AN EXACT STABILITY TEST FOR PLANAR AND MULTI-MODAL PIECEWISE LINEAR SYSTEMS

Yasushi Iwatani [*,1] **Shinji Hara** [*]

[*] *Department of Information Physics and Computing, The University of Tokyo*

Abstract: A necessary and sufficient stability condition for planar and multi-modal piecewise linear systems is derived. The condition is given in terms of poles and zeros of subsystems, and it is computationally tractable. In addition, three numerical examples are illustrated in order to clarify differences between a class of linear time-invariant systems and a class of piecewise linear systems from the view point of stability.

Keywords: Hybrid dynamical systems, Piecewise linear systems, Stability analysis, Poles, Zeros.

1. INTRODUCTION

In the last decade, a lot of techniques have been developed for analysis and controller synthesis for hybrid dynamical systems. Each result depends on the mathematical model that represents behavior of hybrid dynamical systems. One of the typical models is the piecewise linear system (PLS). The system consists of some pairs of linear time-invariant dynamics and a switching rule given by a linear function of the continuous state. Study on PLSs is important as a first step to establish hybrid control theory, because the hybrid dynamics is the simplest in all classes of hybrid dynamical systems.

Unlike linear time-invariant systems, checking stability of a given PLS is a very hard problem due to its hybrid nature. In fact, there exist no systematic ways to check stability of the class of PLSs exactly, although many results have been obtained on stability for several classes of hybrid dynamical systems which include the class of PLSs (see (Decarlo *et al.*, 2000; Lygeros *et al.*, 2003) and the references therein). Most of the results on stability are extensions of Lyapunov's theorem, where we need to show the existence of a Lyapunov function to guarantee the stability. The Lyapunov methods provide not only sufficient conditions but also necessary conditions for stability under hybrid natures. Actually, the converse theorems ensure the existence of a Lyapunov function when the system is asymptotically stable (Michel and Hu, 1999). On the other hand, we must restrict available classes of Lyapunov functions within a class of piecewise quadratic functions (Rantzer and Johansson, 2000; Gonçalves *et al.*, 2003) or a class of sums of squares (Prajna and Papachristodoulou, 2003) to give systematic ways of finding the Lyapunov functions. This makes the stability conditions conservative. Therefore, we need a new approach to get an explicit necessary and sufficient stability condition for the class of PLSs.

In recent years, direct analysis of behavior of hybrid states has led to exact stability tests for several classes of hybrid dynamical systems. Xu

[1] This work was supported in part by the Grants-in-Aid for JSPS fellows No.15003305 and Scientific Research (C) (2) No.14550438 of the Ministry of Education, Culture, Sports, Science and Technology, Japan.

and Antsaklis (2000) derived necessary and sufficient conditions for stabilizability of a class of planar and linear switched systems through an investigation of behavior of the systems. Çamlıbel et al. (2003) provided a necessary and sufficient stability condition for a class of planar and linear complementarity systems. Recently, two necessary and sufficient stability conditions were obtained for a class of planar and bimodal PLSs (Iwatani and Hara, 2004*b*) which includes the class of planar and linear complementarity systems as a special case. In (Iwatani and Hara, 2004*b*), a class of higher-order and bimodal systems has been also treated.

The contribution of this paper is to derive an explicit necessary and sufficient stability condition for planar and multi-modal PLSs. The results in this paper are an extension of one of the results on planar and bimodal PLSs in (Iwatani and Hara, 2004*b*) to multi-modal systems. The extension, however, is not straightforward, since the behaviors of the multi-modal system are not so simple as those of the bimodal system. In other words, we require a new approach that leads to an exact stability condition for multi-modal systems. To this end, we define two concepts, namely transitive mode and weak transitive mode, which properly capture behaviors of the hybrid state. The two concepts characterize a necessary and sufficient stability condition for the systems. We then derive a necessary and sufficient stability condition in terms of poles and zeros of transfer functions of the subsystems, which are naturally defined by state space data of given multi-modal system. It clearly provides a computationally tractable exact stability test. In addition, the condition is equivalent to the results in (Iwatani and Hara, 2004*b*) for the bimodal case. Three numerical examples are finally addressed. They illustrate typical trajectories of PLSs and clarify differences between a class of linear time-invariant systems and a class of PLSs from a view point of stability.

