MEAN AND ENTROPY OF B-SPLINE PDF MODELS: ANALYSIS AND DESIGN [1]

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Abstract: For continuous probability density functions (PDFs) approximated by the B-spline basis functions, the relationships between the B-spline weights, the entropy and the mean have been analyzed in detail. It shows the different characteristics of the entropy with and without mean constraint. A minimum entropy controller subjected to mean constraint is developed by taking the performance function as a Lyapunov function and ensuring the negativeness of its first-order derivative. Simulation examples are included to validate the analysis results and evaluate the closed-loop control performance. *Copyright ©2005IFAC*

Keywords: Dynamic stochastic systems; Probability density function (PDF); B-spline neural network; Minimum entropy; Mean constraint.

1. INTRODUCTION

Under the assumption that the random variables in the system are subjected to Gaussian processes, research on the control of stochastic systems has been focused on the mean and variance control (Astrom, 1970; Astrom and Wittenmark, 1988; Goodwin and Sin, 1984). In practice, however, many non-Gaussian stochastic systems should be considered and some of them require the control of the output probability density function (PDF) or the minimization of the output uncertainties (Wang, 2000). This leads to the use of some new performances, such as the entropy or other high order moment functions (Wang, 2002; Yue and Wang, 2003).

For Gaussian-type of stochastic systems, the equivalence of the entropy control and the variance control has already been proved, which shows that the minimum variance control is a special case of the minimum entropy control (Wang, 2002; Yue and Wang, 2003). However, the entropy is more general in terms of representing the system uncertainties (randomness) because it measures the dispersion of the probability distribution. This explains why the entropy control is important for non-Gaussian systems. The commonly used Shannon's entropy is defined as follows.

Definition 1. For a continuous random variable y with probability density function $\gamma(y)$, the Shannon's entropy of y is defined by

$$H(y) = -\int_{-\infty}^{+\infty} \gamma(y) ln(\gamma(y)) dy$$

It can be seen that the entropy formulation is based on the description of probability density function. In this paper, the output PDF is modelled by the B-spline neural network for the reason that it has been proved to be a mature modelling technique in PDF shaping (Wang, 2000). The aim of the controller design is to minimize the output

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entropy and maintain its mean value at the same time. The mean constraint is embedded to make sure that the output PDF is located in the required place. In order to realize such a design, the relationship between the B-spline weights and the entropy subjected to the mean constraint should be investigated carefully. This motivates the work in this paper.

2. B-SPLINE PDF APPROXIMATION

The B-spline basis functions (Girosi and Poggio, 1990) are commonly adopted to approximate the bounded PDF (Wang, 2000). In the one dimensional case, the B-spline basis functions are defined in an interval $[a, b] \in R$ with $\mathbf{Y} = (\mathbf{y_1}, ..., \mathbf{y_{n+1}})^{\mathbf{T}}$ being a node vector satisfying $a = y_1 < ... < y_{n+1} = b$. A set of normalized B-spline basis functions of order l ($l > 1$ is a positive integer), denoted by $\phi_{i,l}(i = 1, ..., n)$, can be represented as $\phi_{i,l}(y) = \frac{y-y_i}{y_{i+l-1}-y_i}\phi_{i,l-1}(y) + \frac{y_{i+l}-y}{y_{i+l}-y_{i+1}}\phi_{i+1,l-1}(y)$, where $\phi_{i,1}(y) = \begin{cases} 1 & , \quad if \;\; y \in [y_i, y_{i+1}] \\ 0 & , \quad otherwise \end{cases}$. A univariate B-spline basis function of order l, denoted by $B_i(y) = \frac{l}{y_{i+l}-y_i}\phi_{i,l}(y)$, has the following integral characteristics

$$\int_a^b B_i(y)dy = 1 \qquad (1)$$

Suppose a PDF $\gamma(y)$ is continuous with respect to y for all $y \in [a, b]$, there exists the following B-spline approximation of the PDF

$$\gamma(y) = \sum_{i=1}^{n} \omega_i B_i(y) \qquad (2)$$

with the approximation error being ignored for simplicity. Here n is the number of the basis functions selected for approximation, $B_i(y)$ and $\omega_i(\omega_i \geq 0)$ are the basis functions and the corresponding weights. With the PDF constraint $\int_a^b \gamma(y)dy = 1$, it can be obtained from equations (1) and (2) that

$$\sum_{i=1}^{n} \omega_i = 1 \qquad (3)$$

The entropy of y can then be represented as

$$H(y) = -\int_a^b \left(\sum_{i=1}^{n} \omega_i B_i(y)\right) ln \left(\sum_{i=1}^{n} \omega_i B_i(y)\right) dy \qquad (4)$$

When the basis functions are chosen, the PDF and the entropy can also be denoted as $\gamma(\omega_1, \cdots, \omega_n)$ and $H(\omega_1, \cdots, \omega_n)$, respectively.

