# PRODUCTION PLANNING UNDER STOCHASTIC TIME-VARYING DEMAND: FLEXIBLE SERVICE LEVEL APPROACH

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Abstract: Consider stochastic production-inventory control problem in which there are multiple capacity sources, multiple heterogeneous products, non-stationary demand and time-varying costs. If shortage[2] costs can be evaluated the natural objective is to minimize expected production, inventory and shortage costs. Otherwise one can use service level requirements as target inventory while trying to minimize costs. In this paper we present the algorithm for solving optimal production plan with service level requirements. The cost for the deviation from target value can be any differentiable and convex function. However in this paper we study the problem in which deviation cost is related to underlying demand distribution. 

Keywords: Inventory control, production planning, dynamic programming, stochastic control, supply chain management

## 1. INTRODUCTION

This paper concentrates on short term/tactical level planning problems where planning horizons vary between day and few weeks. Planning horizon however depends heavily on industry area and what might be short term planning for some areas is tactical level planning for some others. Aim of those methods considered here is to make decision what products, how much and when we should produce.

Existing capacity constrained production-inventory control problems can be roughly divided in two classes. One is cost minimization where objective is to minimize expected production, holding and shortage costs over planning horizon. This produces optimal solution in terms of costs. However there is one important parameter, shortage cost which one has to approximate.

The other way to formulate the problem is to use target inventories to guarantee reliable supplies. The user sets desired service level as target inventory and minimizes deviation from target inventory. There are two well studied ways to formulate target inventory problem. One uses hard service level constraints while minimizing expected holding cost and production cost. This approach guarantees reliable supplies in every period but there is major drawback in this approach. The possibility that production costs increase significantly is high. The other well known way uses quadratic penalty cost for the deviation from target value.

Most closely related to our work at production - inventory control problem literature are (Bitran and Yanasse, 1984), (Martel *et al.*, 1995), (Sox and Muckstadt, 1996) and (Holt *et al.*, 1960). Bitran and Yanasse (1984) developed a heuristics for multi-period production planning problem with service-

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[2] Shortage costs are caused by late deliveries.

level constraints. All unmet demand is backlogged. Their heuristics are based on deterministic approximation and provide results which have small relative error with high service levels. They consider production, overtime and inventory holding costs. Martel et al. (1995) consider both inventory holding and shortage costs. They use piecewise linear cost approximations and cumulative production quantities over planning horizon. They are therefore able to formulate problem as a static linear programming problem or a mixed integer linear programming problem if set-up costs are included. Sox and Muckstadt (1996) use Lagrangian relaxation to develop a sub-gradient optimization algorithm. Their approach is similar to Martel et al. (1995) in a way that they also use cumulative inventory and production quantities over planning horizon. All demand not met is also backlogged in their study. Sox and Muckstadt (1996) formulation allows non-stationary demand, time-varying costs and multiple production sources with convex costs. Holt et al. (1960) are the first one who presented production - inventory control model with quadratic penalty from target inventory value. The other studies from the rich production-inventory control literature related to our work are (Bookbinder and H'ng, 1986), (Mehrez and Ben-Arieh, 1991), (Ramsay and Rardin, 1983), (Garcia and Smith, 2000) and (Ciarallo *et al.*, 1994).

The paper is organized as follows. In chapter 2. we present problem formulation. Solution algorithm is covered in chapter 3. and computational results in chapter 4.

## 2. PROBLEM FORMULATION

Our model is finite-horizon, discrete-time, production and inventory planning model with multiple heterogeneous products. The model assumes time-varying stochastic non-stationary demand with known demand distributions. It also assumes deterministic time-varying costs, multiple production and buying sources with convex costs. All unmet demand is backlogged. Although our model can handle multiple resources we present our model here as a single "bottleneck" resource model. Production and inventory are modelled as cumulative quantities over planning horizon. Model does not contain set-up time or costs.

