# SYNTHESIS AND SIMULATION OF FRACTIONAL ORTHONORMAL BASES

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Abstract: Although rational orthogonal bases can be used to model any $L_2[0, \infty[$ system, they fail to capture the aperiodic multi-mode behaviour of fractional systems in a limited number of terms. The classical definition of orthogonal Laguerre, Kautz, and GOB functions has been extended for the use of fractional derivatives. An appropriate diagram is thus proposed for simulation. 

Keywords: simulation, orthogonal functions, generalised orthogonal basis, fractional calculus, fractional derivative, dynamic system, identification.

## 1. INTRODUCTION AND MATHEMATICAL BACKGROUND

### *1.1 Context and motivation*

Recently, we proposed an interpolation of Laguerre functions to fractional differentiation orders which keeps them convergent when differentiation orders are non-integer (Aoun *et al.*, 2003*b*). We then generalised the use of fractional orthogonal bases to any number of poles: real or pair complex conjugate. This interpolates the well-known definition of the generalised orthogonal basis to fractional derivatives called FraGOB (Malti *et al.*, 2004).

A major difficulty with fractional models, and therefore fractional bases, is their time-domain simulation. Often, the analytical solution of a model's output is not simple to compute. During the last 20 years numerical algorithms have been developed using either continuous or discrete-time rational models approximating fractional systems: (Oustaloup, 1995; Chen *et al.*, 2003; Aoun *et al.*, 2003*a*). Here, after a reminder of the orthogonalization procedure, we propose a new simulation diagram for evaluating the output of fractional bases.

### *1.2 Representation of fractional systems*

Fractional mathematical models are based on fractional differential equations:

$$y(t) + b_1\mathbf{D}^{\beta_1}y(t) + \cdots + b_{m_B}\mathbf{D}^{\beta_{m_B}}y(t) = a_0\mathbf{D}^{\alpha_0}u(t) + a_1\mathbf{D}^{\alpha_1}u(t) + \cdots + a_{m_A}\mathbf{D}^{\alpha_{m_A}}u(t) \quad (1)$$

where differentiation orders $\beta_1, \ldots, \beta_{m_B}, \alpha_0, \ldots, \alpha_{m_A}$ are allowed to be non-integer positive numbers. The concept of differentiation to an arbitrary order (non-integer),

$$\mathbf{D}^\gamma \triangleq \left(\frac{d}{dt}\right)^\gamma \qquad \forall\gamma \in R_+$$

was defined in the $19^{th}$ century by Riemann and Liouville. They extend differentiation by using not only integer but also non-integer (real or complex) orders. The $\gamma$ fractional order derivative of $x(t)$ is defined as being an integer derivative of order

$m = \lfloor \gamma \rfloor + 1$ ( $\lfloor . \rfloor$ stands for the floor operator) of a non-integer integral of order $1-(m-\gamma)$ (Samko *et al.*, 1993):

$$\mathbf{D}^{\gamma} x(t) \triangleq \frac{1}{\Gamma(m-\gamma)} \left(\frac{d}{dt}\right)^{m} \int_{0}^{t} \frac{x(\tau)\, d\tau}{(t-\tau)^{1-(m-\gamma)}} \tag{2}$$

where $t > 0$, $\gamma > 0$.

A more concise algebraic tool can be used to represent such fractional systems: the Laplace transform. The Laplace transform of a $\gamma$ order derivative ($\gamma \in \mathbb{R}_+$) of a signal $x(t)$ relaxed at $t = 0$ (i.e. all derivatives of $x(t)$ equal 0 when $t < 0$) is given by (Oldham and Spanier, 1974):

$$\mathbf{L}\{\mathbf{D}^{\gamma} x(t)\} = s^{\gamma} X(s)$$

This property permits the fractional differential equation (1), provided all signals $u(t)$ and $y(t)$ are relaxed at $t = 0$, in a transfer function form:

$$F(s) = \frac{\sum_{i=0}^{m_A} a_i s^{\alpha_i}}{1 + \sum_{j=1}^{m_B} b_j s^{\beta_j}} \tag{3}$$

where $(a_i, b_j) \in \mathbb{C}^2$, $(\alpha_i, \beta_j) \in \mathbb{R}_+^2$, $\forall i = 0, 1, \ldots, m_A$, $\forall j = 1, 2, \ldots, m_B$.