When using the B-spline PDF model, the most common case is that all the basis functions are chosen to have the same shape but located at different places within the effective PDF interval. In order to clarify the main results of this work, all the following discussions will be based on this assumption and the basis functions are labelled with $i = 1$ to $i = n$ according to their positions from left to right on the effective PDF interval. In this case, when defining

$$x_i = \int_a^b yB_i(y)dy \qquad (5)$$

there will be $x_1 < x_2 < \cdots < x_n$ (x_i is fixed because $B_i(y)$ is fixed). The mean of y is represented as

$$\mu = \sum_{i=1}^{n} \omega_i x_i \qquad (6)$$

It is obvious that $x_1 \leq \mu \leq x_n$. As such, the PDF, the entropy, the mean and the natural PDF constraint can all be expressed as functions of the B-spline weights.

3. B-SPLINE WEIGHTS AND ENTROPY WITH MEAN CONSTRAINT

Two definitions are introduced to establish the relationship between the B-spline weights and the entropy.

Definition 2. Effective B-spline basis function: A B-spline basis function is called an effective B-spline basis function when its corresponding weight is not zero. Otherwise it is referred as ineffective for the PDF approximation.

Definition 3. Effective PDF interval: the minimum interval which covers all the definition intervals of the effective B-spline basis functions.

The following discussions are based on these two definitions.

Proposition 4. The entropy is a concave function of the B-spline weights.

Proof: Suppose that $\{\omega_i, \omega_j\}$ ($j > i$) are the pair of weights subjected to the change of z. Without loss of generality, it can be assumed that such a change leads to $\omega_i \to \omega_i + z$ and $\omega_j \to \omega_j - z$ as all the weights are subjected to the constraint given by (3). As such, the entropy is related to z by

$\varphi(z) = H(y)|_z = -\int_a^b \gamma(\cdots, \omega_i + z, \cdots, \omega_j - z, \cdots)$

$$ln(\gamma(\cdots,\omega_i+z,\cdots,\omega_j-z,\cdots))dy$$

where $-\omega_i \le z \le \omega_j$. The first order derivative of $\varphi(z)$ can be calculated to give

$$\dot{\varphi}(z) = -\int_a^b (B_i(y)-B_j(y))ln(\gamma(\cdots,\omega_i+z,\cdots,\omega_j-z,\cdots))dy$$

Differentiating $\dot{\varphi}(z)$ with respect to z leads to

$$\ddot{\varphi}(z) = -\int_a^b \frac{(B_i(y)-B_j(y))^2}{\gamma(\cdots,\omega_i+z,\cdots,\omega_j-z,\cdots)}dy \le 0$$

Therefore, $\varphi(z)$ is a concave function of z. This result can be extended to the situation when more than two weights are subjected to change. □

As $\varphi(z)$ is a concave function of z, the minimum point(s) can be achieved either on $z=-\omega_i$ and/or on $z=\omega_j$, which gives the following Corollary.

Corollary 5. When $\{\omega_i,\omega_j\}$ are the pair of weights subjected to change, at least one of the following inequalities should hold in any case:

$$H(...,0,...,\omega_i+\omega_j,...) \le H(...,\omega_i,...,\omega_j,...) \quad (7)$$
$$H(...,\omega_i+\omega_j,...,0,...) \le H(...,\omega_i,...,\omega_j,...) \quad (8)$$

It would be interesting to see the difference between Corollary 5 and the entropy's *grouping* proposition for discrete distribution (Yuan and Kesavan, 1998). For a discrete probability system with states $x_1 < \cdots, < x_n$ and corresponding probability distribution $(p_1,\cdots,p_n)$, the change of moving one state to another can always make the entropy decrease, i.e.,

$$\begin{aligned} &H(p_1,...,0,...,p_i+p_j,...,p_n) \\ &= H(p_1,...,p_i+p_j,...,0,...,p_n) \\ &\le H(p_1,...,p_i,...,p_j,...,p_n). \end{aligned} \quad (9)$$

This is because the entropy of the discrete distribution is only related to the probability distribution. For a continuous random variable, however, the entropy depends not only on the basis functions but also on the weights, the shape of the approximated PDF has a direct impact on the entropy value. The effect of the PDF shape to the entropy can be further illustrated by the following proposition.