Table 1. Variables

| | |
|---|---|
| $X_{it}$ | the cumulative inventory of item $i$ through period $t$, |
| $u_{it}$ | the production amount for product $i$ in period $t$, |
| $y_{it}$ | the rate of capacity usage for item $i$ |
| $C_{1t}\left(\sum_{i=1}^{N} y_{it}u_{it}\right)$ | the cost of resource 1 in period $t$, |
| $h_{it}$ | the unit holding costs for product $i$ in period $t$, |
| $p_{it}$ | the unit shortage costs for product $i$ in period $t$, |
| $ul_{1t}$ | the upper limit for resource 1 in period $t$, |
| $Sl_{it}$ | service level for product $i$ in period $t$ expressed as $0.$ decimal number, f.g. 99% service level would be $0.99$, |
| $F_{it}(x)$ | the c.d.f. of cumulative demand for product $i$ through period $t$, |
| $\bar{F}_{it}(x)$ | $1-F_{it}$ and |
| $f_{it}(x)$ | the p.d.f. of cumulative demand for product $i$ through period $t$. |

We formulate our multi-period production planning problem as follows:

$$\min \sum_{t=1}^{T}\left[\sum_{i=1}^{N}\left[H_{it}\left(X_{it}\right)\right]+C_{1t}\left(\sum_{i=1}^{N} y_{it}u_{it}\right)\right] \qquad (1)$$

Subject to:

$$\sum_{i=1}^{N} X_{iT} = \min\left(\sum_{i=1}^{N} F_{iT}^{-1}\left(Sl_{iT}\right), \sum_{t=1}^{T} ul_{1t}\right)$$

$$0 \le \sum_{i=1}^{N}\left[y_i u_{it}\right] \le ul_{1t} \qquad \forall\, t$$

$$X_{i,t-1} \le X_{it}, \qquad \forall\, i \text{ and } t$$

$$y_i \ge 0 \quad \wedge \quad u_{it} \ge 0$$

The expected holding and shortage costs function for product $i$ in period $t$ is:

$$H_{it}\left(X_{it}\right) = h_{it}\int_{0}^{X_{it}}\left(X_{it}-x\right) f_{it}(x)\,dx + \ldots$$
$$p_{it}\int_{X_{it}}^{\infty}\left(x-X_{it}\right) f_{it}(x)\,dx \qquad (2)$$

The optimal expected holding and shortage costs for product $i$ in period $t$ are used as a reference costs. The objective is to keep inventory levels as close as possible to desired service levels while minimizing the production costs over planning horizon. This formulation allows inventory levels to be under desired service level in some period while still maintaining service level requirements over planning horizon. The production costs $C_{1t}$ in period $t$ can be any convex function. An example of different capacity sources are normal working hours and overtime hours.

Cumulative inventory levels at the end of the planning horizon are required to be at certain level. Therefore it is not necessary to define salvage value for inventory left over after planning horizon.

## 3. SOLUTION ALGORITHM

The algorithm assumes that initial inventory levels $X_{i0}$ are known. Solution can be seen as an optimal open-loop controller (OLC) see f.g. Bertsekas (2000). Optimization is done as a static problem instead of solving computationally intractable dynamic programming problem. Optimal closed-loop controller has better or at least as good performance as open-loop controller has because it calculates optimal control for every possible state over the planning horizon. When OLC is used in rolling horizon basis control strategy is referred as open-loop feedback controller (OLFC) f.g. Bertsekas (2000). The performance of OLFC is at least as good as optimal OLC.

Our approach does not require the user to define shortage costs. However it is necessary to define desired service level for every product $i$ in every period $t$. The approximation of shortage costs is then calculated by using Newsvendor solution see f.g. Winston (1994).

$$\begin{aligned} \frac{p_{it}}{p_{it}+h_{it}} &= Sl_{it} \\ &\vdots \\ p_{it} &= \frac{Sl_{it}*h_{it}}{1-Sl_{it}} \end{aligned} \qquad (3)$$

The dependence between $p$ and $Sl$ has always the same form. One however has to consider the strong nonlinear nature of dependence. When operating with service level values over 90% the increase in relative importance of $p$ is significant.