*Definition 1.* A transfer function $F(s)$ is commensurate of order $\gamma$ iff it can be written as $F(s) = S(s^{\gamma})$, where $S = \frac{T}{R}$ is a rational function with $T$ and $R$, two coprime polynomials.

All differentiation orders are multiples of the commensurate order, and lead to a rational transfer function. Here, the commensurate order is left free to vary in $\mathbb{R}_+^*$. Taking as an example $F(s)$ defined in (3), assuming that $F(s)$ is commensurate of order $\gamma$, and using $F(s) = S(s^{\gamma})$, one can write:

$$S(s) = \frac{T(s)}{R(s)} = \frac{\sum_{m=0}^{m_A} a_m s^{\frac{\alpha_m}{\gamma}}}{1 + \sum_{m=1}^{m_B} b_m s^{\frac{\beta_m}{\gamma}}} \tag{4}$$

All powers of $s$ in (4) are integers. A sufficient condition for $F(s)$ to be commensurate is that all differentiation (or integration) orders belong to the set of rational numbers $\mathbb{Q}$. It covers a wide range of fractional systems.

### *1.3 Stability condition*

Matignon (1998, theorem 2.21 p.150) has established the stability condition of any commensurate explicit fractional model. However, here is a revisited version of his theorem :

*Theorem 2.* A commensurate (of order $\gamma$) transfer function $F(s) = S(s^{\gamma}) = \frac{T(s^{\gamma})}{R(s^{\gamma})}$ is BIBO stable iff

$$0 < \gamma < 2 \tag{5}$$

and for every $s \in \mathbb{C}$ such that $R(s) = 0$

$$|\arg(s)| > \gamma \frac{\pi}{2} \tag{6}$$

□

### *1.4 Fractional transfer functions belonging to $H_2(\mathbb{C}^+)$*

Contrary to rational systems, the stability condition does not guarantee that a fractional transfer function belongs to $H_2(\mathbb{C}^+)$. The $H_2$ norm of fractional systems was extensively studied in (Malti *et al.*, 2003), where it was proven that a fractional transfer function as defined in (3) belongs to $H_2(\mathbb{C}^+)$ iff stability conditions (5) and (6) are satisfied and the difference between numerator and denominator degrees satisfy:

$$\beta_{m_B} - \alpha_{m_A} > \frac{1}{2} \tag{7}$$

Condition (7) will be necessary when choosing fractional generating functions for the orthogonal bases to be synthesised.

### *1.5 Scalar product, orthogonality and rational orthogonal functions*

The classical Laguerre, Kautz, and GOB functions form a complete orthonormal basis in $L_2[0, \infty[$, according to the usual definition of the scalar product (Szego, 1975):

$$\langle l_n(t), l_m(t) \rangle = \int_{0}^{\infty} l_n(t)\, l_m(t)\, dt = \delta_{nm} \tag{8}$$

whose reciprocal in the frequency domain is obtained by Plancherel's theorem:

$$\langle L_n(s), L_m(s) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} L_n(j\omega)\, \overline{L_m(j\omega)} d\omega = \delta_{nm} \tag{9}$$

Any function $f(t) \in L_2[0, \infty[$, thus satisfying:

$$\langle f(t), f(t) \rangle^{\frac{1}{2}} = \|f\|_2 < \infty \tag{10}$$

can be written as a linear combination of these functions:

$$F(s) = \sum_{n=0}^{\infty} a_n L_n(s) \tag{11}$$

$F(s)$ is the Laplace transform of $f(t)$. Usually, (11) is truncated to a given order $N$ which is justified by the fact that Fourier coefficients are convergent as $n$ tends to infinity. $F(s)$ is hence approximated by the finite sum:

$$F(s) \approx F_N(s) = \sum_{n=0}^{N} \theta_n L_n(s) \tag{12}$$

## 2. CONSTRUCTION OF ORTHONORMAL BASES

### 2.1 Gram-Schmidt principle

Given an arbitrary set of functions $\{F_m\}_{m=1}^M$, where $F_m \in H^2(\mathbb{C}^+)$ $\forall m$, orthonormal functions $\{G_m\}_{m=1}^M$ are obtained, according to the Gram-Schmidt orthogonalisation principle, as a linear combination of generating functions $F_m$, $m = 1...M$:

