H_2 CONTROL OF DIFFERENTIAL LINEAR REPETITIVE PROCESSES

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Abstract: Repetitive processes are a distinct class of two-dimensional systems (i.e. information propagation in two independent directions) of both systems theoretic and applications interest. They cannot be controlled by direct extension of existing techniques from either standard (termed 1D here) or two-dimensional (2D) systems theory. Here we give new results on the design of physically based control laws using an H_2 setting. These results are for the sub-class of so-called differential linear repetitive processes which arise in applications areas such as iterative learning control.

Keywords: H_2 control, differential linear repetitive processes, linear matrix inequalities.

1. INTRODUCTION

Repetitive processes are a distinct class of 2D systems of both system theoretic and applications interest. The essential unique characteristic of such a process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for them in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(t)$, $0 \leq t \leq \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(t)$, $0 \leq t \leq \alpha$, $k \geq 0$.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (see, for example, (Rogers and Owens, 1992)). Also in recent years applications have arisen where

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adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control (ILC) schemes (Amann *et al.*, 1996) and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle (Roberts, 2000).

Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass and along the passes. In seeking a rigorous foundation on which to develop a control theory for these processes, it is natural to attempt to exploit structural links which exist between them and other classes of 2D linear systems, such as the work on H_∞ and H_2 based control systems design reported, for example, in (Du and Xie, 2002).

A key distinguishing feature of repetitive processes is that information propagation in one of the independent directions, along the pass, only occurs over a finite duration — the pass length. Moreover, in this paper the subject is so-called differential linear repetitive processes where the dynamics along the pass are governed by a linear matrix differential equation. This means that results for 2D discrete linear systems are not applicable.

The structure of these processes means that there is a natural way to write down control laws for them which can be based on current pass state or output (pass profile) feedback control and feedforward control from the previous pass profile. For example, in the ILC application, one such family of control laws is composed of output feedback control action on the current pass combined with information 'feedforward' from the previous pass (or trial in the ILC context) which, of course, has already been generated and is therefore available for use.

Previous work has established the basic feasibility of this general approach and provided some algorithms for the design of control laws (see, for example, (Galkowski *et al.*, 2003)). The aim of this paper is to consider the use of an H_2 approach to control law design to augment existing results using, for example, an H_∞ setting. We begin in the next section by giving the necessary background results.

Throughout this paper, the null matrix and the identity matrix with appropriate dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ denotes a real symmetric positive definite matrix. Similarly, $M < 0$ denotes a real symmetric negative definite matrix.

2. BACKGROUND

The state space model of a differential linear repetitive process has the following form over $0 \le t \le \alpha$, $k \ge 0$

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + u_{k+1}(t) + D_0 y_k(t) \end{aligned} \tag{1}$$

Here on pass k, $x_k(p)$ is the $n \times 1$ state vector, $y_k(p)$ is the $m \times 1$ pass profile vector and $u_k(p)$ is the $l \times 1$ vector of control inputs.

To complete the process description, it is necessary to specify the boundary conditions i.e. the state initial vector on each pass and the initial pass profile (i.e. on pass 0). The simplest possible choice for these is

$$\begin{aligned} x_{k+1}(0) &= d_{k+1},\ k \ge 0 \\ y_0(t) &= f(t) \end{aligned}$$

where d_{k+1} is $n \times 1$ vector with known constant entries and $f(t)$ is the $m \times 1$ vector whose entries are known functions of t over $0 \le t \le \alpha$. For the purposes of this paper, it is assumed that $d_{k+1} = 0$, $k \ge 0$.

The stability theory (Rogers and Owens, 1992) for linear repetitive processes is based on an abstract model in a Banach space setting which includes all such processes as special cases. In this setting, a bounded linear operator mapping a Banach space into itself describes the contribution of the previous pass dynamics to the current one and the stability conditions are described in terms of properties of this operator. Noting again the unique control problem for these processes, i.e. oscillations that increase in amplitude from pass-to-pass (the k direction in the notation for variables used here), the stability theory is based on ensuring that such a response cannot occur. This is achieved by demanding that the output sequence of pass profiles generated $\{y_k\}_k$ has a bounded input bounded output stability property defined in terms of the norm on the underlying Banach space.