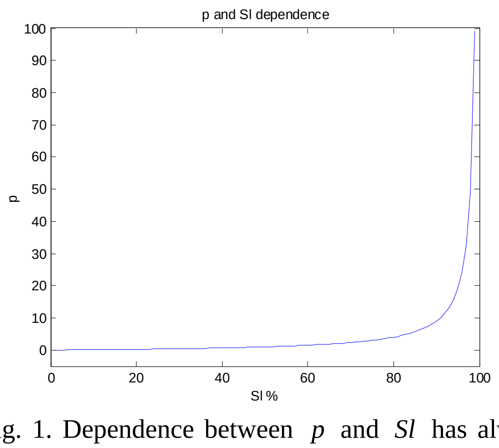


Fig. 1. Dependence between $p$ and $Sl$ has always the same form which does not depend on $h$ (in this case $h=1$).

The cumulative inventory quantity $X_{it}$ for product $i$ and period $t$ is calculated as follows:

$$X_{it} = \sum_{t=1}^{t}\left(u_{it}\right) + X_{i0} \qquad (4)$$

Demand is also modelled as cumulative quantities over planning horizon. Demand for product $i$ in period $t$ is a sum distribution of demand through periods $1,2,...,t$. Dependence in subsequent periods can have any form.

Production cost gradient for product $i$ in period $t$ is

$$\frac{\partial C_{1t}}{\partial u_{it}} = y_{it} C\left(\sum_{i=1}^{N} y_{it}u_{it}\right) \qquad (5)$$

The expected holding and shortage cost gradient is total differential and calculated as follows.

$$\frac{\partial H_{it}\left(X_{it}\right)}{\partial X_{it}} = \sum_{t=t}^{T}\left[h_{it} * F_{it}\left(X_{it}\right) - p_{it}\bar{F}\left(X_{it}\right)\right] \qquad (6)$$

Production in some earlier period will affect cumulative inventory in later periods. When calculating holding and shortage cost gradients one has to take this fact into account. Period $t$ gradient is a combination of period $t$ gradient plus sum of the all later period gradients. This calculation will guarantee the expected holding and shortage cost gradient to be improving direction.

The solution algorithm is based on idea of reasonable initial values, calculation of optimal production quantities separately for every product and calculation of optimal combined production.

The solution algorithm:

*Step 1.* Calculate initial values for every product $i$ in every period $t$, $u_{it} = F_{it}^{-1}\left(Sl_{it}\right) - X_{i,t-1}$.

*Step 2.* Start from product $1,2,...,N$,

1. Set production for other products to zero.
2. Calculate derivatives for all periods.
3. Calculate differences between period $i$ derivative and periods $1,2,...,T$ derivatives.
4. Find maximum difference between any periods. For chosen pair of periods move production from one period to another according to the way difference is calculated.
5. Update derivatives.
6. Continue until maximum difference between derivatives is smaller than some chosen value $\varepsilon$.

*Step 3.*

1. Combine all products and start similar procedure as in *Step 2.* In multi-product case only one product at a time is movable.

The objective of the initial solution is to speed up the optimization procedure. If resources have equal costs and demand is not very volatile the initial solution is close to optimum. Single product optimum solution is required to guarantee optimality when costs and resources vary between periods. In multi-product phase we are interested in smoothing out difference in resource usage between periods. Single product solution is used to simplify multi-product phase calculation. Without single product solution it would be time consuming to calculate not only derivatives for all the control variables but also their effect on other variables derivatives. In this algorithm it is relatively easy to calculate the cost for moving production from one period to another. All existing capacity constraints are replaced with penalty costs see f.g. Rardin (1998). In a case where initial solution exceeds total capacity available in planning horizon it is only necessary to reduce production from products which has the least worsening impact.

