SINGULAR STOCHASTIC MAXIMUM PRINCIPLE

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Abstract: In this paper, an optimal singular stochastic control problem is considered. The state process is described by a non linear stochastic differential equation. The variation of the control variable is bounded. For this model, it is obtained a general stochastic maximum principle by using a time transformation. This is the first version of the stochastic maximum principle that covers nonlinear cases.

Keywords: Stochastic control, singular control, maximum principle.

1. INTRODUCTION

In the last few years, singular stochastic control problems have received considerable attention. The connection between singular control problem and optimal stopping problem has been studied by many authors (Alvarez, 1999; Alvarez, 2001; Boetius, 2001*a*; Boetius, 2001*b*; Boetius and Kohlmann, 1998; Chow *et al.*, 1995; Dufour and Miller, 2004; El Karoui and Karatzas, 1988; El Karoui and Karatzas, 1991; Karatzas, 1983; Karatzas, 1985; Karatzas and Shreve, 1984; Karatzas and Shreve, 1985; Karatzas and Shreve, 1986). Results on the dynamic programming principle can be found in (Boetius, 2001*b*; Haussmann and Suo, 1995*b*; Fleming and Soner, 1993; Zhu, 1992). Sufficient conditions for the existence optimal singular control for general nonlinear models have been obtained in (Dufour and Miller, 2004; Haussmann and Suo, 1995*a*). The authors do not pretend to present here an exhaustive panorama of the literature relative to singular control problems. However, the interested reader may consult (Boetius, 2001*b*) for a survey on stochastic singular control problems including theoretical results and applications.

To the best knowledge of the authors the stochastic maximum principle for singular controls was only considered in (Cadenillas and Haussmann, 1994). In their paper A. Cadenillas and U. Haussmann used a different approach and different hypotheses that are presented now in order to bring to the fore the main differences between their results and ours. In (Cadenillas and Haussmann, 1994), the control process is described by a process $\{\zeta(t)\}$ (see the above discussion) of bounded variation and they do not impose any L^p bounds on the control while we assume that the class of admissible controls $\{v(t)\}$ is such that $v(T) \leq M$ for a constant M. However, in (Cadenillas and Haussmann, 1994) the state process must satisfy a linear stochastic differential equation (the function A and D are assumed to be linear), and the cost function are convex. In

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our work, we suppose that the state process is defined by a general nonlinear stochastic differential equation, and we do not impose a convexity hypothesis on the cost function (see assumptions 1-3 of the next section). In many aspects the results obtained in (Cadenillas and Haussmann, 1994) and here are different and complementary.

In general terms, the approach we used to obtain the maximum principle for singular control problems can be divided in three steps. The first step is to convert with a time transformation the original singular control problem into a classical control problem. In order to be concise, the description of this method is briefly presented, for a complete description see (Dufour and Miller, 2004). The second step is to derive the maximum principle for the auxiliary control problem. It must be stressed the fact that the auxiliary control problem is characterized by a state constraint. The last step consists to recover from the auxiliary maximum principle the original state and control variables by using a time change, thus giving a maximum principle for the singular control problem (see Theorem 5.2). The form of the maximum principle we obtained turns out to be different from the one derived in (Cadenillas and Haussmann, 1994) since the adjoint variables have a singular part, and since the optimal singular control maximizes an Hamiltonian almost surely with respect to the Doleans-Dade measure generated by $\{v(t)\}$ (see the detailed discussion before Theorem 5.2).

Due to lack of space, the proofs of the results are not presented here. The paper is organized as follows. In section 2, we formulate the singular control problem. The time change and the auxiliary control problem is briefly described in section 3. Section 4 deals with the auxiliary maximum principle. In section 5, the main results are obtained and in particular the stochastic maximum principle for singular controls (see Theorem 5.2). In the last section, we make some comments about possible generalizations of our work.

Notation

$\mathbb{N}_N \doteq \{1, \cdots, i, \cdots, N\}$.

For a vector V, V_i denotes the ith component of V. If M is a matrix, M_i denotes a vector given by the ith column of the matrix M, and M_{ij} is the element corresponding to ith row and the jth column. $(^\top)$ denotes the transpose operation. $0_n \in \mathbb{R}^n$ is the zero vector.

