mathcal{P}^{\perp} \tag{8}$$

If we define dy_e such that $span_{\mathcal{K}}\{dy_e\} = span_{\mathcal{K}}$ $\{dy, dz\}+[\dot{\bar{\Omega}}^*]$ in the step 1 of $\mathcal{Q}^*$-Algorithm, then the application of $\mathcal{Q}^*$-Algorithm yields a controlled invariant subspace $\bar{\mathcal{Q}}^*$ of $\mathcal{X}^*$ in $\mathcal{P}^{\perp}$ such that

$$\bar{\mathcal{Q}}^* + \dot{\bar{\mathcal{Q}}}^* = \bar{\mathcal{Q}}^* + span_{\mathcal{K}}\{d\bar{\phi}_{u1}, d\bar{\phi}_{u2}\} \tag{9}$$

For a measurement feedback control $u = u(z, v)$, let $\tilde{\phi}_u(x, v) = \phi_u(x, u(z, v))$, where the ϕ_u satisfied (3).

Theorem 3.1 Under the condition $\bar{\mathcal{Q}}^* = \bar{\Omega}^*$, the DDPMF of system (1) is solvable if and only if there exists a measurement feedback $u = u(z, v)$ such that

$$\begin{aligned} & span_{\mathcal{K}}\{d\tilde{\phi}_u, d\dot{\tilde{\phi}}_u, \cdots, d\tilde{\phi}_u^{(s)}\} \\ & \subset span_{\mathcal{K}}\{d\tilde{\phi}_u, d\dot{\tilde{\phi}}_u, \cdots, d\tilde{\phi}_u^{(s-1)}\} + \Omega^* \\ & \subset \bar{\Omega}^* \ (\text{mod } \mathcal{V}^*) \end{aligned} \tag{10}$$

is satisfied for some $s \leq r = \dim \bar{\Omega}^* - \dim \Omega^*$

The proof is omitted as space limitation.

When condition $\bar{\mathcal{Q}}^* = \bar{\Omega}^*$ is not satisfied, let $\mathcal{O}_s^*$, Ω^*, $\mathcal{Q}^*$, $\bar{\Omega}^*$, $\bar{\mathcal{Q}}^*$ be constructed such that the conditions (6), (7), (9) are satisfied. One has

Theorem 3.2 The DDPMF of system (1) is solvable if there exists a measurement feedback $u = u(z, v)$ satisfying one of the following conditions.

$$\begin{aligned} & \{d\tilde{\phi}_{u1}(x, v)\} \subset \Omega^* + span_{\mathcal{K}}\{dv\} \\ & \{d\tilde{\phi}_{u1}(x, v), d\tilde{\phi}_{u2}(x, v)\} \subset \mathcal{Q}^* + span_{\mathcal{K}}\{dv\} \\ & \{d\tilde{\bar{\phi}}_{u1}(x, v)\} \subset \bar{\Omega}^* + span_{\mathcal{K}}\{dv\} \\ & \{d\tilde{\bar{\phi}}_{u1}(x, v), d\tilde{\bar{\phi}}_{u2}(x, v)\} \subset \bar{\mathcal{Q}}^* + span_{\mathcal{K}}\{dv\} \end{aligned} \tag{11}$$

where $\tilde{\phi}_{u1} = \phi_{u1}(x, u(z, v))$, $\tilde{\phi}_{u2} = \phi_{u2}(x, u(z, v))$ and $\tilde{\bar{\phi}}_{u1} = \bar{\phi}_{u1}(x, u(z, v))$, $\tilde{\bar{\phi}}_{u2} = \bar{\phi}_{u2}(x, u(z, v))$.

If the system is left-invertible, then it easy to show that $\mathcal{Q}^* = \Omega^*$, $\bar{\mathcal{Q}}^* = \bar{\Omega}^*$.

Corollary 3.1 If the nonlinear system (1) is left invertible, the DDPMF of (1) is solvable if and only if there exists a measurement feedback $u = u(z, v)$ such that

$$d\phi_u(x, u(z, v)) \subset \Omega^* + span_{\mathcal{K}}\{dv\} \tag{12}$$

All conditions appeared in Theorems 3.1 and 3.2 are of the form (12). The Algorithm of DDPMF checks if the DDPMF is solvable and finds an exact or an approximate solution with arbitrary accuracy, which satisfies the condition in the form (12), if it is solvable.