$$\mathbf{G} = \Delta * \mathbf{F} \qquad (13)$$

where $\Delta$ is a real-value $M$x$M$ matrix,

$$\mathbf{G} = \left[\, G_1(s)\ G_2(s)\ \cdots\ G_{M-1}(s)\ G_M(s) \,\right]^T$$

and

$$\mathbf{F} = \left[\, F_1(s)\ F_2(s)\ \cdots\ F_{M-1}(s)\ F_M(s) \,\right]^T$$

Since $\{G_m\}_{m=1}^M$ is the set of orthonormal functions

$$\left\langle \mathbf{G}, \mathbf{G}^T \right\rangle = \mathbf{I} \qquad (14)$$

$\mathbf{I}$ denotes an $M$ by $M$ identity matrix. Thus, using (13):

$$\left\langle \mathbf{G}, \mathbf{G}^T \right\rangle = \Delta \left\langle \mathbf{F}, \mathbf{F}^T \right\rangle \Delta^T = I \qquad (15)$$

Then, it is easy to check that

$$\Delta^T \Delta = \left\langle \mathbf{F}, \mathbf{F}^T \right\rangle^{-1}$$

From this quadratic form, $\Delta$, a lower triangular matrix, is computed using the Cholesky decomposition.

$$\Delta = Cholesky\left( \left\langle \mathbf{F}, \mathbf{F}^T \right\rangle^{-1} \right) \qquad (16)$$

Using (13), functions of the orthonormal set are given by:

$$\mathbf{G} = Cholesky\left( \left\langle \mathbf{F}, \mathbf{F}^T \right\rangle^{-1} \right) \times \mathbf{F} \qquad (17)$$

The remaining difficulty is to compute the matrix of scalar products $\left\langle \mathbf{F}, \mathbf{F}^T \right\rangle$ for fractional transfer functions which are known to be multi-value complex functions as soon as non-integer differentiation is involved. Hence, a plane cut is necessary in the complex $s$-plane. A procedure for computing the scalar product of any fractional explicit transfer function is described in (Aoun *et al.*, 2004; Malti *et al.*, 2004).

### 2.2 Fractional Generating functions

The construction of the orthogonal basis starts by choosing a set of generating functions which must not be colinear in the sense of the definition (9) of the scalar product. Each generating function can introduce either a real mode or a complex one. If a generating function introduces a complex mode, then the next generating function must introduce its conjugate so that the impulse response remains a real signal. Once these functions chosen, (17) is applied so that the orthogonal $G_m$ functions are obtained.

*2.2.1. Fractional Laguerre-like generating functions* If a real mode is to be included in the first FraGOB, a generating function is chosen as:

$$F_{m_0}(s) = \frac{1}{(s^\gamma + \lambda_{m_0})^{m_0}} \qquad (18)$$

where

$$m_0 = \left\lfloor \frac{1}{2\gamma} \right\rfloor + 1 \qquad (19)$$

and

$$\lambda_{m_0} \in \mathbb{R}_+^*, \qquad \gamma \in ]0, 2[ \qquad (20)$$

All other generating functions to be included with real value modes are chosen as:

$$F_m(s) = F_{m-1}(s) \frac{1}{(s^\gamma + \lambda_m)} \qquad (21)$$

where

$$\lambda_m \in \mathbb{R}_+^*, \quad \forall m \in \mathbb{N}, \quad m \geq m_0, \quad \gamma \in ]0, 2[ \qquad (22)$$

Conditions announced in (20 and 22) stem from stability theorem 2 and the fact that impulse responses of (18 and 21) are real-value signals.

Condition (19) stems from the fact that each generating function must belong to $H^2(\mathbb{C}^+)$. Hence, applying (7) for the generating function (18) shows that:

$$\gamma m_0 > \frac{1}{2} \qquad (23)$$

Keeping in mind that $m_0$ is integer yields (19).

It is interesting to note that, in the special case where all the $\lambda_m$ modes are chosen to be alike, the multiplicity of the $\lambda_m$ mode is incremented in (21) for every new function, in which case the set of fractional Laguerre generating functions is obtained as illustrated in (Aoun *et al.*, 2003*b*).