For $x \in \mathbb{R}$, x^+ is defined by $x^+ = \dfrac{1}{x}$ if $x \neq 0$; and $x^+ = 0$ if $x = 0$.

Let $\big(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}\big)$ be a filtered probability space, for a an increasing *corlol*, adapted processes, $\{A(t)\}$ the measure defined on $(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F})$ by $E_P\big[\int_0^{+\infty} I_C(s) dA(s)\big]$ for $C \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ is denoted by $\mathcal{M}_A$.

Let $\big(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}\big)$ be a filtered probability space satisfying the usual hypotheses and supporting a standard m-dimensional Brownian motion $\{W_t\}$. Then $\{\mathcal{F}_t^W\}$ denotes the augmentation of the natural filtration generated by $\{W_t\}$.

In order to define the state processes, let us introduce the following data:

- T and M are fixed real numbers.
- K is a subset of $\mathbb{R}^r$.
- ζ is a fixed vector in $\mathbb{R}^n$.
- $B_1(K) \doteq \{x \in K : |x| \leq 1\}$.
- $A : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.
- $B : \mathbb{R} \to \mathbb{R}^{n \times r}$.
- $D : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$.
- $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.
- $N : \mathbb{R} \to \mathbb{R}$ such that $N(x) = (x - T)^2$.

Let us introduce the following notation:

$$\mathcal{A} : \mathbb{R} \times \mathbb{R}^n \times B_1(K) \times [0,1] \longrightarrow \mathbb{R}^{n+2}$$
$$(t,x,u,z) \longrightarrow \begin{pmatrix} 1-z \\ A(t,x)(1-z) + zB(t)u \\ z|u| \end{pmatrix}$$

and

$$\mathcal{D} : \mathbb{R} \times \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R}^{n+2}$$
$$(t,x,z) \longrightarrow \begin{pmatrix} 0 \\ D(t,x)\sqrt{1-z} \\ 0 \end{pmatrix}$$

$(\forall (t,x,u,z,p,q,P) \in \mathbb{R} \times \mathbb{R}^n \times B_1(K) \times [0,1] \times \mathbb{R}^{2+n} \times \mathbb{R}^{(2+n)\times m} \times \mathbb{R}^{n \times n})$,

$$\begin{aligned} \mathcal{H}(t,x,u,z,p,q) &\doteq \mathcal{A}(t,x,u,z)^\top p \\ &\quad + \operatorname{tr}\big[\mathcal{D}(t,x,z)^\top q\big] \\ H(t,x,u,z,p,r,P) &\doteq \mathcal{H}(t,x,u,z,p,r) \\ &\quad + \frac{1}{2}\operatorname{tr}[D(t,x)^\top P D(t,x)](1-z). \end{aligned}$$

$(\forall (t,x,\overline{p},\overline{q}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times m})$,

$$J(t,x,\overline{p},\overline{q}) = A(t,x)^\top \overline{p} + \operatorname{tr}\big[D(t,x)^\top \overline{q}\big]$$

The following assumptions will be used:

Assumption A1. A, B, D, and G are C^2.

Assumption A2. The first and second derivatives of A, B, D and the second derivative of G are bounded. The maps $A(t,x)$, $B(t,x)$, $D(t,x)$ are bounded by $C(1+|t|+|x|)$. The first derivative of $G(w,x)$ is bounded by $C(1+|w|+|x|)$.

Assumption A3. $(\forall x \in \mathbb{R}^n)$, $(\forall (w_1,w_2) \in \mathbb{R} \times \mathbb{R})$ if $w_1 \leq w_2$ then $G(w_1,x) \leq G(w_2,x)$.

In the rest of the paper, the derivative of the function B, will be denoted by B_t, the partial derivatives of the function A (respectively G, $\mathcal{H}$, and J) with respect to the first variable will be denoted by A_t (respectively $\mathcal{H}_t$, and J_t) and with respect to the second variable it will be denoted by A_x (respectively $\mathcal{H}_x$, and J_x). For $j \in \mathbb{N}_m$, D_{jt} (respectively D_{jx}) denotes the partial derivative of the function D_j with respect to the first variable (respectively the second variable). The partial derivative of G with respect to the first variable will be denote by G_w and with respect to the second variable it will be denoted by G_x.