All the proofs of lemmas and theorems in this paper are omitted from this paper for lack of space. They are found in a more detailed technical report (Iwatani and Hara, 2004*a*).

2. PLANAR MULTI-MODAL PIECEWISE LINEAR SYSTEMS

We consider a class of planar and multi-modal PLSs represented by

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x \in \mathbb{S}_1, \\ A_2 x, & \text{if } x \in \mathbb{S}_2, \\ \quad \vdots & \quad \vdots \\ A_m x, & \text{if } x \in \mathbb{S}_m, \end{cases} \tag{1}$$

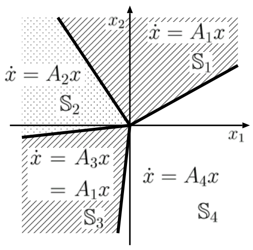

Fig. 1. Planar multi-modal piecewise linear model.

where $A_i \in \mathbb{R}^{2\times 2}$, $\mathbb{S}_i$ is a convex cone of the form $\mathbb{S}_i := \{x \in \mathbb{R}^2 \mid C_i x \geq 0\}$, and $C_i \in \mathbb{R}^{2\times 2}$ $(i = 1, \ldots, m)$. A matrix A_i may be equal to another matrix A_j as seen in Figure 1.

We here introduce two notions, called the proper state space and well-posedness, in order to exclude the cases which are out of our interests.

First, the state space of the system (1) is said to be *proper*, if all the following statements hold:

$$\begin{aligned} &\mathbb{S}_i \neq \mathbb{R}^2, && \forall i, \\ &\mathrm{int}\mathbb{S}_i \neq \emptyset, && \forall i, \\ &\cup_{i=1}^m \mathbb{S}_i = \mathbb{R}^2, && \\ &\mathrm{int}(\mathbb{S}_i \cap \mathbb{S}_j) = \emptyset, && \forall (i,j),\ (i \neq j). \end{aligned}$$

It is clear that they are quite natural and hence they are not restrictive under memoryless nonlinearities.

Second, the system (1) is said to be *well-posed*, if the system has a unique solution for each initial state. Definition of solutions can be chosen from the concepts defined in (Imura and van der Schaft, 2000; Imura, 2002; Imura, 2003). The choice does not influence our results shown below. On the other hand, we do not treat systems with sliding modes. In fact, Theorem 7 provided in Section 5 does not hold under existence of sliding modes.

The solution from an initial state x_0 is denoted by $x(t, x_0)$ where the initial time is always set 0.

Finally, we suppose $\det C_i \neq 0$ for all i throughout this paper. The assumption does not make the resulting stability condition conservative, because each system whose state space is proper satisfies the assumption after additional partition of the state space with the x_1-axis and the x_2-axis. An alternative to avoiding the non-singular assumption is found in (Iwatani and Hara, 2004*a*) where it needs complicated notation.

Remark 1. Consider a class of planar and bimodal PLSs represented by

$$\dot{x} = \begin{cases} A_1 x, \text{ if } cx \geq 0, \\ A_2 x, \text{ if } cx \leq 0, \end{cases} \tag{2}$$

where $c \in \mathbb{R}^{1\times 2}$ (Iwatani and Hara, 2004*b*). It is clear that no bimodal systems of the form (2) satisfy the non-singular assumption.

3. STABILITY ANALYSIS

This section provides the basic concepts of stability analysis for the system (1).

We first investigate trajectories of the system (1). Suppose that the system is well-posed and the state space is proper. Then the system has one of the following two properties: **(T-i)** Infinitely many events occur on each trajectory as illustrated in Figure 2-(i). **(T-ii)** The number of events which take place on each trajectory is finite as depicted in Figure 2-(ii). The two properties are closely connected with transitive modes and weak transitive modes defined as follows.

Definition 2. **(i)** A mode i is said to be *transitive*, if $\forall x_0 \in \mathbb{S}_i \backslash \{0\},\ \exists t > 0,\ x(t, x_0) \notin \mathbb{S}_i$.
(ii) A mode i is said to be *weakly transitive*, if one of the following two statements holds for all $x_0 \in \mathbb{S}_i \backslash \{0\}$:
(a) $\exists t > 0$, $x(t, x_0) \notin \mathbb{S}_i$.
(b) $\lim_{t\to\infty} x(t, x_0) = 0$ and $\forall t \geq 0, x(t, x_0) \in \mathbb{S}_i$.