*2.2.2. Fractional Kautz-like generating functions* Suppose that $(n+1)$ $\lambda_{m_0}, ..., \lambda_{m_0+n}$ modes have been included in $F_{m_0}, F_{m_0+1}, ..., F_{m_0+n}$ and now we wish to include a complex mode $\lambda_{m_0+n+1}$. Then, a conjugate mode must follow ($\lambda_{m_0+n+2} = \overline{\lambda_{m_0+n+1}}$) to obtain a real impulse response. Also, both basis functions $F_{m_0+n+1}$ and $F_{m_0+n+2}$ are replaced by two new basis functions $F'_{m_0+n+1}$ and $F''_{m_0+n+1}$ which have real impulse responses and which are a linear combination of $\frac{1}{(s^\gamma + \lambda_r)^r}$ and $\frac{1}{(s^\gamma + \overline{\lambda_r})^r}$, where $r = m_0 + n + 1$.

The linear combination we are suggesting can be expressed as:

$$\begin{bmatrix} F'_r \\ F''_r \end{bmatrix} = \begin{bmatrix} c_0 & c_1 \\ c'_0 & c'_1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{(s^\gamma + \lambda_r)^r} \\ \dfrac{1}{(s^\gamma + \overline{\lambda_r})^r} \end{bmatrix} F_{r-1} \qquad (24)$$

where $c_0, c_1, c_0', c_1' \in \mathbb{C}$.
As Gram-Schmidt orthogonalisation procedure follows, the only constraint while choosing $c_0, c_1, c_0'$ and $c_1'$ is that both functions $F_r'$ and $F_r''$ have real-value impulse responses which gives the following conditions:

$$c_0 = \overline{c_1} \quad \text{and} \quad c_0' = \overline{c_1'}$$

Four more degrees of freedom are left while choosing the real and imaginary parts of $c_0, c_1, c_0'$ and $c_1'$. Therefore, as a result any of the following transfer functions can be chosen as generating functions of the basis:

$$F_r'(s) = \frac{(\beta s^\gamma + \mu)}{s^{2\gamma} + (\lambda_r + \overline{\lambda_r})s^\gamma + \lambda_r \overline{\lambda_r}} F_{r-1} \qquad (25)$$

$$F_r''(s) = \frac{(\beta' s^\gamma + \mu')}{s^{2\gamma} + \left(\lambda_r + \overline{\lambda_r}\right) s^\gamma + \lambda_r \overline{\lambda_r}} F_{r-1} \qquad (26)$$

where

$$|arg(\lambda_r)| > \gamma \frac{\pi}{2}, \qquad \gamma \in ]0, 2[, \qquad (27)$$

and $(\beta, \mu) \neq d(\beta', \mu'), \forall d \in \mathbb{C}$ and $(\beta, \mu, \beta', \mu') \in \mathbb{R}^4$. Parameters $(\beta, \mu, \beta', \mu')$ can be chosen arbitrarily with the following constraint $(\beta, \mu) \neq d(\beta', \mu'), \forall d \in \mathbb{C}$ i.e. $F_r'(s)$ and $F_r''(s)$ must not be co-linear according to definition (9) of the scalar product. One should, however, keep in mind that an infinite pair of functions can span a plane. Hence, two different choices of $(\beta, \mu, \beta', \mu')$ may lead to different pairs of orthogonal functions $F_r'(s)$ and $F_r''(s)$.

When $r = 1$, the two first functions are :

$$F_1'(s) = \left( \frac{(\beta s^\gamma + \mu)}{s^{2\gamma} + (\lambda_1 + \overline{\lambda_1})s^\gamma + \lambda_1 \overline{\lambda_1}} \right)^{m_0} \qquad (28)$$

$$F_2''(s) = \left( \frac{(\beta' s^\gamma + \mu')}{s^{2\gamma} + \left(\lambda_1 + \overline{\lambda_1}\right) s^\gamma + \lambda_1 \overline{\lambda_1}} \right)^{m_0} \qquad (29)$$

where $m_0$ is such that $F_1'$ and $F_2''$ belong to $H^2(\mathbb{C}^+)$ and $m_0$ is given by (19).

In the special case where all complex conjugate modes $(\lambda_r, \overline{\lambda_r})$ are chosen to be alike, the set of fractional Kautz-like bases is synthesised.