2. PROBLEM STATEMENT

In this section, we formulate the original singular stochastic control problem presented in the introduction using the formulation described in (El Karoui *et al.*, 1987) and in (Haussmann and Lepeltier, 1990).

Definition 2.1. A singular control is defined by the following term:
$C \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, \{u(t), v(t)\}, \{W(t)\}, \{x(t)\})$ where

(i) $(\Omega, \mathcal{F}, P)$ is a complete probability space with a right continuous complete filtration $\{\mathcal{F}_t\}$.
(ii) $\{W(t)\}$ is a standard m-dimensional $\{\mathcal{F}_t\}$-Brownian motion.
(iii) $\{u(t), v(t)\}$ is a $B_1(K) \times \mathbb{R}_+$-valued, *corlol*, $\{\mathcal{F}_t\}$-progressively measurable process such that $\{v(t)\}$ is increasing and satisfies
$$v(T) \leq M. \tag{1}$$
(iv) $\{x(t)\}$ is an $\mathbb{R}^n$-valued, corlol $\{\mathcal{F}_t\}$-progressively measurable process such that $(\forall t \in [0,T])$
$$\begin{aligned} x(t) \doteq& \zeta + \int_0^t A(s, x(s))ds \\ &+ \int_{[0,t]} B(s)u(s)dv(s) \\ &+ \int_0^t D(s, x(s))dW(s), \end{aligned}$$
and $x(0-) = \zeta$.

We write $\mathfrak{C}$ for the set of controls satisfying the previous conditions.
The cost is given by
$$J[C] \doteq E_P\left[G\Big(\int_0^T |u(s)|dv(s), x(T)\Big)\right]. \tag{2}$$
The set $\mathfrak{C}^a$ of admissible controls is defined by
$$\mathfrak{C}^a \doteq \{C \in \mathfrak{C} : J[C] < \infty\}. \tag{3}$$

The singular control problem is defined by the minimization of $J[C]$ on $\mathfrak{C}^a$. Assuming the existence of an optimal singular control $\widetilde{C}$, the aim of the paper is to derive necessary conditions for $\widetilde{C}$ to be optimal in terms of variational inequalities (see the maximum principle presented in Theorem 5.2).

3. THE AUXILIARY CONTROL PROBLEM

In this section, it is shown that the original singular control problem can be converted into a classical control problem by using a time transformation (see Propositions 3.2, 3.3, and 3.5). We used the technique previously described in (Dufour and Miller, 2004). These results are presented here with minimal details in order to be concise.
Assume the existence of an optimal singular control denoted by
$\widetilde{C} \doteq \Big(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}, \{\widehat{\mathcal{F}}_t\}, \{\widetilde{u}(t), \widetilde{v}(t)\}, \{\widehat{W}(t)\}, \{\widetilde{x}(t)\}\Big)$.
such that $\{\widetilde{u}(t), \widetilde{v}(t)\}$ is $\{\widehat{\mathcal{F}}_t^{\widehat{W}}\}$-progressively measurable.
The existence problem for singular stochastic control has been already studied under general hypotheses in many papers (see for example (Dufour and Miller, 2004) and (Haussmann and Suo, 1995*a*) and the references therein). With the next Proposition, we show how it is possible to construct an optimal singular control $\widehat{C}$ satisfying $\widehat{v}(T) = M$ from the optimal singular control $\widetilde{C}$.