Proposition 6. If the effective PDF interval becomes narrower, there must exist a reduced number of weights which makes the entropy to decrease.

Proof: When the effective PDF interval is decreased, it means that at least one effective basis function was taken off from the end(s) of the effective PDF interval. Taking $\omega_n = 0$ as an example, Proposition 6 says that there exist a set of a_i ($a_i + \omega_i \ge 0$) such that

$$H(\omega_1+a_1,\cdots,\omega_{n-1}+a_{n-1},0) < H(\omega_1,\cdots,\omega_n) \quad (10)$$

where $\sum_{i=1}^{n-1} a_i = \omega_n$. For the original PDF approximation, assume that ω_{n-1} and ω_n are subjected to change and denote $\eta = \omega_{n-1}+\omega_n$. From Corollary 5, it can be seen that the inequality (10) can be proved if the following inequality

$$H(\omega_1,\ldots,\eta,0) < H(\omega_1,\ldots,0,\eta) \quad (11)$$

is satisfied. Denote

$$H_1 = H(\omega_1,\ldots,\eta,0),$$
$$H_2 = H(\omega_1,\ldots,0,\eta),$$
$$\alpha = \int_a^b \eta B_{i-1}(y)ln(\eta B_{i-1}(y))dy,$$
$$\beta = \int_a^b \eta B_i(y)ln(\eta B_i(y))dy,$$

then H_1 and H_2 can be represented as

$$H_1 = H(\omega_1,\ldots,0,0) - \delta_1 - \delta_2 - \alpha$$
$$H_2 = H(\omega_1,\ldots,0,0) - \delta_1 - \beta$$

where $\delta = \delta_1 + \delta_2$, $\delta_1 = -\int_a^b S_1(y)lnS_1(y)dy$ and $\delta_2 = -\int_a^b S_2(y)lnS_2(y)dy$.

As shown in Fig.1, S_1 is the overlap area of $\omega_{n-2}B_{n-2}(y)$ and $\eta B_n(y)$, S_2 is the overlap area of $\omega_{n-2}B_{n-2}(y)$ and $\eta B_{n-1}(y)$ subtracting S_1. It is obvious that $\delta_1 > 0$ and $\delta_2 > 0$. Also, when the basis functions have the same shape, α is equal to β. Therefore, $H_1 < H_2$, the inequality (11) is satisfied. Proposition 6 is thus proved. □

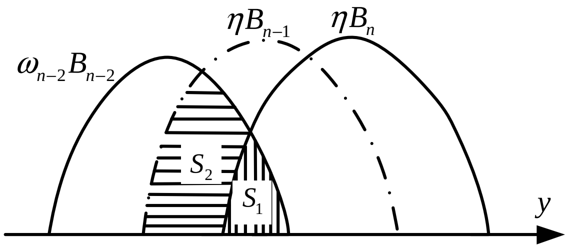

Fig. 1. Illustration of S_1 and S_2 for the proof of Proposition 6.

By applying Proposition 6 and Corollary 5 to the B-spline PDF model repeatedly, the following conclusion on minimum entropy can be made.

Corollary 7. The entropy is minimum when there is only one effective B-spline basis function.

The above propositions and corollaries about the entropy of continuous distributions show the existence of minimum entropy under certain conditions. Next, the minimum entropy subjected to the mean constraint will be discussed.

Theorem 8. Assume that the PDF $\gamma(y)$ is approximated by three pre-specified B-spline basis functions, the following two conditions

(a) the entropy is minimum

(b) the mean is kept unchanged

can be satisfied at the same time by reducing the number of the effective B-spline basis functions to be two or one.

Proof: The following two situations can be discussed separately.

(1) When $\mu = x_i$, the PDF can be approximated by the single basis function $B_i(y)$ with $\omega_i = 1$ so as to keep the mean value to be x_i. The minimum entropy is achieved in this case according to Corollary 7.