In actual fact, two distinct forms of stability can be defined in this setting which are termed asymptotic stability and stability along the pass respectively. The former requires this property with respect to the (finite and fixed) pass length and the latter uniformly, i.e. independent of the pass length. Asymptotic stability guarantees the existence of a so-called limit profile defined as the strong limit as $k \to \infty$ of the sequence $\{y_k\}_k$ and for the processes under consideration here this limit profile is described by a 1D differential linear systems state space model with state matrix $A_{lp} := A + B_0(I_m - D_0)^{-1}C$. Hence it is possible for asymptotic stability to result in a limit profile which is unstable as a 1D differential linear system, e.g. $A = -1, B = 0, B_0 = 1 + \beta, C = 1, D = 0, D_0 = 0$, where $\beta > 0$ is a real scalar. Stability along the pass prevents this from happening by demanding that the stability property be independent of the pass length, which can be analyzed mathematically by letting $\alpha \to \infty$.

Several equivalent sets of conditions for stability along the pass are known but here it is one expressed in terms of the 2D transfer function matrix description of the process dynamics, and hence 2D characteristic polynomial, which is the basic starting point. Since the state on pass 0 plays no role, it is convenient to re-label the state vector as $x_{k+1}(t) \mapsto x_k(t)$ (keeping of course the same interpretation). Also define the pass-to-pass shift operator as z_2 applied e.g. to $y_k(t)$ as follows

$$y_k(t) := z_2 y_{k+1}(t)$$

and for the along the pass dynamics we use the Laplace transform variable s. Note here that it is routine to argue that finite pass length does not cause a problem provided the variables considered are suitably extended from $[0, \alpha]$ to $[0, \infty]$ and we assume that this has been done in what follows.

Let $Y(s, z_2)$ and $U(s, z_2)$ denote the results of applying these transforms to the sequences $\{y_k\}_k$ and $\{u_k\}_k$ respectively. Then the process dynamics can be written as

$$Y(s, z_2) = G(s, z_2)U(s, z_2)$$

where the 2D transfer function matrix $G(s, z_2)$ is given by

$$G(s, z_2) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} sI - A & -B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix}^{-1} \begin{bmatrix} B \\ D \end{bmatrix} \tag{2}$$

The 2D characteristic polynomial is given by

$$\mathcal{C}(s, z_2) := \det\left(\begin{bmatrix} sI - A & -B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix}\right)$$

and it has been shown elsewhere (Rogers and Owens, 1992) that stability along the pass holds if, and only if,

$$\mathcal{C}(s, z_2) \neq 0 \tag{3}$$

in $\mathcal{U}(s, z_2) := \{(s, z) : \mathrm{Re(s)} \geq 0,\ |\mathrm{z_2}| \leq 1\}$.

It also possible to use this 2D transfer function matrix description to conclude that stability along the pass requires each frequency component of the initial profile (and hence on each subsequent pass) to be attenuated from pass-to-pass. In 1D control systems theory and design, the H_2 norm of the system, i.e. the average gain over all frequencies, is a very powerful analysis and control law design tool. Hence it is to be expected that a suitably defined H_2 norm on the 2D transfer function matrix will play a similar role and the development of this physically motivated idea is the subject of this paper. We will also require the following result (which allows for LMI based computations).

Introduce the Lyapunov function for these processes as

$$V(k, t) = x_{k+1}^T(t)P_1 x_{k+1}(t) + y_k^T(t)P_2 y_k(t)$$

and associated increment

$$\begin{aligned}\Delta V(k, t) = {} & \dot{x}_{k+1}^T(t)P_1 x_{k+1}(t) + x_{k+1}^T(t)P_1 \dot{x}_{k+1}(t) \\ & + y_{k+1}^T(t)P_2 y_{k+1}(t) - y_k^T(t)P_2 y_k(t)\end{aligned}$$

where $P_1 > 0$ and $P_2 > 0$. Then we have the following result whose proof is a routine extension of results in, for example, (Galkowski *et al.*, 2003).

Lemma 1. A differential linear repetitive process described by (1) is stable along the pass if

$$\Delta V(k, t) < 0 \tag{4}$$

We will also require the following result which allows for LMI based computations.