Proposition 3.1. The control $\widehat{C}$ defined by
$$\widehat{C} \doteq \Big(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}, \{\widehat{\mathcal{F}}_t\}, \{\widehat{u}(t), \widehat{v}(t)\}, \{\widehat{W}(t)\}, \{\widehat{x}(t)\}\Big)$$
where
$$\begin{aligned} \widehat{v}(t) &= \widetilde{v}(t) I_{[0,T[} + (M - \widetilde{v}(T) + \widetilde{v}(t)) I_{[T,+\infty[}, \\ \widehat{u}(t) &= \widetilde{u}(t) I_{[0,T[} + \widetilde{u}(t)\left[\frac{\widetilde{v}(T) - \widetilde{v}(T-)}{M - \widetilde{v}(T-)}\right. \\ & \left. I_{[T,+\infty[\times\{\widetilde{v}(T)<M\}} + I_{[T,+\infty[\times\{\widetilde{v}(T)=M\}}\right], \end{aligned}$$
is optimal. Moreover, $\widehat{v}(T) = M$, and $\{\widehat{u}(t), \widehat{v}(t)\}$ is a $\{\widehat{\mathcal{F}}_t^{\widehat{W}}\}$-progressively measurable process.

Now, we will work with the optimal control $\widehat{C}$ for technical reasons. However, a general stochastic maximum principle will be derived in terms of the optimal control $\widetilde{C}$ giving the full generality to our result (see Theorem 5.2).

Proposition 3.2. Denote the process $\big\{t + \widehat{v}(t)\big\}$ by $\{\widehat{\Gamma}(t)\}$. Let $\{\eta^*(t)\}$ be the right inverse of $\{\widehat{\Gamma}(t)\}$. Then, $\{\eta^*(t)\}$ is a continuous time change such that the probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}, \{\widehat{\mathcal{F}}^{\widehat{W}}_{\eta^*(t)}\})$ satisfies the usual hypotheses. Moreover, there exists a $[0,1]$-valued, $\{\widehat{\mathcal{F}}_t^{\widehat{W}}\}$-progressively measurable process $\{\widehat{z}(t)\}$ such that $\widehat{v}(t) = \int_{[0,t]} \widehat{z}(s) d\widehat{\Gamma}(s)$.

Define the $B_1(K) \times [0,1]$-valued, $\{\widehat{\mathcal{F}}^{\widehat{W}}_{\eta^*(t)}\}$- progressively measurable process $\{(\alpha^*(t), \theta^*(t))\}$ by $\alpha^*(t) = \widehat{u}(\eta^*(t))$ and $\theta^*(t) = \widehat{z}(\eta^*(t))$. Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}, \{\widetilde{\mathcal{F}}_t\})$ be a filtered probability space satisfying the usual hypotheses and supporting a standard m-dimensional Brownian motion $\{\widetilde{V}_t\}$. Define by $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$, the usual augmentation of the filtered probability space $\{\widehat{\Omega}\times\widetilde{\Omega}, \widehat{\mathcal{F}}\otimes\widetilde{\mathcal{F}}, \widehat{P}\otimes\widetilde{P}, \widehat{\mathcal{F}}^{\widehat{W}}_{\eta^*(t)}\otimes\widetilde{\mathcal{F}}_t\}$. A random variable $\widehat{X}$ defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ may be viewed as defined on $(\Omega, \mathcal{G}, Q)$ by setting $\overline{X}(\widehat{\omega}, \widetilde{\omega}) = \widehat{x}(\widehat{\omega})$ for $(\widehat{\omega}, \widetilde{\omega}) \in \widehat{\Omega}\times\widetilde{\Omega}$ and similarly for a random variable defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$. Consequently, let us introduce on $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$ the following processes: $\overline{\alpha}(t, \widehat{\omega}, \widetilde{\omega}) \doteq \alpha^*(t, \widehat{\omega})$,

$$\begin{aligned}&\overline{\theta}(t, \widehat{\omega}, \widetilde{\omega}) \doteq \theta^*(t, \widehat{\omega}), \quad \overline{\eta}(t, \widehat{\omega}, \widetilde{\omega}) \doteq \eta^*(t, \widehat{\omega}),\\ &\overline{\xi}(t, \widehat{\omega}, \widetilde{\omega}) \doteq \xi^*(t, \widehat{\omega}), \quad \overline{\mu}(t, \widehat{\omega}, \widetilde{\omega}) \doteq \mu^*(t, \widehat{\omega}),\\ &\overline{W}(t, \widehat{\omega}, \widetilde{\omega}) \doteq \widehat{W}(t, \widehat{\omega}), \quad \widetilde{W}(t, \widehat{\omega}, \widetilde{\omega}) \doteq \widetilde{V}(t, \widetilde{\omega}).\end{aligned}$$