We first discuss the relationship between transitive modes and the property (T-i). To this end, we define a set of vectors on the boundary $\partial\mathbb{S}_i$ as follows:

$$\mathbb{B}_i := \{x_0 \in \partial\mathbb{S}_i \backslash \{0\} \mid \exists \varepsilon > 0,\ \forall t \in [0, \varepsilon), x(t, x_0) \in \mathbb{S}_i\},$$

where the set $\mathbb{B}_i$ is called the *inward boundary* (see Figure 3-(i)). Then, we can show that a mode i is transitive, if and only if $\mathbb{B}_i \neq \emptyset$ and

$$\exists \tau_i > 0,\quad \forall x_0 \in \mathbb{B}_i,\ \{x(\tau_i, x_0) \in \partial\mathbb{S}_i \text{ and } \forall t \in (0, \tau_i),\ x(t, x_0) \in \mathrm{int}\mathbb{S}_i\}, \quad (3)$$

(see Figure 3-(ii)). This implies that (T-i) holds if and only if all modes are transitive. Suppose that all modes are transitive. Then

$$\forall x_0 \in \mathbb{R}^2,\ \exists \rho \in \mathbb{R},\ \rho x_0 = x\left(\sum_{i=1}^{m} \tau_i, x_0\right), \quad (4)$$

where each τ_i is defined by (3). Clearly, the origin is globally asymptotically stable if and only if $\rho < 1$ for all x_0.

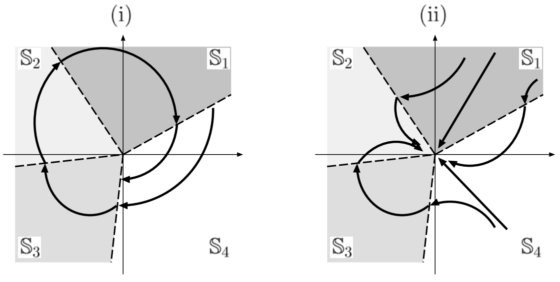

Fig. 2. Trajectories of the system (1)

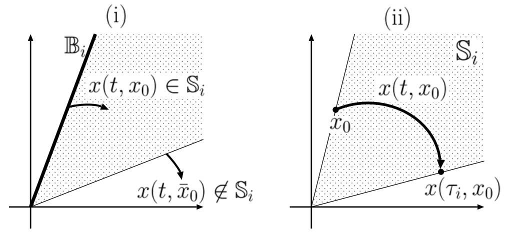

Fig. 3. (i) The thick line represents the inward boundary $\mathbb{B}_i$. (ii) An illustration of (3).

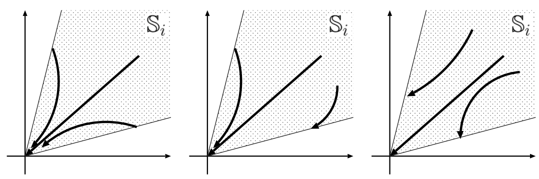

Fig. 4. Trajectories in weak transitive modes.

We then focus on the weak transitive modes. Trajectories in three possible weak transitive modes can be seen in Figure 4. Every weak transitive mode has the following three features which are immediate from Definition 2: (i) A mode i is weakly transitive, if i is transitive. (ii) The origin is not stable, if there exists a mode i which is not weakly transitive. (iii) There exists at least one weak transitive mode, if the origin is asymptotically stable and (T-ii) holds.

Using the terminologies of transitive modes and weak transitive modes, we can show the following theorem which provides a necessary and sufficient condition for the system (1) to be stable.

Theorem 3. Consider the system (1). Suppose that the system is well-posed and the state space is proper. Then the following statements hold.

(i) Suppose that all modes are transitive. Then the origin is globally asymptotically stable, if and only if $x(\sum_{i=1}^{m} \tau_i, x_0) < x_0$ holds for all $x_0 \in \mathbb{R}^2 \backslash \{0\}$.
(ii) Suppose that there exists a mode which is not transitive. Then the origin is globally asymptotically stable, if and only if all modes are weakly transitive.