(2) When $\mu \neq x_i$, from the PDF and mean constraints (3) and (6), ω_1 and ω_2 are related to ω_3 linearly as follows

$$\omega_1 = \frac{x_2 - \mu}{x_2 - x_1} + \frac{x_3 - x_2}{x_2 - x_1}\omega_3 \qquad (12)$$

$$\omega_2 = \frac{\mu - x_1}{x_2 - x_1} - \frac{x_3 - x_1}{x_2 - x_1}\omega_3 \qquad (13)$$

As denoted in Section 2, all the weights should be nonnegative, therefore,

$$\omega_3 \in \left[\frac{\mu - x_2}{x_3 - x_2}, \frac{\mu - x_1}{x_3 - x_1}\right] \; when \; \mu > x_2 \qquad (14)$$

$$\omega_3 \in \left[0, \frac{\mu - x_1}{x_3 - x_1}\right] \; when \; \mu < x_2 \qquad (15)$$

According to Proposition 4, the entropy is a concave function of ω_3. Therefore, when $\mu < x_2$, the minimum entropy will be achieved either on $\omega_3 = 0$ or $\omega_3 = \frac{\mu - x_1}{x_3 - x_1}$. In the former case $\{B1(y), B_2(y)\}$ are the effective basis functions; in the latter case, $\{B1(y), B_3(y)\}$ are the the effective basis functions.

Similarly, when $\mu > x_2$, the minimum entropy will be achieved either on $\omega_3 = \frac{\mu - x_2}{x_3 - x_2}$ or $\omega_3 = \frac{\mu - x_1}{x_3 - x_1}$. It's easy to see that $\{B_2(y), B_3(y)\}$ are the effective basis functions in the former case and $\{B1(y), B_3(y)\}$ are the the effective basis functions in the latter case. □

Based on Theorem 8, the conditions to realize the minimum entropy subjected to mean constraint can be extended to the PDF approximation with n basis functions.

Theorem 9. For any B-spline PDF approximation systems consisting of n pre-specified basis functions, the minimum entropy subjected to the mean constraint (6) can only be realized when the PDF is approximated by one or two effective basis functions.

Proof: Suppose that there are more than two effective basis functions corresponding to the PDF of the minimum entropy, for example, they are $\omega_i > 0$, $\omega_j > 0$ and $\omega_k > 0$. By applying Theorem 8 to the weights $(\frac{\omega_i}{\omega}, \frac{\omega_j}{\omega}, \frac{\omega_k}{\omega})$ with $\omega = \omega_i + \omega_j + \omega_k$, the entropy relating to such a set of weights may be further reduced under the mean constraint. This is contradictive to the minimum entropy assumption.

When $\mu = x_i$, the PDF of the minimum entropy is approximated by the single effective B-spline basis function $B_i(y)$ with $\omega_i = 1$.

When $x_{i-1} < \mu < x_i$, the minimum entropy will be obtained by either the most unbalanced distribution or the distribution of the narrowest effective PDF interval. The former case corresponds to the effective basis functions $\{B_{i-1}(y), B_n(y)\}$ or $\{B_1(y), B_i(y)\}$, the latter PDF is approximated by $\{B_{i-1}(y), B_i(y)\}$. The minimum entropy is chosen from the three cases. Accordingly, the two weights are determined by the PDF and mean constraints (3) and (6). □

Theorems 8 and 9 provide the basic criterions for the design of minimum entropy controller under the mean constraint.

The univariate B-spline basis functions satisfying (1) are used through out the above discussions. However, it is not a necessary condition, other types of B-spline basis functions can also be used by replacing ω_i with $\omega_i / \int_a^b B_i(y)dy$. Moreover, for the major results presented so far, the assumption that all the basis functions should have the same shape is not a necessary condition either. This assumption is made only to simplify the presentation. As such, most of the results can be applied to more general B-spline PDF models.

4. MINIMUM ENTROPY CONTROL SUBJECTED TO MEAN CONSTRAINT

Consider the following state-space B-spline dynamic model (Wang, 2000)

$$\begin{cases} \dot{\mathbf{V}} = \mathbf{A}\mathbf{V} + \mathbf{B}\mathbf{u} \\ \gamma(y, \mathbf{u}) = \mathbf{C}(y)\mathbf{V} + L(y) \end{cases}$$

where $\gamma(y, \mathbf{u})$ is the output PDF of the system approximated by n univariate B-spline basis functions; $\mathbf{A} \in R^{(n-1)\times(n-1)}$ and $\mathbf{B} \in R^{(n-1)\times m}$ are the parameter matrices; $\mathbf{V} \in R^{(n-1)\times 1}$ is the weights vector; $\mathbf{u} \in R^m$ is the control input; $L(y) \in R^{1\times 1}$ and $\mathbf{C}(y) \in R^{1\times(n-1)}$ are decided by the basis functions.