Proposition 3.3. On $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$, the process $\{\overline{V}(t)\}$ defined by

$$\begin{aligned}\overline{V}(t) \doteq &\int_0^t \sqrt{(1-\overline{\theta}(s))^+}d\overline{W}(\overline{\eta}(s))\\ &+ \int_0^t \sqrt{1-(1-\overline{\theta}(s))(1-\overline{\theta}(s))^+}d\widetilde{W}(s)\end{aligned}$$

is a standard m-dimensional $\{\mathcal{G}(t)\}$-Brownian motion.

On the probability space $(\Omega, \mathcal{G}, Q)$, define the filtration $\mathcal{J}_t \doteq \widehat{\mathcal{F}}^{\widehat{W}}_{\eta^*(t)} \otimes \{\emptyset, \widetilde{\Omega}\}$. The set of auxiliary control $\mathcal{E}$ is the set of $\{\mathcal{J}_t\}$-progressively measurable processes defined on $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$ and taking their value in $B_1(K)\times[0,1]$. For any $\{(\alpha(t), \theta(t))\}$ in $\mathcal{E}$, the auxiliary state process $(\eta(t), \xi(t), \mu(t))$ is defined on $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$ by

$$\begin{aligned}\eta(t) \doteq &\int_0^t (1-\theta(s))ds,\\ \xi(t) \doteq &\zeta + \int_0^t A(\eta(s), \xi(s))(1-\theta(s))ds\\ &+ \int_0^t B(\eta(s))\alpha(s)\theta(s)ds\\ &+ \int_0^t D(\eta(s), \xi(s))\sqrt{(1-\theta(s))}d\overline{V}(s),\\ \mu(t) \doteq &\int_0^t |\alpha(s)|\theta(s)ds.\end{aligned}$$

Note that for any $\{(\alpha(t), \theta(t))\}$ in $\mathcal{E}$, the previous system admits a unique solution. Moreover, we have $E_Q\Big[G\big(\mu(T+M), \xi(T+M)\big)\Big] < \infty$. The associated cost functional is defined by

$$\mathcal{M}[\alpha, \theta] \doteq E_Q\Big[G\big(\mu(T+M), \xi(T+M)\big)\Big]. \quad (4)$$

Definition 3.4. The set of admissible auxiliary control $\mathcal{E}_{ad}$ is defined by the set of processes $\{(\alpha(t), \theta(t))\} \in \mathcal{E}$ such that the corresponding auxiliary state process $\{(\eta(t), \xi(t), \mu(t))\}$ satisfies the following constraint

$$E_Q\Big[N\big(\eta(T+M)\big)\Big] = 0. \quad (5)$$

The auxiliary control problem is to minimize the cost (4) over $\mathcal{E}_{ad}$.

Proposition 3.5. The auxiliary control process $\{(\overline{\alpha}(t), \overline{\theta}(t))\}$ is optimal. Moreover, $\{(\overline{\alpha}(t), \overline{\theta}(t))\}$ and the corresponding optimal auxiliary state $\{(\overline{\eta}(t), \overline{\xi}(t), \overline{\mu}(t))\}$ are $\{\mathcal{J}_t\}$-progressively measurable processes.

4. THE MAXIMUM PRINCIPLE FOR THE AUXILIARY CONTROL PROBLEM

In this section, a maximum principle is obtained for the auxiliary singular control problem.