Assume that $dz \cap \Omega^* = 0$ and write $\Omega^* = span_{\mathcal{K}}\{d\xi\}$. If (12) is satisfied, then ϕ_u must be a vector function of ξ, z and u. Since $\phi_u = (\phi_1, \cdots, \phi_m)^{\tau}$ is affine in u, each ϕ_i, $i \in \underline{m}$, can be written as

$$\phi_i = \phi_{i0}(\xi, z) + \sum_{j=1}^{m} \phi_{ij}(\xi, z) u_j \tag{13}$$

where ϕ_{ij}, $j = 0, 1, \cdots, m$, are locally analytic functions.

With an abuse of notation, let $\{z\}$ be the vector $z = (z_1, z_2, \cdots, z_{p_2})^{\tau}$, $\{z\}^2$ be a column vector containing all the entries $\{z_i z_j;\ i \leq j,\ i, j \in \underline{p_2}\}$, $\{z\}^3$ be a column vector containing all the entries $\{z_i z_j z_k;\ i \leq j \leq k,\ i, j, k \in \underline{p_2}\}$ and then $\{z\}^l$, for $l = 1, 2, \cdots$, are defined in the similar way. Then, we can expand the locally analytic functions ϕ_{ij} into a Taylor series locally as follows.

$$\phi_{ij}(z) = \beta_{ij0} + \beta_{ij1}\{z\} + \beta_{ij2}\{z\}^2 + \cdots, \tag{14}$$

where $i \in \underline{m}$, $j = 0, 1, \cdots, m$, each β_{ijk} is a row vector of appropriate dimension with the entries being functions of ξ.

Let the output feedback be written as

$$u_j(z, v) = \psi_{j0}(z) + \sum_{l=1}^{m} \psi_{jl}(z) v_l,\ j \in \underline{m} \tag{15}$$

Then ψ_{jl}, for $j \in \underline{m}$ and $l = 0, 1, \cdots, m$, are analytic functions of z around some operation point. Each ψ_{jl} can be expanded into a Taylor series around the operation point as

$$\psi_{jl}(z) = \gamma_{jl0} + \gamma_{jl1}\{z\} + \gamma_{jl2}\{z\}^2 + \cdots, \tag{16}$$

where $j \in \underline{m}, l = 0, 1, \cdots, m$, each γ_{jlk} is a row vector of appropriate dimension over $\mathcal{R}$. Thus to define an output feedback control in the form (15) for the DDPMF is equivalent to define the coefficient vectors $\gamma := \{\gamma_{jlk}, j \in \underline{m}, l = 0, 1, \cdots, m, k = 1, 2, 3, \cdots\}$ in the form (16).

Substitute (16), (15) and (14) into (13) for each $i \in \underline{m}$, we obtain

$$\begin{aligned} \tilde{\phi}_i &= \phi_{i0}(\xi, z) + \sum_{j=1}^{m} \phi_{ij}(\xi, z)(\psi_{j0}(z) \\ &\quad + \sum_{l=1}^{m} \psi_{jl}(z) v_l) \\ &= \tilde{\phi}_{i0}(z) + \sum_{l=1}^{m} \tilde{\phi}_{il}(z) v_l \end{aligned} \tag{17}$$

where $\tilde{\phi}_{ij}(z) = \tilde{\beta}_{ij0} + \tilde{\beta}_{ij1}\{z\} + \tilde{\beta}_{ij2}\{z\}^2 + \tilde{\beta}_{ij3}\{z\}^3 + \cdots, j = 0, 1, \cdots, m$, and coefficient vectors $\tilde{\beta}_{ijk}$ are functions of ξ and γ.

Condition (12) implies that in $\tilde{\phi}_i$ all the coefficients $\{\tilde{\beta}_{ijk}; k > 0\}$ of $\{z\}, \{z\}^2, \{z\}^3, \cdots$ are zero. Thus, the solvability condition (12) is equivalent to if one can find the **real coefficient vector set** $\{\gamma_{jls}\}$ such that $\tilde{\beta}_{ijk}(\xi, \gamma) = 0$ for $k \geq 1$. If (12) does not hold under any feedback, then in some steps $\underline{\tilde{\beta}_{ijk}(\xi, \gamma) \neq 0}$ holds for any real coefficients $\{\gamma_{jls}\}$. If (12) is solvable, then by properly choosing the coefficients γ_{jls} we can obtain an exact solution or an approximate solution with sufficient accuracy for the DDPMF.

When $dz \cap \Omega^* \neq 0$, we decompose the space $span_{\mathcal{K}}\{dz\}$ into two integrable subspaces, i.e. $span_{\mathcal{K}}\{d\hat{\xi}\} \in \Omega^*$ and $span_{\mathcal{K}}\{d\hat{z}\}$ with $d\hat{z} \cap \Omega^* = 0$, such that $span_{\mathcal{K}}\{dz\} = span_{\mathcal{K}}\{d\hat{z}, d\hat{\xi}\}$.

