RANGE CONTROL MPC APPROACH FOR TWO-DIMENSIONAL SYSTEM [1]

Jirka Roubal * **Vladimír Havlena** **

* *Department of Control Engineering, Faculty of Electrical Engineering, Czech Technical University in Prague*
Karlovo náměstí 13, 121 35 Prague 2, Czech Republic
E-mail: roubalj@control.felk.cvut.cz

** *Honeywell Prague Laboratory, Honeywell Intl.*
Pod vodárenskou věží 4, 182 08 Prague 8, Czech Republic
E-mail: havlena@htc.honeywell.cz

Abstract: This paper deals with model predictive control of a distributed parameters system which is described by a linear two-dimensional (dependent on two spatial directions) parabolic partial differential equation. This partial differential equation is transformed to the discrete state space description using the finite difference approximation. A model with a large dimension is obtained and has to be reduced for an advanced control design. The balanced truncation method is used for the model dimension reduction. For this low dimension model, the range control approach is applied.

Keywords: Distributed Parameters, Partial Differential Equations, Reduced-order Models, Modelling, Predictive Control.

1. INTRODUCTION

There are many industrial processes which have distributed parameters behaviour. Consequently, these processes cannot be modelled by lumped inputs and/or lumped outputs models for correct representation.

This paper deals with two-dimensional dynamic processes (systems with parameters dependent on two spatial directions) which can be described by lumped inputs and distributed output models. These models can be mathematically described by partial differential equations (PDE) (Long, C. A., 1999). Unlike ordinary differential equations, the PDEs contain, in addition, derivatives with respect to spatial directions. Consequently, the partial differential equations lead to more accurate models but their complexity is larger.

The dynamic behaviour of the distributed parameters system, which is described by the PDE, can be approximately described by a finite-dimensional model, for example, by using the finite difference method (Babuška, I. *et al.*, 1966). Then the ordinary differential equation model with large dimension is obtained and can be used for a finite-dimensional controller design. Unfortunately, for online solving of an optimization problem, e.g. the model predictive control approach, the large model dimension introduces a problem for the control design. Therefore a model reduction method has to be used.

[1] The authors acknowledge financial support of Honeywell Prague Laboratory (96/95017/13135) and grants 102/03/H116 and 102/05/2075 of the Grant Agency of the Czech Republic.

Variables of every real process have certain limits given by laws of physics. Unlike the classical control law, the model predictive control (MPC) considers explicitly the future implication of current control action. This approach enables us to include the constraints on inputs/outputs to the control algorithm (Maciejowski, J. M., 2002).

In (Havlena, V. and Findejs, J., 2005), the range control approach is described for lumped inputs and lumped outputs systems. The main idea of the range control concept is to replace the set point (reference) by low and high limits. This methodology leads to very stable and robust control because the manipulated inputs do not compensate the high-frequency component of the noise. In this paper, this concept is applied for the distributed parameters system.

The paper is organized as follows. In section 2, the distributed parameters model for the finite controller design is developed. In section 3, the balanced truncation method is shortly described. In section 4, the basic idea of model predictive control with the range control approach is described. In section 5, this methodology is applied to a heat transfer process as a demonstration example.

2. DISTRIBUTED PARAMETER PROCESS DESCRIPTION

In this section, the model of a heat transfer process described by a linear two-dimensional parabolic PDE (Long, C. A., 1999) is developed for the finite-dimensional controller design. At first, the stationary PDE is transformed to a linear equation system using the finite difference approximation (Babuška, I. *et al.*, 1966). Then the implicit scheme (Babuška, I. *et al.*, 1966) and this equation system are used for the transformation of the evolutionary PDE to a linear dynamic discrete system.

2.1 Stationary Partial Differential Equation

For the surface thermal conductivity λ [W/K] independent on the temperature Θ [K] and a surface heat source f [W/m^2], the heat transfer process in the stationary case can be described by a parabolic PDE

$$-\lambda\left(\frac{\partial^2\Theta(x,y)}{\partial x^2}+\frac{\partial^2\Theta(x,y)}{\partial y^2}\right)= \qquad (1)$$
$$=-\lambda\,\Delta\,\Theta(x,y)=f(x,y),$$

where Δ is the Laplace operator. The unknown temperature Θ must satisfy equation (1) on an open set $\Omega=(0,L_1)\times(0,L_2)$ and boundary condition on $\partial\Omega$ ($\partial\Omega$ means the boundary of set Ω).