## 4. COMPUTATIONAL RESULTS

The performance of the algorithm is illustrated with two examples. Data contains demand distributions, service level requirements, production costs, holding costs, resource consumption and available capacity from two different sources. We first test the performance of the algorithm when there are two low capacity periods (first capacity source) in the middle of the planning horizon. In second test first capacity source is linearly increasing. Capacity from second source is unlimited in both cases.

To simplify the analysis we set for every period $t$ and every product $i$ the capacity consumption for producing one unit to be 1 capacity unit. The production costs from capacity source 1 is always 10$/unit and from capacity source 2 30$/unit. Planning horizon is 10 periods and there are 5 products to produce. The holding costs are randomly selected from uniform distribution. For products $1,2,\ldots,5$ they are $[1.44, 2.62, 3.79, 4.74$ and $5.18]$ \$.

We assume the demand to be normally distributed with mean 500. Standard deviation is selected so that probability of negative demand occurring is very low. We also assume that it is not possible to substitute product's $i$ demand with other products. Demand for consecutive periods is assumed to be independent from each other. Standard deviations are from uniform distribution in range $[10,150]$.

Table 3. Standard deviations(Std) for cumulative demand.

| | Product | | | | |
|---|---|---|---|---|---|
| Period | 1 | 2 | 3 | 4 | 5 |
| 1 | 89.50 | 145.92 | 67.67 | 39.00 | 63.76 |
| 2 | 150.66 | 208.27 | 132.78 | 102.70 | 123.39 |
| 3 | 151.76 | 240.59 | 141.02 | 142.08 | 125.52 |
| 4 | 178.73 | 250.96 | 158.15 | 154.97 | 126.40 |
| 5 | 179.54 | 263.33 | 211.66 | 179.47 | 158.57 |
| 6 | 192.04 | 266.35 | 236.57 | 193.82 | 184.95 |
| 7 | 199.14 | 284.53 | 239.89 | 194.49 | 185.35 |
| 8 | 239.14 | 289.76 | 271.66 | 194.98 | 185.76 |
| 9 | 239.45 | 323.75 | 288.81 | 202.26 | 189.33 |
| 10 | 266.73 | 342.49 | 290.24 | 202.61 | 210.58 |

In these illustrations only capacity changes but in practise it is also common that demand has trends and seasonal components. It is however possible to show how the algorithm works by only making available capacity change over time.

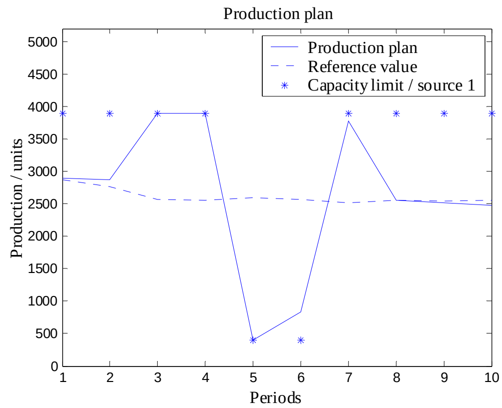


Fig. 4. Aggregated production plan with 80% service level. Capacity is scaled so that there is same amount of free capacity from source 1 over planning horizon in 80% and 95% examples.

Since the initial stock $X_{i0}$ for every product $i$ is zero the reference value is production plan without considering limited capacity from source 1. The algorithm moves production to both directions, backward and forward. The total cost for reference plan are $3.99*10^5$ \$ and for optimized plan $3.35*10^5$ \$.