With an abuse of notation, we denote $\{z\} = \{\hat{\xi}, \hat{z}\}$, where $\{\hat{\xi}\} = (\xi_1, \xi_2 \cdots, \xi_{n_1'})^\tau$ is a subvector of $\xi^\tau := (\hat{\xi}^\tau, \bar{\xi}^\tau)$, and $\{\hat{z}\} = (z_1, \cdots, z_{q'})^\tau$ with $n_1' + q' = q$. Further denote $\{z\} = \{\hat{\xi}, \hat{z}\}$, $\{z\}^2 = \{\hat{\xi}, \hat{z}\}^2$, $\{z\}^3 = \{\hat{\xi}, \hat{z}\}^3, \cdots$.

Thus, (13), (14), (15) and (16) can be rewritten as follows.

$$\phi_i = \phi_{i0}(\xi, \hat{z}) + \sum_{i=1}^{m} \phi_{ij}(\xi, \hat{z}) u_j \tag{18}$$

For $j = 0, 1, \cdots, m$

$$\begin{aligned} &\phi_{ij}(\xi, \hat{z}) \\ &= \beta_{ij0} + \beta_{ij1}\{\hat{z}\} + \beta_{ij2}\{\hat{z}\}^2 + \cdots \end{aligned} \tag{19}$$

where the coefficients β_{ijk} are functions of ξ. The feedback control can be written as

$$u_j(z, v) = \psi_{j0}(\hat{\xi}, \hat{z}) + \sum_{l=1}^{m} \psi_{jl}(\hat{\xi}, \hat{z}) v_l \tag{20}$$

$$\begin{aligned} &\psi_{jl}(z) \\ &= \gamma_{jl0} + \gamma_{jl1}\{\hat{\xi}, \hat{z}\} + \gamma_{jl2}\{\hat{\xi}, \hat{z}\}^2 + \cdots \end{aligned} \tag{21}$$

where each γ_{jlk} is a real coefficient vector. ψ_{jl} can be written in terms of $\{\hat{z}\}$, $\{\hat{z}\}^2, \{\hat{z}\}^3, \cdots$ as

$$\begin{aligned} &\psi_{jl}(z) \\ &= \gamma_{jl0} + \bar{\gamma}_{jl1}(\hat{\xi})\{\hat{z}\} + \bar{\gamma}_{jl2}(\hat{\xi})\{\hat{z}\}^2 + \cdots \end{aligned} \tag{22}$$

Substitute (22), (20) and (19) into (18). Then (18) has a representation of Taylor's series in terms of $\{\hat{z}\}$, $\{\hat{z}\}^2, \{\hat{z}\}^3, \cdots$. Compare the coefficients of $\{\hat{z}\}$ $\{\hat{z}\}^2, \{\hat{z}\}^3, \cdots$. The solvability condition (12) for the DDPMF implies that all the coefficients of $\{\hat{z}\}$, $\{\hat{z}\}^2, \{\hat{z}\}^3$ are zero with v being considered as a parameter vector. This is equivalent to finding the real coefficients $\{\gamma_{jlk}\}$ to satisfy a set of equations.

The following example is from (Xia and Moog, 1999), by which we shall illustrate how to check (12) using Taylor expansion.

Example 5.1

$$\begin{aligned} &\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3 \sin x_2 + u \cos x_2 \\ &\dot{x}_3 = w, \quad y = x_1, \quad z = x_3 \end{aligned}$$

It is left-invertible system and by Ω^*-Algorithm one has $\mathcal{O}_s^* = \Omega^* = span_{\mathcal{K}}\{dx_1, dx_2\}$ and $\phi = \ddot{y} = x_3 \sin x_2 + u \cos x_2 = \phi_0 + \phi_1 u$. Notice that $x_3 = z$ and $\phi_1 \neq 0$ at $z = x_3 = 0$, let

$$\begin{aligned} u = u(z, v) &= (\gamma_{00} + \gamma_{01} z + \gamma_{02} z^2 + \cdots) \\ &+(\gamma_{10} + \gamma_{11} z + \gamma_{12} z^2 + \cdots) v \end{aligned}$$

where all $\{\gamma_{ij}\}$ are real numbers. Substitute $u = u(z, v)$ into $\phi(x, u)$ one obtains

$$\begin{aligned} \tilde{\phi} = \phi(x, u(z, v)) &= \gamma_{00} \cos x_2 + (\sin x_2 \\ &+\gamma_{11} \cos x_2) z + \gamma_{12} \cos x_2 z^2 + \cdots \\ &+(\gamma_{10} + \gamma_{11} z + \gamma_{12} z^2 + \cdots) \cos x_2 v \end{aligned}$$

Condition (12) implies that the following equations must hold.

$$\gamma_{11} = \gamma_{12} = \cdots = 0, \quad \sin x_2 + \gamma_{11} \cos x_2 = 0, \ \cdots$$

But they have no real solution for γ_{11}. Thus, the system has no solution for its DDPMO.