In this paper, the boundary condition which specifies the temperature gradient on the boundary $\partial\Omega$ is described by the following statement

$$-\lambda\frac{\partial\Theta(x,y)}{\partial n}=\alpha\big(\Theta(x,y)-\Theta_s(x,y)\big), \qquad (2)$$

where n is unit normal vector, α [W/(mK)] is an external heat transfer coefficient and Θ_s is the surrounding temperature. Note that equation (2) is known as Newton (or the third kind) boundary condition (Long, C. A., 1999).

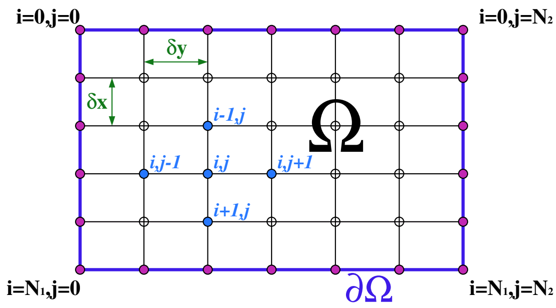

Figure 1. The mesh on the set Ω

For the transformation of PDE (1) with boundary condition (2) to the finite dimensional model, the set Ω is covered by an imaginary mesh so that the values of mesh points satisfy $\mathbf{\Theta}_{i,j}=\Theta(i\,\delta x,j\,\delta y)$ on the closed set Ω, $\mathbf{F}_{i,j}=f(i\,\delta x,j\,\delta y)$ on the open set Ω and $\mathbf{F}_{i,j}=\mathbf{\Theta}_s(i\,\delta x,j\,\delta y)$ on the set $\partial\Omega$, where $\delta x=L_1/N_1$ and $\delta y=L_2/N_2$ are the grid sizes of the imaginary mesh and i, j are row and column indices, respectively (see Figure 1). Matrix $\mathbf{\Theta}$ is the matrix of the temperature values in the mesh points and matrix $\mathbf{F}$ represents the heat source f and the surrounding temperature Θ_s.

Using the second order finite difference approximation (Babuška, I. *et al.*, 1966), the distributed parameters system, equations (1) and (2), can be obtained as a linear equation system

$$\mathbf{P}\boldsymbol{\theta}=\mathbf{f}\,, \qquad (3)$$

where

$$\boldsymbol{\theta}=\begin{bmatrix}\mathbf{\Theta}(:,0)\\ \mathbf{\Theta}(:,1)\\ \vdots\\ \mathbf{\Theta}(:,N_2)\end{bmatrix},\qquad \mathbf{f}=\begin{bmatrix}\mathbf{F}(:,0)\\ \mathbf{F}(:,1)\\ \vdots\\ \mathbf{F}(:,N_2)\end{bmatrix},$$

where $\mathbf{\Theta}(:,0)$ means the zero column of the matrix $\mathbf{\Theta}$, $\mathbf{\Theta}(:,1)$ the first column and so on. Note that the square matrix $\mathbf{P}$ contains $(N_1+1)\times(N_2+1)$ rows and its structure and derivation can be found in (Roubal, J. *et al.*, 2004).

2.2 Evolutionary Partial Differential Equation

In the non stationary case, PDE (1) can be written as

$$\rho\,c_0\frac{d\Theta(x,y,t)}{dt}-\lambda\,\Delta\,\Theta(x,y,t)=f(x,y,t), \qquad (4)$$

where ρ [kg/m^2] is the surface density of the medium and c_0 [Ws/kgK] is its thermal capacity. In this case, the unknown temperature profile $\Theta(x, y, t)$, dependent on time t, must satisfy, for an initial condition $\Theta(x, y, t_0) = \Theta_{init}(x, y)$, equation (4) on the open set Ω and boundary condition (2) on $\partial\Omega$ for all time horizon $t \in \langle t_0, t_{end} \rangle$.

Using equation (3) and the implicit discretization scheme (Babuška, I. *et al.*, 1966) with a sampling period δt, evolutionary PDE (4) with Newton boundary condition (2) can be approximated as

$$\boldsymbol{\theta}(k{+}1) = \mathbf{M}\boldsymbol{\theta}(k) + \mathbf{N}\mathbf{f}(k), \qquad \boldsymbol{\theta}(k_0) = \boldsymbol{\theta}_{init}, \qquad (5)$$

$$\mathbf{M} = \left(\mathbf{I} + \frac{\delta t}{\rho\, c_0}\mathbf{P}\right)^{-1}, \quad \mathbf{N} = \left(\mathbf{I} + \frac{\delta t}{\rho\, c_0}\mathbf{P}\right)^{-1} \cdot \frac{\delta t}{\rho\, c_0},$$

where $\mathbf{I}$ is the identity matrix with the corresponding dimension. More details can be found in (Roubal, J. *et al.*, 2004).