Theorem 3 will be written in terms of poles and zeros of subsystems in Section 5. The stability condition in Theorem 7 shown there is computationally tractable as seen in an algorithm described below the theorem.

4. TRANSFER FUNCTION REPRESENTATION

This section introduces a 2×2 matrix transfer function for each mode. Each function is the Laplace transform of initial value responses for two initial states on $\partial\mathbb{S}_i$. The poles and zeros of the functions characterize a necessary and sufficient stability condition in the next section.

We first partition C_i^{-1} as $[z_{i1}, z_{i2}] := C_i^{-1}$. Every vector z_{ij} is on $\partial\mathbb{S}_i$. We then consider a transfer function of the form

$$\begin{aligned} T_i(s) &= C_i(sI - A_i)^{-1}C_i^{-1} \\ &= \frac{1}{s^2 + \alpha_i s + \beta_i}\begin{bmatrix} s+\gamma_{i11}, & \gamma_{i12} \\ \gamma_{i21}, & s+\gamma_{i22} \end{bmatrix}. \end{aligned} \tag{5}$$

Clearly, $T_i(s)$ represents the Laplace transform of initial value responses whose initial values are on $\partial\mathbb{S}_i$. We can easily check if z_{ij} is on the inward boundary $\mathbb{B}_i$ as follows.

Proposition 4. Consider the system (1). Assume $\det C_i \neq 0$. Then $z_{ij} \in \mathbb{B}_i$ holds, if and only if $\gamma_{i\bar{j}j} \geq 0$ where $\bar{j} \in \{k \in \{1,2\} \mid k \neq j\}$.

The following definition chooses two special zeros associated with an initial state on $\mathbb{B}_i$ from γ_{ijk} $(j = 1, 2,\ k = 1, 2)$ in $T_i(s)$, if it exists:

$$\gamma_i := \gamma_{ip_ip_i}, \quad \delta_i := \gamma_{iq_ip_i}, \tag{6}$$

where

$$p_i := \begin{cases} 1, \text{ if } \gamma_{i21} \geq 0, \\ 2, \text{ otherwise}, \end{cases} \tag{7}$$

$$q_i \in \{j \in \{1,2\} \mid j \neq p_i\}. \tag{8}$$

Note that z_{ip_i} is on the inward boundary $\mathbb{B}_i$, if $\mathbb{B}_i \neq \emptyset$. Conversely, $\mathbb{B}_i = \emptyset$, if $\delta_i < 0$.

5. AN EXPLICIT NECESSARY AND SUFFICIENT STABILITY CONDITION

An explicit necessary and sufficient stability condition for the system (1) is given in terms of poles and zeros of $T_i(s)$, where we often omit the index i which expresses a mode from symbols to simplify the notation.

The following lemma provides a necessary and sufficient condition for a mode i to be transitive or weakly transitive.

Lemma 5. Consider the system (1) and assume $\det C_i \neq 0$. Then the following statements hold.

(i) The mode i is transitive, if and only if it holds that

$$\beta > \begin{cases} \dfrac{\alpha^2}{4}, & \text{if } \alpha \leq 2\gamma, \\ \gamma\alpha - \gamma^2, & \text{if } \alpha \geq 2\gamma. \end{cases} \tag{9}$$

(ii) The mode i is weakly transitive, if and only if it holds that

$$\begin{cases} \beta > \dfrac{\alpha^2}{4}, & \text{if } \delta \geq 0 \text{ and } \alpha \leq 2\underline{\gamma}, \\ \beta > \underline{\gamma}\alpha - \underline{\gamma}^2, & \text{if } \delta \geq 0 \text{ and } \alpha \geq 2\underline{\gamma}, \\ \beta < \overline{\gamma}\alpha - \overline{\gamma}^2, & \text{if } \delta < 0, \end{cases} \tag{10}$$

where $\overline{\gamma} := \max(0, \gamma)$ and $\underline{\gamma} := \min(0, \gamma)$.

The following lemma plays an important role to make ρ defined in (4) or equivalently (i) in Theorem 3 explicit.