Theorem 4.1. There exists unique solutions of the following backward stochastic differential equations defined on the filtered probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}, \{\widehat{\mathcal{F}}^{\widehat{W}}_{\eta^*(t)}\})$:

$$\begin{aligned}dp^*(t) = &\, q^*(t)d\widehat{W}(\eta^*(t))\\ &- \begin{pmatrix}\mathcal{H}_t(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), \widetilde{q}^*(t))\\ \mathcal{H}_x(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), \widetilde{q}^*(t))\\ 0\end{pmatrix}dt\end{aligned}$$

with $\widetilde{q}^*(t) = \sqrt{1-\theta^*(t)}q^*(t)$,

$$p^*(T+M) = -\begin{pmatrix}0\\ G_x\big(\mu^*(T+M), \xi^*((T+M)\big)\\ G_w\big(\mu^*(T+M), \xi^*(T+M)\big)\end{pmatrix}$$

and

$$\begin{aligned}dP^*(t) = &-A_x(\eta^*(t), \xi^*(t))^\top P^*(t)(1-\theta^*(t))dt\\ &- P^*(t)A_x(\eta^*(t), \xi^*(t))(1-\theta^*(t))dt - (1-\theta^*(t))\\ &\sum_{j=1}^m [D_{jx}(\eta^*(t), \xi^*(t))]^\top P^*(t)D_{jx}(\eta^*(t), \xi^*(t))dt\\ &- (1-\theta^*(t))\sum_{j=1}^m\Big([D_{jx}(\eta^*(t), \xi^*(t))]^\top Q^{*j}(t)\\ &+ Q^{*j}(t)D_{jx}(\eta^*(t), \xi^*(t))\Big)dt\\ &- \mathcal{H}_{xx}(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), \widetilde{q}^*(t))dt\\ &+ \sum_{j=1}^m Q^{*j}(t)d\widehat{W}(\eta^*(t))\end{aligned}$$

with $P^*(T+M) = -G_{xx}\big(\mu^*(T+M), \xi^*(T+M)\big)$.

Now we give the maximum principle for the auxiliary control problem.

Theorem 4.2. For all $(\alpha, \theta) \in B_1(K)\times[0,1]$

$$\begin{aligned}&H(\eta^*(t), \xi^*(t), \alpha, \theta, p^*(t), r^*(t), P^*(t))\\ &\le H(\eta^*(t), \xi^*(t), \alpha^*(t), \theta^*(t), p^*(t), r^*(t), P^*(t))\end{aligned}$$

$\lambda\otimes\widehat{P}-$a.s. on $[0, T+M]\times\widehat{\Omega}$, with

$r^*(t) \doteq \Big[q^*(t) - TP^*(t)D(\eta^*(t), \xi^*(t))\Big]\sqrt{1-\theta^*(t)}$,

where the matrix T is defined by $\Big(0_n\; I_n\; 0_n\Big)^\top$.

5. THE SINGULAR MAXIMUM PRINCIPLE AND ADJOINT VARIABLES

It is possible to obtain the adjoint variables for the original control problem by using a time change. The interesting feature of these adjoint variables is that the first component is the solution of a singular backward equation. Finally, the stochastic maximum principle for the original singular control problem in the general case is obtained and presented in Theorem 5.2.

Definition 5.1. Let $C \in \mathfrak{C}^a$ be a singular control $C \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, \{u(t), v(t)\}, \{W(t)\}, \{x(t)\})$ such that $\{u(t), v(t)\}$ is $\{\mathcal{F}_t^W\}$-progressively measurable. If there exist $\left(\left(\{p^i(t)\}, \{q^i(t)\}\right)_{i\in\mathbb{N}_3}, \{P(t)\}, \left(\{Q^j(t)\}\right)_{j\in\mathbb{N}_m}\right)$ solutions of the following backward stochastic differential equations