Lemma 1. (Galkowski *et al.*, 2003) A differential repetitive process described by (1) is stable along the pass if there exist matrices $P_1 > 0$ and $P_2 > 0$ such that the following LMI is feasible

$$\begin{bmatrix} -P_2 & P_2 C & P_2 D_0 \\ C^T P_2 & A^T P_1 + P_1 A & P_1 B_0 \\ D_0^T P_2 & B_0^T P_1 & -P_2 \end{bmatrix} < 0 \tag{5}$$

Finally, we need the following signal space.

Definition 2. The L_2 norm of the $q \times 1$ vector $w_k(t)$ defined over the real interval $0 \leq t \leq \infty$ and the integers $0 \leq k \leq \infty$, written as $\{[0, \infty], [0, \infty]\}$ for ease of notation, is given by

$$\|w\|_2 = \sqrt{\sum_{k=0}^{\infty} \int_0^{\infty} w_k(t)^T w_k(t)\, dt} \tag{6}$$

and w_k is said to be a member of $L_2^q\{[0, \infty], [0, \infty]\}$, or L_2^q for short, if $||w_k||_2 < \infty$.

3. THE H_2 NORM AND STABILITY

Using the 1D case as motivation, consider a single input stable along the pass process (note again that this can be analyzed mathematically by letting the pass length $\alpha \to \infty$) and let the $m \times 1$ vector $g(k, t)$ denote the response to an impulse, denoted by $\delta(k, t)$ applied at $t = 0$ on pass k. Then, by invoking Parseval's theorem in the along pass direction on each pass and summing over the pass index, the H_2 norm is given by

$$\|G\|_2 = \sqrt{\|g(k,t)\|_2^2} = \sqrt{\sum_{k=0}^{\infty}\int_0^{\infty} g^T(k,t)g(k,t)dt} \tag{7}$$

To extend this definition to vector-valued inputs, introduce

$$u_k^h(t) = \delta(k,t)e^h$$

where e^h is the $l \times 1$ vector whose entries are zero except for a unit entry in position h, $1 \leq h \leq l$. Then we have that

$$\|G\|_2 = \sqrt{\sum_{h=1}^{l}\sum_{k=0}^{\infty}\int_0^{\infty} (g^h)^T(k,t)g^h(k,t)dt}$$

To determine $g^h(k,t)$, first introduce

$$\xi^h(k,t) = \begin{bmatrix} \dot{x}_{k+1}^h(t) \\ y_{k+1}^h(t) \end{bmatrix}, \ \zeta^h(k,t) = \begin{bmatrix} x_{k+1}^h(t) \\ y_k^h(t) \end{bmatrix}$$

and

$$\widehat{A}_1 = \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \widehat{A}_2 = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix}, \Omega = \begin{bmatrix} B \\ D \end{bmatrix}, \Psi^T = \begin{bmatrix} C^T \\ D_0^T \end{bmatrix}$$

Then (for any value of α)

$$\begin{aligned} \xi^h(k,t) &= (\widehat{A}_1 + \widehat{A}_2)\zeta^h(k,t) + \Omega\delta(k,t)e^h \\ &= \begin{cases} \widehat{\Omega}_h + \widehat{A}_2\zeta^h(k,t), & \text{for } k=0,\ 0 \leq t \leq \alpha \\ (\widehat{A}_1 + \widehat{A}_2)\zeta^h(k,t), & \text{for } k>0,\ 0 \leq t \leq \alpha \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} g^h(k,t) &= \Psi\zeta^h(k,t) + D\delta(k,t)e^h \\ &= \begin{cases} \widehat{D}_h + D_0 y_k^h(t), & \text{for } k=0,\ 0 \leq t \leq \alpha \\ \Psi\zeta^h(k,t), & \text{for } k>0,\ 0 \leq t \leq \alpha \\ 0, & \text{for otherwise} \end{cases} \end{aligned}$$

where $\widehat{D}_h$ and $\widehat{\Omega}_h$ denote h-th column of the matrices D and Ω respectively.

The following is the first major result of this paper and gives a sufficient condition for stability along the pass together with an upper bound on the H_2 norm of the 2D transfer function matrix.