Lemma 6. Consider the system (1). Suppose that $\det C_i \neq 0$ and (9) holds for i. Define τ according to (3). Then, $x(\tau, z_p) = \eta z_q$ holds, where [2]

$$\eta = \begin{cases} \dfrac{\delta}{\sqrt{\beta - \alpha\gamma + \gamma^2}} \exp\left(\dfrac{-\alpha\theta}{\sqrt{4\beta - \alpha^2}}\right), & \text{if } \alpha^2 < 4\beta, \\ \dfrac{2\delta}{\alpha - 2\gamma} \exp\left(\dfrac{-\alpha}{\alpha - 2\gamma}\right), & \text{if } \alpha^2 = 4\beta, \\ \delta \exp\left(\dfrac{\lambda_2 \log|\lambda_2 + \gamma| - \lambda_1 \log|\lambda_1 + \gamma|}{\lambda_1 - \lambda_2}\right), & \text{if } \alpha^2 > 4\beta, \end{cases} \tag{11}$$

$$\theta = \mathrm{Arccos}\left(\frac{\alpha - 2\gamma}{2\sqrt{\beta - \alpha\gamma + \gamma^2}}\right),$$

$$\lambda_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2},\ \lambda_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}.$$

Note that η is well-defined under (9).

We are now ready to give an explicit necessary and sufficient stability condition in terms of poles and zeros of $T_i(s)$.

Theorem 7. Consider the system (1). Suppose that the system is well-posed, the state space is proper and $\det C_i \neq 0$ for all i. Then the following two statements hold true.

(i) Suppose (9) holds for all i. Then the origin is globally asymptotically stable, if and only if it holds that

$$\prod_{i=1}^{m} (\eta_i \|z_{iq_i}\|) < \prod_{i=1}^{m} \|z_{ip_i}\|. \tag{12}$$

(ii) Suppose that there exists a mode i such that (9) does not hold. Then the origin is globally asymptotically stable, if and only if (10) holds for all i.

The following algorithm based on Theorem 7 leads to an exact stability test for the system (1).

Algorithm 1.

[2] For $x \in [-1, 1]$, $\mathrm{Arccos}(x)$ expresses the principal value of the inverse cosine of x, i.e. $\mathrm{Arccos}(x) \in [0, \pi]$.

Step 1) Check if the state space of a given system is proper and the system is well-posed (see (Imura, 2002; Imura, 2003; Imura and van der Schaft, 2000) for well-posedness conditions). If yes, go to Step 2. If not, the algorithm ends.
Step 2) Check if $\det C_i \neq 0$ for all i. If not, partition the state space with the x_1-axis and the x_2-axis.
Step 3) Check if (9) holds for all i. If yes, then go to Step 4-i. If not, then go to Step 4-ii.
Step 4)
- **(i)** Check if (12) is satisfied. If yes, the origin is globally asymptotically stable. If not, the origin is not globally asymptotically stable.
- **(ii)** Check if (10) holds for all i. If yes, the origin is globally asymptotically stable. If not, the origin is not globally asymptotically stable.

Remark 8. The bimodal system (2) is globally asymptotically stable, if and only if all the following inequalities holds (Iwatani and Hara, 2004*b*):

$$\beta_i > \max(0, -\frac{|\alpha_i|\alpha_i}{4}), \qquad i = 1, 2,$$
$$\frac{|\alpha_1|\alpha_1}{\beta_1} + \frac{|\alpha_2|\alpha_2}{\beta_2} > 0.$$

Algorithm 1 may lead to the same stability condition, which implies that the result in this paper is an extension of the result in (Iwatani and Hara, 2004*b*) when $x \in \mathbb{R}^2$. However, it requires further partition of the state space, and hence we need a lot of work to derive the stability condition from Algorithm 1. On the other hand, another approach provided in (Iwatani and Hara, 2004*a*) leads to the stability condition more easily than Algorithm 1.

6. NUMERICAL EXAMPLES

This section illustrates three typical numerical examples in order to clarify differences between trajectories of linear time invariant systems and PLSs from a view point of stability.