$$
\begin{aligned}
p^1(t) &= \int_t^T A_t(s,x(s))^\top p^2(s)ds \\
&+ \int_{]t,T]} p^2(s)^\top B_t(t)u(s)dv(s) \\
&+ \int_t^T \sum_{j=1}^m D_{jt}(t,x(t))^\top q_j^2(s)dt \\
&- \int_t^T q^1(s)dW(s) \\
p^2(t) &= -G_x\left(\int_0^T |u(s)|dv(s), x(T)\right) \\
&+ \int_t^T A_x(s,x(s))^\top p^2(s)ds \\
&+ \int_t^T \sum_{j=1}^m D_{jx}(s,x(s))^\top q_j^2(s)dt \\
&- \int_t^T q^2(s)dW(s) \\
p^3(t) &= -G_w\left(\int_0^T |u(s)|dv(s), x(T)\right) \\
&- \int_t^T q^3(s)dW(s) \\
P(t) &= -G_{xx}\left(\int_0^T |u(s)|dv(s), x(T)\right) \\
&+ \int_t^T \Big[A_x(s,x(s))^\top P(s) - P(s)A_x(s,x(s))\Big]ds \\
&+ \sum_{j=1}^m \int_t^T D_{jx}(s,x(s))^\top P(s) D_{jx}(s,x(s))ds \\
&+ \int_t^T J_{xx}(s,x(s),p^2(s),q^2(s))ds \\
&+ \sum_{j=1}^m \int_t^T \big[D_{jx}(s,x(s))^\top Q^j(s) \\
&+ Q^j(s)D_{jx}(s,x(s))\big]ds - \int_t^T \sum_{j=1}^m Q^j(s)dW(s).
\end{aligned}
$$

$\left(\left(\{p^i(t)\}, \{q^i(t)\}\right)_{i\in\mathbb{N}_3}, \{P(t)\}, \left(\{Q^j(t)\}\right)_{j\in\mathbb{N}_m}\right)$ are then called the adjoint variables associated to the control C. They are said unique if the solutions of the previous equations are unique.

By using a time transformation, we show that from $\left(\{p^*(t)\}, \{q^*(t)\}, \{P^*(t)\}, \left(\{Q^{*j}(t)\}\right)_{j\in\mathbb{N}_m}\right)$ we can obtain the adjoint variables for the original optimal control. Here we present a maximum principle given in terms of three variational inequalities but not in the integral from as in the work by A. Cadenillas and U. Haussmann (Cadenillas and Haussmann, 1994). The first two variational inequalities result from a time change of the variational inequality of the auxiliary control problem. The first one is given with respect to the measure of Doleans-Dade generated by the absolutely continuous part of $\{\widetilde{v}(t)\}$. Note that it does not depend directly on $\{\widetilde{v}(t)\}$ but on the derivative $\frac{d\widetilde{v}}{dt}(t)$. The second one is given with respect to the measure of Doleans-Dade generated by the singular part of $\{\widetilde{v}_t\}$. The last inequality can be interpreted as a necessary condition for the size of the jumps of $\{\widetilde{v}_t\}$. It is different from the first two ones because it is not obtain from a time change of the variational inequality of the auxiliary control problem.

Theorem 5.2. Assume the existence of an optimal singular control denoted by
$\widetilde{C} \doteq \left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}, \{\widehat{\mathcal{F}}_t\}, \{\widetilde{u}(t), \widetilde{v}(t)\}, \{\widehat{W}(t)\}, \{\widetilde{x}(t)\}\right).$
such that $\{\widetilde{u}(t), \widetilde{v}(t)\}$ is $\{\widehat{\mathcal{F}}_t^{\widehat{W}}\}$-progressively measurable. Denote by $\{\widetilde{\tau}_i\}_{i\in\mathbb{N}^*}$ the sequence of $\{\widehat{\mathcal{F}}_t^{\widehat{W}}\}$-stopping times which exhausts the jumps of $\{\widetilde{v}(t)\}$. Then, for all $(u,z) \in B_1(K)\times[0,1]$

$$
\begin{aligned}
&[zu - \widetilde{z}(t)\widetilde{u}(t)]^\top B(t)^\top \widetilde{p}^2(t) \\
&+ [z|u| - \widetilde{z}(t)|\widetilde{u}(t)|]\widetilde{p}^3(t) \\
&+ \mathrm{tr}\big[D(t,\widetilde{x}(t))^\top \widetilde{r}(t)\big]\big(\sqrt{1-z} - \sqrt{1-\widetilde{z}(t)}\big) \\
&+ [\widetilde{z}(t) - z]\Big[\frac{1}{2}\,\mathrm{tr}\big[D(t,\widetilde{x}(t))^\top \widetilde{P}(t) D(t,\widetilde{x}(t))\big] \\
&+ \widetilde{p}^1 + A(t,\widetilde{x}(t))^\top \widetilde{p}^2(t)\Big] \le 0, \mathcal{M}_{\widetilde{v}^{ac}} - \text{a.s. on}
\end{aligned}
$$