3. MODEL REDUCTION METHOD

The accuracy of model (5) increases with decreasing grid sizes δx and δy. Unfortunately, for the advanced controller design such as the predictive controller, a low dimension model is needed. In this section, one reduction method is shortly described.

3.1 Model Reduction by Balanced Truncation

There are infinitely many different state space realizations for a given transfer function. But some realizations are more useful in control design. One of these realizations is the balanced realization which gives balanced Gramians for controllability $\mathbf{W}_c$ and observability $\mathbf{W}_o$ (Zhou, K. *et al.*, 1996). In addition, these Gramians are equal to the diagonal matrix $\boldsymbol{\Sigma}$

$$\mathbf{W}_c = \mathbf{W}_o = \boldsymbol{\Sigma} = \mathrm{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_n).$$

Note that the decreasingly ordered numbers,

$$\sigma_1 \ \geq \ \sigma_2 \ \geq \ \ldots \ \geq \ \sigma_n \ \geq 0,$$

are called the *Hankel singular values* of the system.

We suppose $\sigma_r \gg \sigma_{r+1}$ for some $r \in \langle 1; n)$. Then the balanced realization implies that the states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_n$ are less controllable and observable than the states corresponding to $\sigma_1, \ldots, \sigma_r$. The states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_n$ have smaller influence on the input/output behaviour of the system. Therefore, truncating the "less controllable and observable" states will not lose much information about the system input/output behaviour and the dimension of the model can be significantly reduced.

3.2 Reduced Model for the Control Design

The reduced model for control of evolution partial differential equation (4) with Newton boundary condition (2) can be written as

$$\mathbf{x}(k{+}1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{E}\mathbf{z}(k), \quad \mathbf{x}(k_0) = \mathbf{x}_0\,,$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)\,, \qquad (6)$$

where $\mathbf{x}$ is a state of the model, $\mathbf{y}$ is its output (temperature in several points on the set Ω), $\mathbf{u}$ is its input (manipulated variable), $\mathbf{z}$ represents the surrounding temperature profile (measurable disturbance) and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ are state matrices.

4. MODEL PREDICTIVE CONTROL

4.1 Output Prediction

For model (6), the output prediction for the prediction horizon T_p can be written in a compact form

$$\hat{\mathbf{y}}_k = \mathbf{V}\mathbf{x}(k) + \mathbf{T}\hat{\mathbf{z}}_k + \mathbf{S}\hat{\mathbf{u}}_k = \tilde{\mathbf{y}}_k + \mathbf{S}\hat{\mathbf{u}}_k\,, \quad (7)$$

where $\tilde{\mathbf{y}}_k = \mathbf{V}\mathbf{x}(k) + \mathbf{T}\hat{\mathbf{z}}_k$ and $\hat{\mathbf{y}}$, $\hat{\mathbf{u}}$ and $\hat{\mathbf{z}}$ are the output, inputs and disturbance prediction, respectively

$$\hat{\mathbf{y}}_k = \begin{bmatrix} \mathbf{y}(k) \\ \mathbf{y}(k{+}1) \\ \vdots \\ \mathbf{y}(k{+}T_p{-}1) \end{bmatrix}, \ \hat{\mathbf{u}}_k = \begin{bmatrix} \mathbf{u}(k) \\ \mathbf{u}(k{+}1) \\ \vdots \\ \mathbf{u}(k{+}T_p{-}1) \end{bmatrix}, \ \hat{\mathbf{z}}_k = \begin{bmatrix} \mathbf{z}(k) \\ \mathbf{z}(k{+}1) \\ \vdots \\ \mathbf{z}(k{+}T_p{-}1) \end{bmatrix}$$

and the prediction matrices are

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{T_p-1} \end{bmatrix}, \ \mathbf{S} = \begin{bmatrix} \mathbf{D} & & & \\ \mathbf{CB} & \mathbf{D} & & \\ \mathbf{CAB} & \mathbf{CB} & \mathbf{D} & \\ \vdots & & \ddots & \ddots \\ \mathbf{CA}^{T_p-2}\mathbf{B} & & \mathbf{CB} & \mathbf{D} \end{bmatrix}.$$

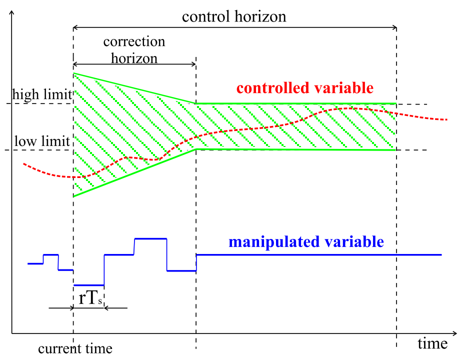

Figure 2. Control variables specification in time domain – the funnel for a controlled variable and the manipulated variable with reduced degrees of freedom and a constant value at the end of the correction horizon

Note that matrix $\mathbf{T}$ is similar to matrix $\mathbf{S}$ with matrix $\mathbf{D}$ replaced by zero matrix $\mathbf{0}$ and matrix $\mathbf{B}$ replaced by matrix $\mathbf{E}$.