Many control algorithms have been developed for the shaping of the output PDF (Wang, 1999, 2000; Wang, *et al.*, 2001). When the target PDF is not available, the controller design can be made to minimize the output entropy so as to reduce the output randomness. However, without locating the output PDF properly, the minimization of the entropy does not make sense in practice. As such, it is important to specify a mean constraint for the system output when solving the minimum entropy problem. In this context, the performance function is selected as follows

$$J = (\mu - \mu_g)^2 + \mathbf{u}^T\mathbf{Q}\mathbf{u} - \int_a^b \gamma(y,\mathbf{u})ln(\gamma(y,\mathbf{u}))dy \qquad (16)$$

in which the first term characterizes the difference between the output mean μ and the target mean μ_g; the second term represents a constraint on the control input with $\mathbf{Q} = \mathbf{Q}^T > 0$ being a pre-specified weighting matrix; The last term is the Shannon's entropy of the system output. By minimizing this performance function, the closed-loop system can be illustrated by Fig 2, where the outer loop realizes the mean tracking and the inner loop is the minimum entropy controller.

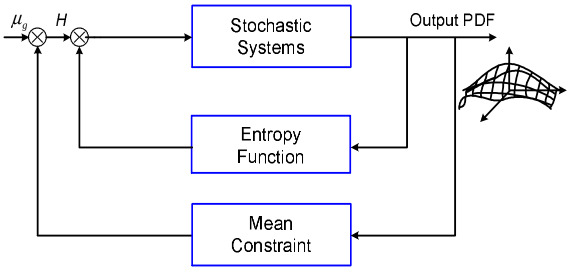

Fig. 2. The closed-loop entropy and mean control system

At this stage, the aim of the controller design is to find a $\mathbf{u}$ such that $J = min$. This can be achieved by selecting a $\mathbf{u}$ so that the performance function J decreases monotonically, leading to $\frac{dJ}{dt} < 0$. The idea is similar to the approach used in (Wang *et al.*, 2001) where the performance function was taken as a Lyapunov function for the closed-loop system. For this purpose, the first order derivative of J can be calculated to give

$$\frac{dJ}{dt} = -\int_a^b \frac{\partial\gamma(y,\mathbf{u})}{\partial\mathbf{V}}\dot{\mathbf{V}}(ln(\gamma(y,\mathbf{u}))+1)dy + \frac{\partial(\mu-\mu_g)^2}{\partial\mathbf{V}}\dot{\mathbf{V}} + \mathbf{u}^T\mathbf{Q}\dot{\mathbf{u}} < 0 \qquad (17)$$

From the above equation the control $\mathbf{u}$ can be obtained to read,

$$\dot{\mathbf{u}} = (\mathbf{u}^T\mathbf{Q})^{-1}(-\lambda\,|\mu-\mu_g| - \frac{\partial(\mu-\mu_g)^2}{\partial\mathbf{V}}\dot{\mathbf{V}} + \int_a^b \frac{\partial\gamma(y,\mathbf{u})}{\partial\mathbf{V}}\dot{\mathbf{V}}(ln(\gamma(y,\mathbf{u}))+1)dy \qquad (18)$$

where $\lambda > 0$. Because $\frac{dJ}{dt} < 0$, the closed-loop stability is naturally guaranteed.

5. SIMULATION ANALYSIS

The following B-spline basic functions are used for the PDF approximation:

$$B_1(y) = \tfrac{1}{2}y^2I_1 + (-y^2+3y-\tfrac{3}{2})I_2 + \tfrac{1}{2}(y-3)^2I_3$$
$$B_2(y) = \tfrac{1}{2}(y-1)^2I_2 + (-y^2+5y-\tfrac{11}{2})I_3 + \tfrac{1}{2}(y-4)^2I_4$$
$$B_3(y) = \tfrac{1}{2}(y-2)^2I_3 + (-y^2+7y-\tfrac{23}{2})I_4 + \tfrac{1}{2}(y-5)^2I_5$$

where $I_i = \begin{cases} 1 & , \quad y\in[i-1,i] \\ 0 & , \quad otherwise \end{cases}$ $i = 1,\cdots,5$. It can be seen from (5) that $x_1 = 1.5$, $x_2 = 2.5$ and $x_3 = 3.5$.

When ω_2 is fixed to be $\omega_2 = -0.1294$, changing ω_1 with $\omega_1 \in [0.2147, 0.9147]$ and ω_3 correspondingly, the entropy change is shown in Fig. 3. It is a concave curve of ω_1.