Theorem 1. A differential linear repetitive process described by (1) is stable along the pass and has H_2 norm bound $\gamma > 0$, i.e. $\|G\|_2 < \gamma$, if there exist matrices $P_1 > 0$ and $P_2 > 0$ such that the following LMIs hold

$$\begin{bmatrix} -P_2 & P_2C & P_2D_0 \\ C^TP_2 & A^TP_1 + P_1A + C^TC & P_1B_0 + C^TD_0 \\ D_0^TP_2 & B_0^TP_1 + D_0^TC & -P_2 + D_0^TD_0 \end{bmatrix} < 0 \tag{8}$$

and

$$\begin{aligned} &\text{trace}\left(\alpha D^TD + \alpha B^TP_1B + \alpha D^TP_2D\right) \\ &+ \text{trace}\left(\Psi^TP_2\Psi\int_0^{\alpha} f(t)f(t)^Tdt\right) \\ &+ \text{trace}\left(D_0^TD_0\int_0^{\alpha} f(t)f(t)^Tdt\right) < \gamma^2 \end{aligned} \tag{9}$$

PROOF. It is straightforward to see that if (8) holds then

$$\begin{bmatrix} -P_2 & P_2C & P_2D_0 \\ C^TP_2 & A^TP_1 + P_1A & P_1B_0 \\ D_0^TP_2 & B_0^TP_1 & -P_2 \end{bmatrix} + \begin{bmatrix} 0 \\ C^T \\ D_0^T \end{bmatrix}\begin{bmatrix} 0 & C & D_0 \end{bmatrix} < 0$$

Also since the second term on the left hand side of the above inequality is clearly non-negative definite, it follows immediately from (5) that stability along the pass holds.

To establish the H_2 performance, we use the stability along the pass condition of Lemma 1, i.e. (4), and introduce

$$\begin{aligned} \Delta V^h(k,t) &= (\dot{x}_{k+1}^h)^T(t)P_1x_{k+1}^h(t) \\ &+ (x_{k+1}^h)^T(t)P_1\dot{x}_{k+1}^h(t) \\ &+ (y_{k+1}^h)^T(t)P_2y_{k+1}^h(t) \\ &- (y_k^h)^T(t)P_2y_k^h(t) \end{aligned}$$

and note that

$$\begin{aligned} \Delta V^h(k,t) = (\zeta^h)^T(k,t)\Big(&\widehat{A}_1^TP + P\widehat{A}_1 \\ &+ \widehat{A}_2^TR\widehat{A}_2 - R\Big)\zeta^h(k,t) \end{aligned}$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & P_2 \end{bmatrix}$$

If stability along the pass holds then

$$\sum_{k=0}^{\infty}\int_0^{\infty} \Delta V^h(k,t)dt = \sum_{k=0}^{\infty}\int_0^{\alpha} \Delta V^h(k,t)dt = 0 \tag{10}$$

Furthermore, based (3) and (10) we have

$$\begin{aligned} \sum_{k=0}^{\infty}\int_0^{\infty} \Delta V^h(k,t) &= \int_0^{\alpha} \widehat{\Omega}_h^T(P+R)\widehat{\Omega}_h dt \\ &+ \int_0^{\alpha} (\zeta^h)^T(0,t)\widehat{A}_2^T(P+R)\widehat{A}_2\zeta^h(0,t)dt \\ &+ \sum_{k=0}^{\infty}\int_0^{\infty} (\zeta^h)^T(k,t)\Big(\widehat{A}_1^TP + P\widehat{A}_1 \\ &\quad + \widehat{A}_2^TR\widehat{A}_2 - R\Big)\zeta^h(k,t) \end{aligned}$$