$[0,T]\times\widehat{\Omega}$, where $\widetilde{z}(t) = \dfrac{\frac{d\widetilde{v}}{dt}(t)}{1+\frac{d\widetilde{v}}{dt}(t)}$,
$\widetilde{r}(t) \doteq \Big[\widetilde{q}^2(t) - \widetilde{P}(t)D(t,\widetilde{x}(t))\Big]\sqrt{1-\widetilde{z}(t)}$, and

$$
\begin{aligned}
&[zu - \widetilde{u}(t)]^\top B(t)^\top \widetilde{p}^2(t) + [z|u| - |\widetilde{u}(t)|]\widetilde{p}^3(t) \\
&+ (1-z)\Big[\frac{1}{2}\,\mathrm{tr}\big[D(t,\widetilde{x}(t))^\top \widetilde{P}(t) D(t,\widetilde{x}(t))\big] \\
&+ \widetilde{p}^1(t) + A(t,\widetilde{x}(t))^\top \widetilde{p}^2(t)\Big] \le 0
\end{aligned}
$$

$\mathcal{M}_{\widetilde{v}^s}$ − a.s. on $[0,T]\times\widehat{\Omega}$, and for all $i \in \mathbb{N}$ and $\gamma \in [0,1]$

$$\left[B(\widetilde{\tau}_i)(zu - \widetilde{u}(\widetilde{\tau}_i))\right]^\top \widetilde{p}^2(\widetilde{\tau}_i) + (z|u| - |\widetilde{u}(\widetilde{\tau}_i)|)\widetilde{p}^3(\widetilde{\tau}_i)$$
$$+ (1-z)\Bigg\{\widetilde{p}^1(\widetilde{\tau}_i-) - \gamma\Big[B(\widetilde{\tau}_i)\widetilde{u}(\widetilde{\tau}_i)\Delta\widetilde{v}(\widetilde{\tau}_i)\Big]^\top \widetilde{p}^2(\widetilde{\tau}_i)$$
$$+ A\Big(\widetilde{\tau}_i, \widetilde{x}(\widetilde{\tau}_i-) + \gamma B(\widetilde{\tau}_i)\widetilde{u}(\widetilde{\tau}_i)\Delta\widetilde{v}(\widetilde{\tau}_i)\Big)^\top \widetilde{p}^2(\widetilde{\tau}_i)$$
$$+ \frac{1}{2}\,\mathrm{tr}\Big[D\Big(\widetilde{\tau}_i, \widetilde{x}(\widetilde{\tau}_i-) + \gamma B(\widetilde{\tau}_i)\widetilde{u}(\widetilde{\tau}_i)\Delta\widetilde{v}(\widetilde{\tau}_i)\Big)^\top \widetilde{P}(\widetilde{\tau}_i)$$
$$D\Big(\widetilde{\tau}_i, \widetilde{x}(\widetilde{\tau}_i-) + \gamma B(\widetilde{\tau}_i)\widetilde{u}(\widetilde{\tau}_i)\Delta\widetilde{v}(\widetilde{\tau}_i)\Big)\Big]\Bigg\} \le 0,$$

$\widehat{P}$ − a.s. on $\{\widetilde{\tau}_i \le T\}$ where $\Big(\big(\{\widetilde{p}^i(t)\}, \{\widetilde{q}^i(t)\}\big)_{i\in\mathbb{N}_3}, \{\widetilde{P}(t)\}, \big(\{\widetilde{Q}^j(t)\}\big)_{j\in\mathbb{N}_m}\Big)$ are the adjoint variables associated $\widetilde{C}$.

6. CONCLUSION

Our work can be generalized in several directions: a running cost can be added to the definition of $J[C]$ (with no convexity hypothesis) and a classical control process can be added in the dynamic of the state (for example in A, D and into the running cost if it exists) as in (Cadenillas and Haussmann, 1994). Soft constraints with the same form of the cost G may also be added to the model, see page 855 in (Haussmann and Lepeltier, 1990) for constraints of these types in classical control problems.

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