Example 5.2 Consider the following non-invertible nonlinear system.

$$\begin{aligned} &\dot{x}_1 = x_2 + x_3 u_1, \quad \dot{x}_2 = \cos x_5 \cdot u_2, \\ &\dot{x}_3 = -x_4^2 x_2^2 + x_2^2 u_3, \quad \dot{x}_4 = x_3^2 + w, \\ &\dot{x}_5 = x_1 x_2 \\ &y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad z = x_4 \end{aligned} \tag{23}$$

We have $\mathcal{O}_s^* = span_{\mathcal{K}}\{dx_1, dx_2\}$, $\Omega^* = span_{\mathcal{K}}\{dx_1, dx_2, dx_5\}$, $\Omega_z^* = 0$. By $\mathcal{Q}^*$-Algorithm we have $\mathcal{Q}^* = span_{\mathcal{K}}\{dx_1, dx_2, dx_3, dx_5\}$. Thus, $\bar{\mathcal{Q}}^* \neq \bar{\Omega}^*$.

To check if the first condition in Theorem 3.2 is satisfied, we let $\phi_u(x, u) = \begin{pmatrix} x_2 + x_3 u_1 \\ \cos x_5 \cdot u_2 \end{pmatrix}$ and $u = u(x_4, v) = \begin{pmatrix} u_1(x_4, v) \\ u_2(x_4, v) \end{pmatrix}$.

It is seen that

$$\begin{aligned} d\phi_u(x, u(x_4, v)) &= d \begin{pmatrix} x_2 + x_3 u_1(x_4, v) \\ \cos x_5 \cdot u_2(x_4, v) \end{pmatrix} \\ &\in span\{dx_1, dx_2, dv\} \end{aligned}$$

is not satisfied under any output feedback control.

We further check the second condition of Theorem 3.2, where $\phi_{u1}(x, u) = \begin{pmatrix} x_2 + x_3 u_1 \\ \cos x_5 \cdot u_2 \end{pmatrix}$ and $\phi_{u2} = -x_4^2 x_2^2 + x_2^2 u_3$.

Let

$$\begin{aligned} u &= u(z, v) \\ &= (\gamma_{00} + \gamma_{01} x_4 + \gamma_{02} x_4^2 + \gamma_{03} x_4^3 \cdots) \\ &+(\gamma_{10} + \gamma_{11} x_4 + \gamma_{12} x_4^2 + \gamma_{13} x_4^3 \cdots) v \end{aligned}$$

where

$\gamma_{ij} := (\gamma_{ij1}, \gamma_{ij2}, \gamma_{ij3}), i = 0, 1, j = 0, 1, 2, \cdots$.

As $\mathcal{Q}^* = span_{\mathcal{K}}\{dx_1, dx_2, dx_3, dx_5\}$, to meet the second condition of Theorem 3.2 one has to find a roper measurement output feedback in the form $u = u(x_4, v)$ such that in left part of the condition contains no dx_4. A simple calculation shows that if we let

$$\begin{aligned} &u_1 = v_1, \\ &u_2 = v_2, \\ &u_3 = u_3(x_4, v), \\ &= (\gamma_{013}x_4 + \gamma_{023}x_4^2 + \gamma_{033}x_4^3 \cdots) \\ &+(\gamma_{103} + \gamma_{113}x_4 + \gamma_{123}x_4^2 + \gamma_{133}x_4^3 \cdots)v_3, \end{aligned} \tag{24}$$

then, by substituting (24) into equation $\phi_{u2} = -x_2^2x_4^2 + x_2^2u_3$ and comparing the coefficients of x_4 of the formula

$$\tilde{\phi}_{u2} = -x_2^2x_4^2 + x_2^2u_3(x_4, v)$$

one gets $\gamma_{023} = \gamma_{103} = 1$ and for the others $\gamma_{ijk} = 0$. It yields a solution $u_1 = v_1$, $u_2 = v_2$ and $u_3 = x_4^2 + v_3$ for the DDPMF of (23).

4. CONCLUSION

We studied solvability of the DDPMF for MIMO nonlinear systems which may not be square and left invertible. Using the differential vector space framework, we have constructed some subspaces of the nonlinear system and used the subspaces to present necessary and sufficient conditions for the DDPMF. A computational method is further presented for checking the solvability condition and finding a solution for the DDPMF.

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