Given (7), we also have that

$$\begin{aligned} \|G\|_2^2 &= \sum_{h=1}^{l}\sum_{k=0}^{\infty}\int_0^{\infty} g^{hT}(k,t)g^h(k,t)dt \\ &= \sum_{h=1}^{l}\Big(\int_0^{\alpha} \widehat{D}_h^T\widehat{D}_h dt + \int_0^{\alpha} (f^h)^T(t)D_0^TD_0f^h(t)dt \\ &+ \sum_{k=0}^{\infty}\int_0^{\infty} \zeta^{hT}(k,t)\Psi^T\Psi\zeta^h(k,t)dt\Big) \end{aligned}$$

and also, using (10),

$$
\begin{aligned}
\|G\|_2^2 = & \sum_{h=1}^{l}\left(\sum_{k=0}^{\infty}\int_0^{\infty}\Delta V^h(k,t)dt\right.\\
&+\alpha\widehat{D}_h^T\widehat{D}_h+\int_0^{\alpha}(f^h)^T(t)D_0^TD_0f^h(t)dt\\
&\left.+\sum_{k=0}^{\infty}\int_0^{\infty}\zeta^{hT}(k,t)\Psi^T\Psi\zeta^h(k,t)dt\right)
\end{aligned}
\tag{11}
$$

Routine manipulations now show that (11) is equivalent to

$$
\begin{aligned}
\|G\|_2^2 = & \sum_{h=1}^{l}\Big(\alpha\widehat{D}_h^T\widehat{D}_h+\alpha B_h^TP_1B_h+\alpha D_h^TP_2D_h\\
&+\int_0^{\alpha}(\zeta^h)^T(0,t)\Psi^TP_2\Psi\zeta^h(0,t)dt\\
&+\int_0^{\alpha}(f^h)^T(t)D_0^TD_0f^h(t)dt\\
&+\sum_{k=0}^{\infty}\int_0^{\infty}(\zeta^h)^T(k,t)\Big(\Psi^T\Psi+\widehat{A}_1^TP\\
&+P\widehat{A}_1+\widehat{A}_2^TR\widehat{A}_2-R\Big)\zeta^h(k,t)dt\Big)
\end{aligned}
$$

Further routine transformations now leads to

$$
\begin{aligned}
\|G\|_2^2 = & \operatorname{trace}\left(\alpha D^TD+\alpha B^TP_1B+\alpha D^TP_2D\right)\\
&+\operatorname{trace}\left(\Psi^TP_2\Psi\int_0^{\alpha}f(t)f(t)^Tdt\right)\\
&+\operatorname{trace}\left(D_0^TD_0\int_0^{\alpha}f(t)f(t)^Tdt\right)\\
&+\sum_{k=0}^{\infty}\int_0^{\infty}\zeta^T(k,t)\Big(\Psi^T\Psi+\widehat{A}_1^TP+P\widehat{A}_1\\
&+\widehat{A}_2^TR\widehat{A}_2-R\Big)\zeta(k,t)dt
\end{aligned}
$$

It now follows immediately from this last expression that (8) and (9) imply that $\|G\|_2<\gamma$ holds and the proof is complete. ■

Note that the H_2 norm bound here can be minimized using the following linear objective minimization problem

$$
\begin{aligned}
&\min_{P_1>0,P_2>0}\quad \mu\\
&\text{subject to (8) and (9) with } \mu=\gamma^2
\end{aligned}
\tag{12}
$$

4. STATIC H_2 CONTROL

Some applications areas will clearly require the design of control laws which guarantee stability along the pass and also have the maximum possible disturbance attenuation (here as measured by an H_2 norm). Here we will show how to address this question in an H_2 setting, for which we now give the relevant background.

The process state space model considered has the following form over $0\le t\le\alpha$, $k\ge 0$

$$
\begin{aligned}
\dot{x}_{k+1}(t) =& Ax_{k+1}(t)+Bu_{k+1}(t)+B_0y_k(t)\\
&+B_{11}w_{k+1}(t)\\
y_{k+1}(t) =& Cx_{k+1}(t)+Du_{k+1}(t)+D_0y_k(t)\\
&+B_{12}w_{k+1}(t)
\end{aligned}
\tag{13}
$$

where $w_{k+1}(t)$ is an $r\times 1$ disturbance vector which belongs to L_2^r (i.e. the model of the previous section with disturbance terms added to the state and pass profile vector updating equations). Also it is easy to see that stability along the pass for such a process is governed by the 2D characteristic polynomial condition of Section 2, i.e. by (3).