Example 9. Consider a four-modal system with

$$A_1 = A_3 = \begin{bmatrix} \sigma, & 1 \\ -\omega^2, & \sigma \end{bmatrix}, \; A_2 = A_4 = \begin{bmatrix} 1, & \pi \\ -\pi, & 1 \end{bmatrix},$$
$$C_1 = -C_4 = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}, C_2 = -C_3 = \begin{bmatrix} -1, & 0 \\ 0, & 1 \end{bmatrix}. \quad (13)$$

Each set $\mathbb{S}_i$ expresses the i-th quadrant. The aim here is to derive a necessary and sufficient stability condition of the system in terms of $\sigma \in \mathbb{R}$ and $\omega \in \mathbb{R}_+$ via Algorithm 1. As a result, Algorithm 1 shows that the origin is asymptotically stable, if and only if $e^{1+\frac{\sigma\pi}{\omega}} < \omega^2$ holds.

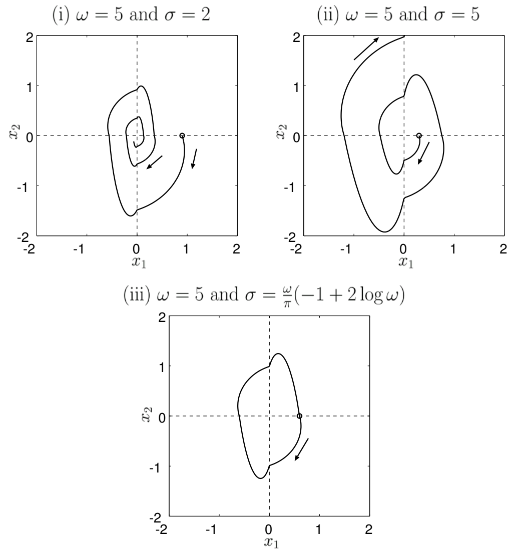

Fig. 5. Trajectories of Example 9.

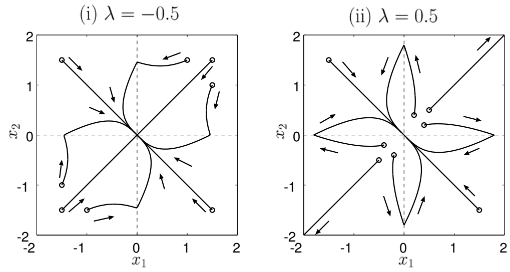

Fig. 6. Trajectories of Example 10.

Typical trajectories of the system are illustrated in Figure 5. It is seen from Figure 5–(i) that the origin is asymptotically stable, while all constituent matrices A_i are unstable: This phenomenon has been pointed out in (Branicky, 1998). Figure 5-(iii) implies that every trajectory is a closed orbit when $e^{1+\frac{\sigma\pi}{\omega}} = \omega^2$ holds. This confirms the proposed stability test is exact.

Example 10. Consider a four-modal system with

$$A_1 = A_3 = \tfrac{1}{2}\begin{bmatrix} \lambda+1, & \lambda-1 \\ \lambda-1, & \lambda+1 \end{bmatrix},$$
$$A_2 = A_4 = \begin{bmatrix} -2, & -1 \\ -1, & -2 \end{bmatrix},$$

and C_i (i=1,2,3,4) defined by (13). Assume $\lambda < 1$, which guarantees well-posedness of the system. It is seen that (9) does not hold for any i. Thus, let us investigate (10). Consequently, Algorithm 1 shows that the system is stable, if and only if $\lambda < 0$. Typical trajectories can be seen in Figure 6. We have a remark for the unstable case, i.e. $0 \leq \lambda < 1$. Every trajectory converges to the origin, if the initial state is not on the line through $[1, 1]^\top$ and $-[1, 1]^\top$ (Iwatani and Hara, 2004*a*). In other words, $\lim_{t\to\infty} x(t, x_0) = 0$, a.e. $x_0 \in \mathbb{R}^2$, even if the origin is unstable as illustrated in Figure 6–(ii).

Example 11. Consider a three-modal system with

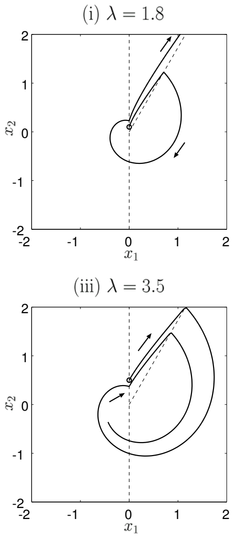

Fig. 7. Trajectories of Example 11.