**Remark** *Completeness of the FraGOB is yet to be proven. However, the completeness of the fractional Laguerre basis is proved in (Malti* et al.*, 2004). Consequently, when conditions (22) are satisfied and all the poles are chosen to be alike:*

$$\lambda_{m_0} = \lambda_{m_0+1} = \lambda_{m_0+2} = \ldots = \lambda_\infty, \qquad (30)$$

*and* $m_0 = \lfloor \frac{1}{2\gamma} \rfloor + 1$*, the fractional Laguerre basis is dense in* $H^2(\mathbb{C}^+)$*. Therefore, it can be used to model any finite energy fractional system.*

## 3. NUMERICAL SIMULATION OF FRACTIONAL GENERALISED ORTHONORMAL BASES

Since the filters of the fractional bases are irrational transfer functions, some simulation methods are now introduced for this class of transfer function.

### *3.1 Simulation of irrational transfer functions*

Simulation of fractional systems is complicated due to their long memory behavior as shown by Oustaloup (1995). Many methods have been developed (Lin, 2001; Aoun *et al.*, 2003*a*; Chen *et al.*, 2003). They are mainly based on the approximation of a fractional model by either a rational discrete-time or a rational continuous-time model. Here, we mention some methods based on discrete-time models. The reader can find however a more detailed presentation in (Aoun *et al.*, 2003*a*).

In these methods, the fractional differentiator $s^\gamma$ is substituted by its discrete-time equivalent.

$$s^\gamma \rightarrow \psi\left(z^{-1}\right) \qquad (31)$$

As a result, a discrete-time transfer function is obtained:

$$\frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s^\gamma + b_2 s^{2\gamma} + \ldots + b_{n_B} s^{n_B \gamma}}{a_0 + a_1 s^\gamma + a_2 s^{2\gamma} + \ldots + a_{n_A} s^{n_A \gamma}}, \qquad (32)$$

$$\equiv \frac{Y\left(z^{-1}\right)}{U\left(z^{-1}\right)}, \qquad (33)$$

$$\equiv \frac{b_0 + b_1 \psi\left(z^{-1}\right) + \ldots + b_{n_B}\left(\psi\left(z^{-1}\right)\right)^{n_B}}{a_0 + a_1 \psi\left(z^{-1}\right) + \ldots + a_{n_A}\left(\psi\left(z^{-1}\right)\right)^{n_A}}. \qquad (34)$$

$\psi\left(z^{-1}\right)$, the discrete mapping of the Laplace operator $s$, can be computed using various approximation methods. The most common are Euler's, Tustin's, Simpson's, or Al-Alaoui's (Al-Alaoui, 1994; Vinagre *et al.*, 2000; Oustaloup, 1995). These analogue-to-digital open-loop design methods lead to irrational z-transforms which are then approximated either by a truncated Taylor series expansion or a continuous fraction expansion. The obtained digital model can then be simulated using a classical implementation structure: direct, parallel, cascade, lattice, ...
***Euler:***

$$\begin{aligned}\psi\left(z^{-1}\right) &= \left(\frac{1 - z^{-1}}{Ts}\right)^\gamma \\ &= \left(\frac{1}{Ts}\right)^\gamma \left(1 - \gamma z^{-1} + \frac{\gamma(\gamma-1)}{2} z^{-2} + ...\right) \\ &= \left(\frac{1}{Ts}\right)^\gamma \sum_{k=0}^{\infty} \left((-1)^k \tbinom{\gamma}{k} z^{-k}\right) \qquad (35)\end{aligned}$$

***Tustin:***

$$\psi\left(z^{-1}\right) = \left(\frac{2}{Ts}\right)^{\gamma}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^{\gamma}$$
$$= \left(\frac{2}{Ts}\right)^{\gamma}\left(1-2\gamma z^{-1}+2\gamma^{2}(\gamma-1)z^{-2}+...\right) \tag{36}$$

***Simpson:***

$$\psi\left(z^{-1}\right) = \left(\frac{3}{Ts}\right)^{\gamma}\left(\frac{\left(1-z^{-1}\right)\left(1+z^{-1}\right)}{1+4z^{-1}+z^{-2}}\right)^{\gamma}$$
$$= \left(\frac{3}{Ts}\right)^{\gamma}\left(1-4\gamma z^{-1}+2\gamma(4\gamma+3)z^{-2}+...\right) \tag{37}$$

***Al-Alaoui:***

$$\psi\left(z^{-1}\right) = \left(\frac{8}{7Ts}\right)^{\gamma}\left(\frac{1-z^{-1}}{1+\frac{z^{-1}}{7}}\right)^{\gamma}$$
$$= \left(\frac{8}{7Ts}\right)^{\gamma}\left(1-\frac{8\gamma}{7}z^{-1}+\frac{-24\gamma+32\gamma^{2}}{49}z^{-2}+...\right)$$

### *3.2 Simulation Diagram of FraGOB*

Let $F$ be a dynamic system approximated using a FraGOB :

$$F(s) = \sum_{m=m_0}^{M}\theta_m G_m(s) \tag{38}$$

The time response of $F$ can be evaluated by simulating directly the $G_m$ filters. The main drawback of this method is that each $G_m$ is simulated separately. Given the complexity of the $G_m$ expression, specially when $m$ increases, simulation is slow and hard to achieve.