The control law employed is given by

$$
u_{k+1}(t)=\begin{bmatrix}K_1 & K_2\end{bmatrix}\begin{bmatrix}x_{k+1}(t)\\ y_k(t)\end{bmatrix}
\tag{14}
$$

where K_1 and K_2 are appropriately dimensioned matrices to be designed. The corresponding closed loop process state space model is

$$
\begin{aligned}
\dot{x}_{k+1}(t) =& (A+BK_1)x_{k+1}(t)\\
&+(B_0+BK_2)y_k(t)+B_{11}w_{k+1}(t)\\
y_{k+1}(t) =& (C+DK_1)x_{k+1}(t)\\
&+(D_0+DK_2)y_k(t)+B_{12}w_{k+1}(t)
\end{aligned}
\tag{15}
$$

The problem considered here is as follows: for a given $\gamma>0$, find a control law of the form (14) for the process (13) such that the closed loop process is stable along the pass and the H_2 norm of the 2D transfer function matrix between the disturbance vector and the current pass profile, denoted here by $G_d(s,z_2)$ and computed by replacing $\begin{bmatrix}B^T & D^T\end{bmatrix}^T$ by $\begin{bmatrix}B_{11}^T & B_{12}^T\end{bmatrix}^T$ in (2), is bounded by γ, i.e. $||G_d||_2<\gamma$ — also termed the H_2 disturbance rejection bound.

The following result gives a solution to this problem together with an algorithm for designing the control law.

Theorem 2. Suppose that a control law of the form (14) is applied to a differential linear repetitive process described by (13). Then the resulting closed loop process (15) is stable along the pass and has prescribed H_2 disturbance rejection bound $\gamma>0$ if there exist matrices $W_1>0$, $W_2>0$, N_1, N_2, and X such that the following LMIs hold

$$
\begin{bmatrix}
-W_2 & CW_1+DN_1 & D_0W_2+DN_2 & 0\\
N_1^TD^T+W_1C^T & W_1A^T+AW_1+N_1^TB^T+BN_1 & B_0W_2+BN_2 & W_1C^T\\
N_2^TD^T+W_2D_0^T & W_2B_0^T+N_2^TB^T & -W_2 & W_2D_0^T\\
0 & CW_1 & D_0W_2 & -I
\end{bmatrix}<0
\tag{16}
$$

and

$$
\operatorname{trace}(X)+\operatorname{trace}(D_0^T D_0 \Upsilon+\alpha B_{12}^T B_{12}) < \gamma^2
$$

$$
\begin{bmatrix} X & B_{11}^T & B_{12}^T & \Pi^{\frac{1}{2}} \\ B_{11} & \alpha^{-1}W_1 & 0 & 0 \\ B_{12} & 0 & \alpha^{-1}W_2 & 0 \\ \Pi^{\frac{1}{2}} & 0 & 0 & W_2 \end{bmatrix} > 0 \tag{17}
$$

where

$$
\Pi = \int_0^\alpha \Psi^T f(t) f^T(t) \Psi dt, \; \Upsilon = \int_0^\alpha f(t) f^T(t) dt \tag{18}
$$

and X is additional symmetric matrix of compatible dimensions. If these conditions hold, the control law matrices K_1 and K_2 are given by $N_1 W_1^{-1}$ and $N_2 W_2^{-1}$ respectively.

PROOF. Interpreting (5) in terms of the closed loop process yields

$$
\begin{bmatrix} -P_2 & P_2C + P_2DK_1 & P_2D_0 + P_2DK_2 \\ K_1^T D^T P_2 + C^T P_2 & \Lambda_1 & P_1B_0 + P_1BK_1 + C^T D_0 \\ K_2^T D^T P_2 + D_0^T P_2 & B_0^T P_1 + K_2^T B^T P_1 + D_0^T C & -P_2 + D_0^T D_0 \end{bmatrix} < 0
$$

where $\Omega^T = \begin{bmatrix} B_{11}^T & B_{12}^T \end{bmatrix}$ and

$$
\Lambda_1 = A^T P_1 + P_1 A + K_1^T B^T P_1 + P_1 B K_1 + C^T C
$$

Now set $W_1 = P_1^{-1}$, $W_2 = P_2^{-1}$, and then pre- and post- multiply both sides of this last inequality by $\operatorname{diag}\{W_2, W_1, W_2\}$ to obtain