Fig. 4 shows the PDFs with two local minimum entropies when all the three weights are adjustable and the mean is fixed to be 1.8. The solid line corresponds to the weights $[0.9147, -0.1294, 0.2147]$ with $H = 1.0554$. The dashed line corresponds to the weights $[0.7, 0.3, 0]$ with $H = 1.0136$. From Theorem 9, it can be found that the minimum entropy is obtained when the effective B-spline basis functions are $B_1(y)$ and $B_2(y)$. The simulation results support Theorem 8.

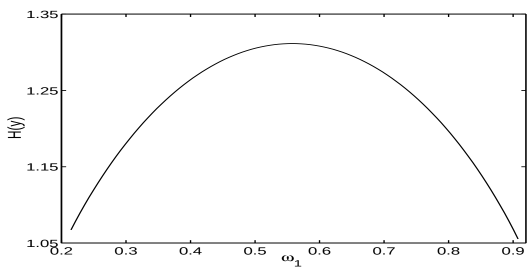

Fig. 3. The entropy vs. weights under mean constraint

Now consider the dynamic simulation. The initial condition of the three weights is given to be $[0.3000, 0.2000, 0.5000]$. The dynamical model is as follows:

$$\dot{\mathbf{V}} = \begin{pmatrix} 0 & 1 \\ -0.8 & -2.5 \end{pmatrix}\mathbf{V} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u$$

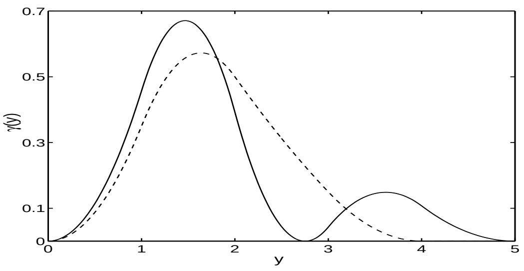

Fig. 4. PDFs of two local minimum entropy subjected to mean=1.8

Assume that ω_1 and ω_2 are the independent weights, then the initial condition of the weights vector is $\mathbf{V} = [0.3\ 0.2]^T$. The mean value is set to be 1.5. From Theorem 8, it is known that the minimum entropy will be obtained when ω_1 is the only effective B-spline basis function. H is calculated to be 0.7193 in this example. The controller is designed by (18). Figs. 5-7 demonstarte the simulation results.

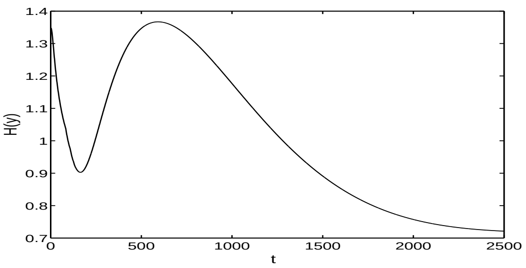

Fig. 5. The response of the entropy

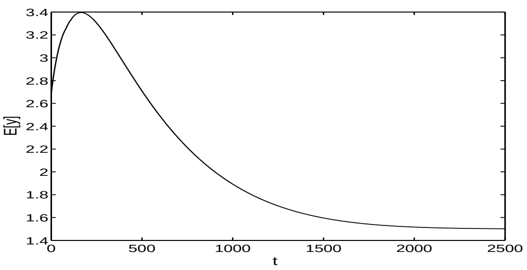

Fig. 6. The response of the mean

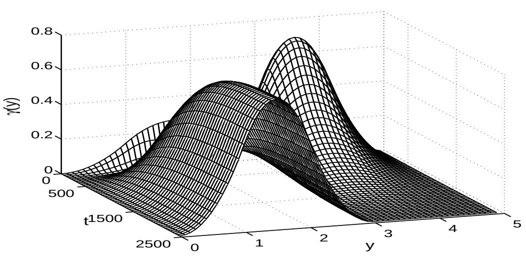

Fig. 7. The 3D plot of the PDF development

It can be seen that the entropy converges to the minimum value and the mean converges to the expected value. In Fig. 7, the related 3D plot is given illustrating the output PDF evolution during the control process.

6. CONCLUSIONS

This paper presents several characteristics about the entropy of continuous PDF which is approximated by the B-spline basis functions. It's the first time that the relationship between the B-spline weights and the entropy be investigated for this type of stochastic control systems. In comparison with the discrete probability system whose entropy is only a function of the probability distribution, the entropy of the continuous PDF system is related to the basis functions, the corresponding weights and the effective PDF interval. The study shows the situations under which the minimum entropy can be obtained while keeping the mean constraint. It leads to a new controller design by taking the entropy and the output mean as separate terms in the selected performance function.

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