To avoid such problems, since $G_m$ orthonormal filters are linear combinations of $F_m$ generating functions, and since each generating function $F_m$ is a product of $F_{m-1}$ by a fractional mode, one can use simulation diagram (1). The coefficients $\delta_{i,j}$ are elements of the matrix which stems from the Cholesky decomposition (16).

## 4. EXAMPLE

Let $\gamma = 0.8$ be the fractional order and all eigenvalues of the basis (2, $2e^{\pm j\frac{\pi}{3}}$, 1 and 0.5).

Since $\gamma = 0.8$, the index of the first generating basis is $m_0 = \lfloor\frac{1}{2\times 0.8}\rfloor + 1 = 1$. Then, by applying (18), $F_1$ is given by :

$$F_1(s) = \frac{1}{s^{0.8}+2}$$

The second and the third functions include two complexe conjugate modes. Then, $F_3$ and $F_4$ are obtained with (28) and (29). In this example, we fixed arbitrarily $\beta = \mu' = 0$ and $\beta' = \mu = 1$ so that the two functions are not co-linear.

$$F_2(s) = \frac{F_1(s)}{s^{1.6}+0.5s^{0.8}+4}$$
$$F_3(s) = \frac{s^{0.8}F_1(s)}{s^{1.6}+0.5s^{0.8}+4}$$

The forth and the fifth generating functions introduce respectively the two reals modes 1 and 0.5. $F_4$ and $F_5$ are then given by (21).

$$F_4(s) = \frac{F_3(s)}{s^{0.8}+1} \quad \text{and} \quad F_5(s) = \frac{s^{0.8}F_4(s)}{s^{0.8}+0.5}$$

The scalar product matrix $\left\langle \mathbf{F}, \mathbf{F^T}\right\rangle$ is computed using the algorithm developed in (Aoun *et al.*, 2004; Malti *et al.*, 2004):

$$\left\langle \mathbf{F}, \mathbf{F^T}\right\rangle = \begin{bmatrix} 341.53 & 19.58 & 43.57 & 12.40 & 5.20 \\ 19.58 & 6.32 & 2.11 & 2.78 & 2.12 \\ 43.58 & 2.11 & 12.27 & 3.28 & 0.16 \\ 12.40 & 2.78 & 3.28 & 2.19 & 0.78 \\ 5.20 & 2.12 & 0.16 & 0.78 & 1.01 \end{bmatrix} 10^{-3}. \tag{39}$$

The matrix $\mathbf{\Delta}$ is then obtained according to (16):

$$\mathbf{\Delta} = \begin{bmatrix} 1.71 & 0 & 0 & 0 & 0 \\ 0.79 & -13.87 & 0 & 0 & 0 \\ -1.61 & 0.91 & 12.23 & 0 & 0 \\ -1.16 & 21.10 & 13.94 & -50.29 & 0 \\ -0.03 & -23.14 & 1.96 & 5.06 & 60.65 \end{bmatrix}. \tag{40}$$

The vectors of the orthonormal basis are computed by applying formula (17).

$$G_1(s) = \frac{1.71}{s^{0.8}+2}$$
$$G_2(s) = \frac{0.79s^{1.6}+1.59s^{0.8}-10.69}{\left(s^{0.8}+2\right)\left(s^{1.6}+0.5s^{0.8}+4\right)}$$
$$G_3(s) = \frac{-1.61s^{1.6}+9.01s^{0.8}-5.54}{\left(s^{0.8}+2\right)\left(s^{1.6}+0.5s^{0.8}+4\right)}$$
$$G_4(s) = \frac{-1.16s^{2.4}+10.45s^{1.6}-22.23s^{0.8}+16.45}{\left(s^{0.8}+1\right)\left(s^{0.8}+2\right)\left(s^{1.6}+0.5s^{0.8}+4\right)}$$
$$G_5(s) = \frac{-0.03s^{3.2}+1.85s^{2.4}-15.37s^{1.6}+29.24s^{0.8}-11.63}{(s^{0.8}+0.5)(s^{0.8}+1)\left(s^{0.8}+2\right)\left(s^{1.6}+0.5s^{0.8}+4\right)}$$