$$
\begin{bmatrix} -W_2 & CW_1 + DK_1W_1 & D_0W_2 + DK_2W_2 \\ W_1K_1^T D^T + W_1C^T & \Lambda_2 & \Lambda_3^T \\ W_2K_2^T D^T + W_2D_0^T & \Lambda_3 & -W_2 + W_2D_0^T D_0 W_2 \end{bmatrix} < 0
$$

where

$$
\begin{aligned}
\Lambda_2 &= W_1A^T + AW_1 + W_1K_1^T B^T + BK_1W_1 + W_1C^T C W_1 \\
\Lambda_3 &= W_2B_0^T + W_2K_2^T B^T + W_2D_0^T C W_1
\end{aligned}
$$

An obvious application of the Schur's complement formula to the left-hand side of this last inequality and setting $N_1 = K_1W_1$ and $N_2 = K_2W_2$ now yields (16).

Note now that

$$
\operatorname{trace}(P_2\Pi) = \operatorname{trace}\left(\Pi^{\frac{1}{2}} P_2 \Pi^{\frac{1}{2}}\right)
$$

where Π is defined in (18). Furthermore, (9) interpreted for this case gives

$$
\begin{aligned}
&\operatorname{trace}(\alpha B_{12}^T B_{12} + D_0^T D_0 \Upsilon) \\
&+\operatorname{trace}\left(\begin{bmatrix} B_{11}^T & B_{12}^T & \Pi^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \alpha W_1 & 0 & 0 \\ 0 & \alpha W_2 & 0 \\ 0 & 0 & W_2 \end{bmatrix}^{-1} \begin{bmatrix} B_{11} \\ B_{12} \\ \Pi^{\frac{1}{2}} \end{bmatrix} \right) < \gamma^2
\end{aligned} \tag{19}
$$

which is equivalent to (17). To see this, introduce a new matrix variable X and note that

$$
\begin{aligned}
&\begin{bmatrix} I & -\alpha^{\frac{1}{2}} B_{11}^T W_1^{-1} & -\alpha^{\frac{1}{2}} B_{12}^T W_2^{-1} & -\Pi^{\frac{1}{2}} W_2^{-1} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \\
&\times \begin{bmatrix} X & \alpha^{\frac{1}{2}} B_{11}^T & \alpha^{\frac{1}{2}} B_{12}^T & Pi^{\frac{1}{2}} \\ \alpha^{\frac{1}{2}} B_{11} & W_1 & 0 & 0 \\ \alpha^{\frac{1}{2}} B_{12} & 0 & W_2 & 0 \\ \Pi^{\frac{1}{2}} & 0 & 0 & W_2 \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ -\alpha^{\frac{1}{2}} W_1^{-1} B_{11} & I & 0 & 0 \\ -\alpha^{\frac{1}{2}} W_2^{-1} B_{12} & 0 & I & 0 \\ -W_2^{-1}\Pi^{\frac{1}{2}} & 0 & 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \Xi & 0 & 0 & 0 \\ 0 & W_1 & 0 & 0 \\ 0 & 0 & W_2 & 0 \\ 0 & 0 & 0 & W_2 \end{bmatrix}
\end{aligned} \tag{20}
$$

where

$$
\Xi = X - \alpha B_{11}^T W_1^{-1} B_{11} - \alpha B_{12}^T W_2^{-1} B_{12} - \Pi^{\frac{1}{2}} W_2 \Pi^{\frac{1}{2}}
$$

Also block (1,1) of the matrix on the right-hand side of (20) (i.e Ξ) implies that

$$
X > \alpha B_{11}^T W_1^{-1} B_{11} + \alpha B_{12}^T W_2^{-1} B_{12} + \Pi^{\frac{1}{2}} W_2^{-1} \Pi^{\frac{1}{2}}
$$

and the proof is completed by an obvious application of the Schur's complement formula. ■

The H_2 disturbance rejection bound γ in the LMI of (17) can be minimized by using linear objective minimization algorithm as per (12).

5. CONCLUSIONS

This paper has developed substantial new results on the control of differential linear repetitive processes in an H_2 setting. Overall, these results strongly suggest that when fully developed this approach to the analysis and control of these processes will provide a very powerful bank of theory/design algorithms to take forward to the applications domain.

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