$$A_1 = \frac{1}{2}\begin{bmatrix} \lambda+1, & \lambda-1 \\ \lambda-1, & \lambda+1 \end{bmatrix}, A_2 = A_3 = \begin{bmatrix} -1, & \pi \\ -\pi, & -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1, & 0 \\ -\sqrt{3}, & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1, & 0 \\ \sqrt{3}, & -1 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} -1, & 0 \\ 0, & 0 \end{bmatrix}.$$

Assume $\lambda > 1$ to guarantee well-posedness of the system. By Algorithm 1, the origin is asymptotically stable, if and only if it holds that

$$\lambda > \frac{\frac{11}{6} + \log \frac{\sqrt{3}+1}{2}}{\frac{11}{6} + \log \frac{\sqrt{3}-1}{2}} \simeq 2.59. \qquad (14)$$

Note that λ is one of the eigenvalues of A_1. Therefore, (14) implies that an eigenvalue of A_1 must be greater than the value defined by the right hand side of (14). Moreover, the greater the value of λ is, the faster the state converges to the origin as illustrated in Figure 7. Roughly speaking, the more unstable the subsystem is, the more stable the hybrid system is, in this case.

7. CONCLUSION

In this paper, we have derived a necessary and sufficient condition for planar and multi-modal PLSs to be stable. The condition is given in terms of poles and zeros of subsystems, and it is computationally tractable. Also, we have shown three numerical examples which provide typical trajectories of PLSs. They clarify differences between a class of linear time invariant systems and a class of PLSs from a view point of stability.

There still remain several open problems on stability of PLSs to be addressed in the future, although we have established some basic tools for stability analysis in this paper. In particular, we need to discuss stability analysis for the higher-order case. A necessary condition and a sufficient condition for stability of higher-order and bimodal systems are addressed in (Iwatani and Hara, 2004*b*).

REFERENCES

Branicky, M. (1998). Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Auto. Contr.* **43**(4), 475–482.

Çamlıbel, M. K., W. P. M. H. Heemels and J. M. Schumacher (2003). Stability and controllability of planar bimodal linear complementarity systems. In: *Proc. of IEEE CDC.*

Decarlo, R., M. Branicky, S. Petersson and B. Lennartson (2000). Perspective and results on the stability and stabilizability of hybrid systems. *Proc. of the IEEE* **88**, 1069–1082.

Gonçalves, J. M., A. Megretski and A. Dahleh (2003). Global analysis of piecewise linear systems using impact maps and surface Lyapunov functions. *IEEE Trans. Auto. Contr.* **48**(12), 2089–2106.

Imura, J. (2002). Classification and stabilizability analysis of bimodal piecewise affine systems. *Int. J. Robust and Nonlinear Contr.* **12**, 897–926.

Imura, J. (2003). Well-posedness analysis of switch-driven hybrid systems. *IEEE Trans. Auto. Contr.* **48**(11), 1926–1935.

Imura, J. and A. J. van der Schaft (2000). Characterization of well-posedness of piecewise linear systems. *IEEE Trans. Auto. Contr.* **45**(9), 1600–1619.

Iwatani, Y. and S. Hara (2004*a*). Exact stability tests for planar and multi-modal piecewise linear systems. *Technical Report of The Univ. Tokyo, METR2004-38,* `http://www.keisu.t.u-tokyo.ac.jp/Research/METR/2004/METR04-38.pdf`.

Iwatani, Y. and S. Hara (2004*b*). Stability tests based on eigenvalue loci for piecewise linear systems. In: *Proc. of ACC.*

Lygeros, J., K. H. Johansson, S. N. Simic, J. Zhang and S. S. Sastry (2003). Dynamical properties of hybrid automata. *IEEE Trans. Auto. Contr.* **48**(1), 2–17.

Michel, A. N. and B. Hu (1999). Towards a stability theory of general hybrid dynamical systems. *Automatica* **35**, 371–384.

Prajna, S. and A. Papachristodoulou (2003). Analysis of switched and hybrid systems. In: *Proc. of ACC.*

Rantzer, A. and M. Johansson (2000). Piecewise linear quadratic optimal control. *IEEE Trans. Auto. Contr.* **45**(4), 629–637.

Xu, X. and P. J. Antsaklis (2000). Stabilization of second-order LTI switched systems. *Int. J. Contr.* **73**(14), 1261–1279.