4.2 Quadratic Program Problem Formulation

Consider process (7) and constraints on the controlled variables and their rates of change

$$\hat{\mathbf{y}}_L \le \hat{\mathbf{y}}_k \le \hat{\mathbf{y}}_H, \quad \Delta\hat{\mathbf{y}}_L \le \Delta\hat{\mathbf{y}}_k \le \Delta\hat{\mathbf{y}}_H \tag{8}$$

and constraints on the manipulated variables and their rates of change

$$\hat{\mathbf{u}}_L \le \hat{\mathbf{u}}_k \le \hat{\mathbf{u}}_H, \quad \Delta\hat{\mathbf{u}}_L \le \Delta\hat{\mathbf{u}}_k \le \Delta\hat{\mathbf{u}}_H. \tag{9}$$

The basic idea of the *range control* approach (HAVLENA, V. and FINDEJS, J., 2005) is to replace the setpoint for the controlled variable $\mathbf{y}$ by a set range which is defined by the sequence of low and high limits $\hat{\mathbf{y}}_L$ and $\hat{\mathbf{y}}_H$, see Figure 2. Then the optimality criterion can be expressed as a quadratic programming problem

$$\min_{\mathbf{u},\mathbf{w}} \frac{1}{2}\left\{\|\tilde{\mathbf{y}}_k + \mathbf{S}\hat{\mathbf{u}}_k - \hat{\mathbf{w}}_k\|^2_{\mathbf{Q}_y} + \|\Delta\hat{\mathbf{u}}\|^2_{\mathbf{Q}_u}\right\} \tag{10}$$

$$\text{subject to} \quad \hat{\mathbf{y}}_L \le \hat{\mathbf{w}}_k \le \hat{\mathbf{y}}_H \tag{11}$$

and constraints (9).

For the penalization of control increments in criterion (10), the increments of control can be obtained as

$$\Delta\hat{\mathbf{u}}_k = \mathbf{\Phi}\hat{\mathbf{u}}_k - \tilde{\mathbf{u}}_k\,, \tag{12}$$

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{I} & & & \\ -\mathbf{I} & \mathbf{I} & & \\ & \ddots & \ddots & \\ & & -\mathbf{I} & \mathbf{I} \end{bmatrix}, \quad \tilde{\mathbf{u}}_k = \begin{bmatrix} \mathbf{u}(k-1) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{I}$ is the identity matrix with the corresponding dimension. Then the optimization criterion (10) can be written as

$$\min_{\mathbf{u},\mathbf{w}} \frac{1}{2}\left\| \begin{bmatrix} \mathbf{S} & -\mathbf{I} \\ \mathbf{\Phi} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_k \\ \hat{\mathbf{w}}_k \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{y}}_k \\ -\tilde{\mathbf{u}}_k \end{bmatrix} \right\|^2_{\begin{bmatrix} \mathbf{Q}_y & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_u \end{bmatrix}}, \tag{13}$$

$$\hat{\mathbf{y}}_L \le \hat{\mathbf{w}}_k \le \hat{\mathbf{y}}_H, \tag{14}$$

$$\hat{\mathbf{u}}_L \le \hat{\mathbf{u}}_k \le \hat{\mathbf{u}}_H, \tag{15}$$

$$\Delta\hat{\mathbf{u}}_L + \tilde{\mathbf{u}}_k \le \mathbf{\Phi}\hat{\mathbf{u}}_k \le \Delta\hat{\mathbf{u}}_H + \tilde{\mathbf{u}}_k. \tag{16}$$

To reduce the dimension and computational requirements of the QP problem, the number of independent moves of manipulated variables may be reduced (HAVLENA, V. and FINDEJS, J., 2005), see Figure 2. The *receding horizon* approach is implemented. The idea of this strategy is to find the optimal control sequence on the prediction horizon T_p. Then for feedback control, only the first element of the optimal control sequence is applied to the plant and the optimal problem is recalculated for a new measured data (MACIEJOWSKI, J. M., 2002).