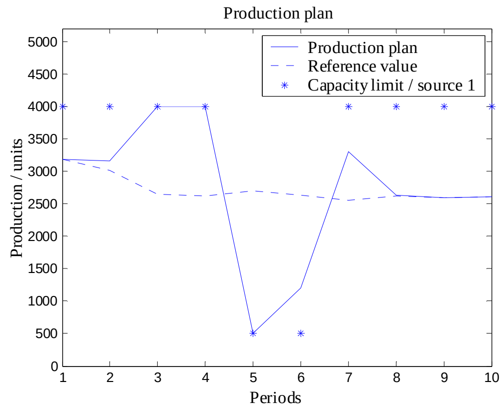


Fig. 3. Aggregated production plan with 95% service level.

The difference in production plans between 80% case and 95% case is only because of difference in shortage costs. Higher service level means higher shortage costs when other parameters are fixed. As a result from this it is not so likely to move production forward in 95% service level case. The total cost for reference plan are $4.29*10^5$ \$ and for optimized plan are $3.71*10^5$ \$.

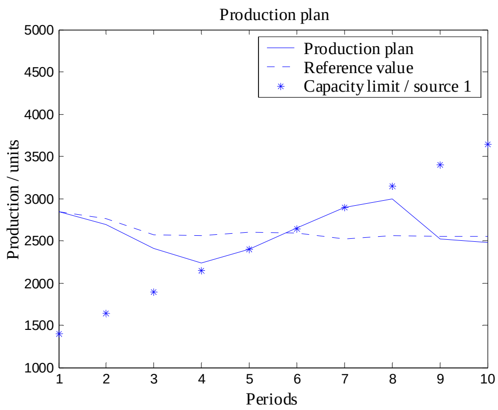


Fig. 4. Aggregated production plan with 80% service level. Capacity is scaled so that there is same amount of free capacity from source 1 over planning horizon in 80% and 95% examples.

Linearly increasing capacity from first source moves production forward in both cases. It is also necessary to point out that reference plan is higher when service level requirement is 95%. The total cost for reference plan are $3.88*10^5$ \$ and optimized plan are $3.79*10^5$ \$.

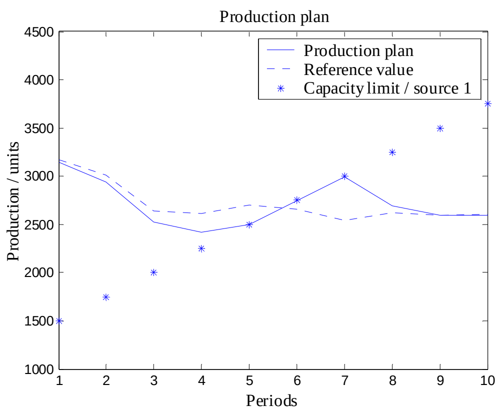


Fig. 5. Aggregated production plan with 95% service level.

It might sometimes be desirable to use high service level requirements but one has to take into account the consequences. Higher service level means higher average stock and often higher production costs. The total cost for reference plan are $4.25*10^5$ \$ and optimized plan are $4.17*10^5$ \$.

## 5. CONCLUSION

In this paper we propose a solution algorithm for target inventory problems. When shortage costs are difficult to evaluate more practical approach is to use service level requirements as a target inventory. Problem formulation allows cost for deviation from reference value to be any differentiable and convex function. The Results shows that algorithm has capability to find production plan which has lower costs than reference plan has.

Future research directions include rolling horizon tests as well as multi-echelon studies. This kind of algorithm would benefit networks in which reliability is essential. By using cumulative quantities it is possible to formulate problem as a static optimization problem instead of solving complex dynamic programming problem. It is obvious that dynamic programming solution will perform better than this kind of algorithm in rolling horizon manner. However this kind of formulation has also potential in practical applications.

Development of algorithm in which set-up time is included will broaden the class of potential applications. Also the case where unmet demand in current period will not move to next period makes model more usable. Capacity planning is one of the major tasks in tactical level production planning. By using this type of formulation it is possible to make capacity planning more effective.

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