The diagram of figure (1) is used to simulate the step response of the five orthonogonal functions between 0 and $T_f = 30s$ with the sampling period $T_s = 0.1s$. The Laplace variable is substituted with Euler's approximation (35). As the input signal is null for negative time, the series in (35) is truncated to the number of input samples $\frac{T_f}{T_s} = 300$. The step responses of the orthogonal functions are plotted on figure 2.

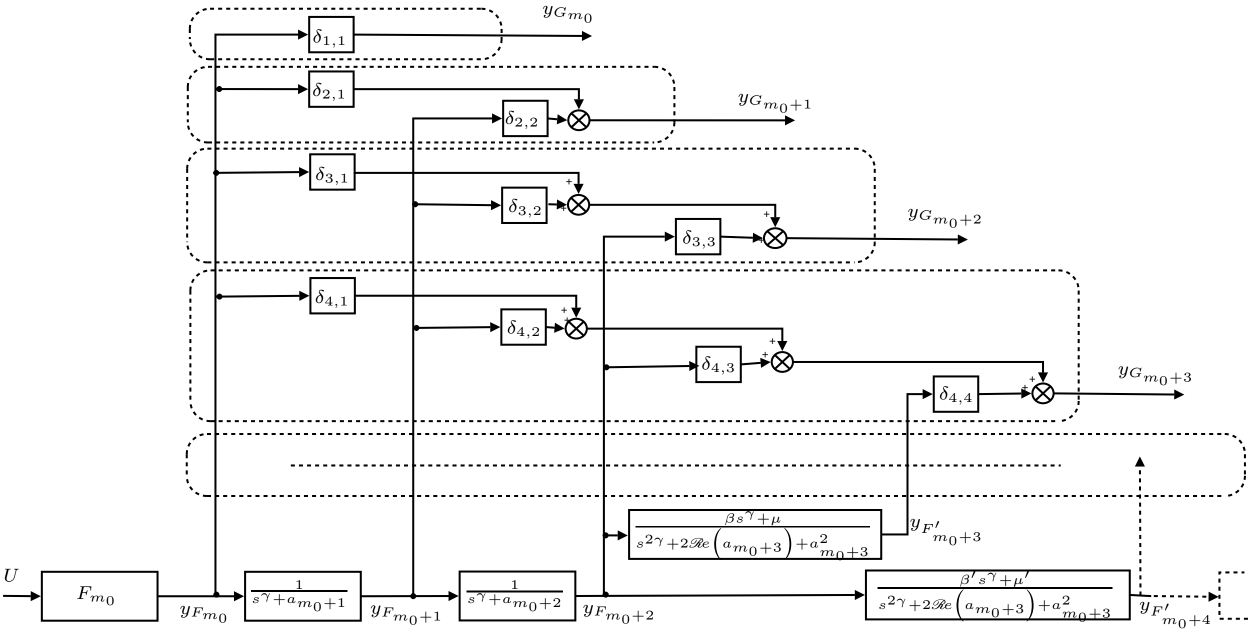

Fig. 1. Simulation diagram of FraGOB

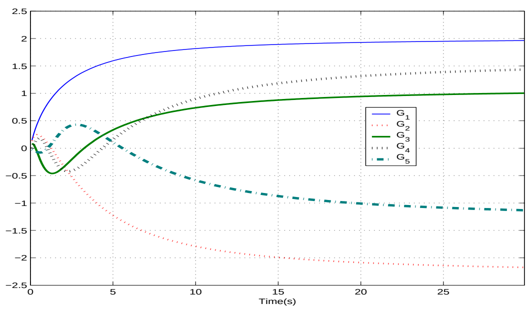

Fig. 2. Step responses of $G_1$, $G_2$, $G_3$, $G_4$ and $G_5$

## 5. CONCLUSION

A new simulation diagram of fractional systems approximated on continuous-time FraGOB (Fractional Generalised Orthogonal basis) is presented. Numerical simulation of the elementary transfer functions of the diagram uses classical approximations such as Euler, Tustin, Simpson, and Al-Alaoui.

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