5. DEMONSTRATION EXAMPLE

Consider a heat transfer process in a furnace where $L_1 = L_2 = 0.9$ m described by equation (4) with constants $\lambda = 51$ W/K, $\rho = 2500$ kg/m^2, $c_0 = 1259$ Ws/(kg K) and $\alpha = 1.14$ W/(mK). The grid sizes are $\delta x = \delta y = 0.02$ m and the sampling period is $\delta t = 300$ s.

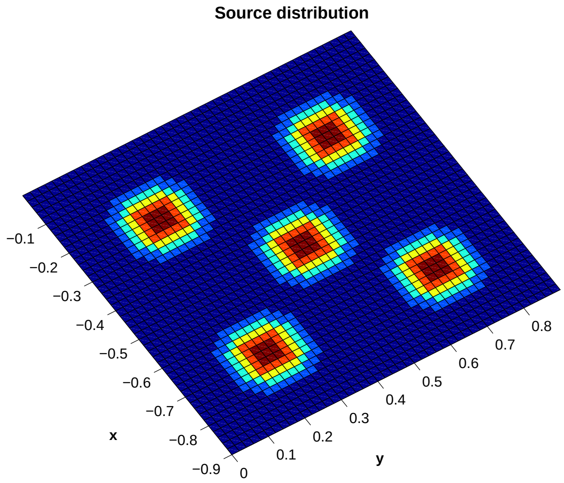

Figure 3. Heat source distribution $f(x, y)$

The heat source distribution $f(x, y)$ is shown in Figure 3. We consider that the sample system has five lumped inputs (manipulated variables) and the surface temperature is measured in 64 points which are uniformly distributed over the area Ω. The inputs can take the values $u \in \langle 0; 5\rangle$.

Figure 4a presents the steady-state temperature distribution (system state) for the unit step as inputs signal (see Figure 3) and the surrounding temperature $\Theta_s = 340$ K. Figure 4b shows the system output $\mathbf{y}$ – temperature in 64 measurement points.

Figure 5a shows the Hankel singular values of our system. From this figure it follows that the system contains one singular value which is greater than 100 (red point in Figure 5a), five singular values which are greater than 10 (red and green points in Figure 5a), nine singular values which are greater than 1 (red, green and blue points in Figure 5a) etc.

In this example, the balanced truncation is used and the number of states of the reduced order model is chosen as $r = 13$. Figure 5b shows time response of Frobenius norm (HORN, R. A. and JOHNSON, CH. R., 1985) of the model output error. From figure it follows that the Frobenius norm reaches a steady state value. Note that the input signal of the system is unit step. The balanced truncation of this model for other numbers of states are compared in (ROUBAL, J. *et al.*, 2004).

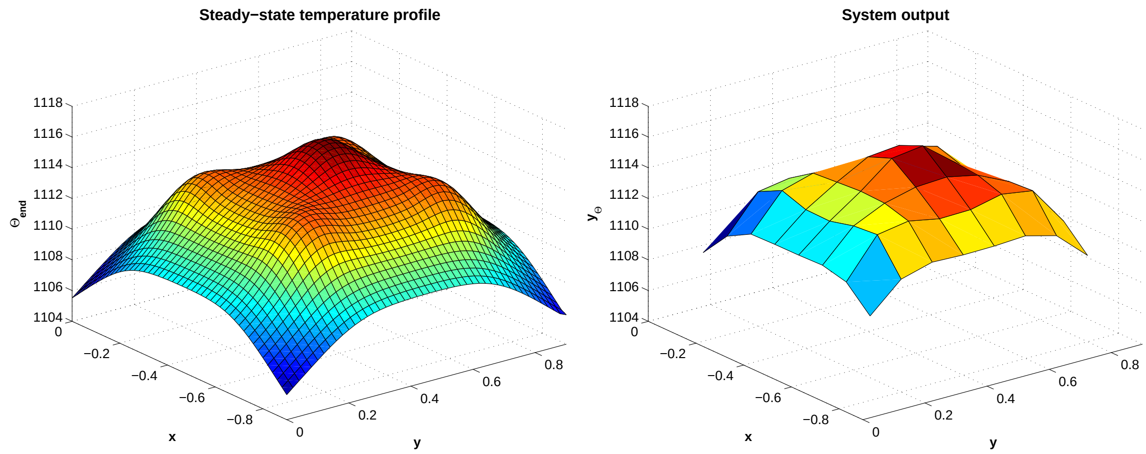

Figure 4. (a) Steady-state temperature distribution $\Theta(x, y)$; (b) System output $\mathbf{y}$

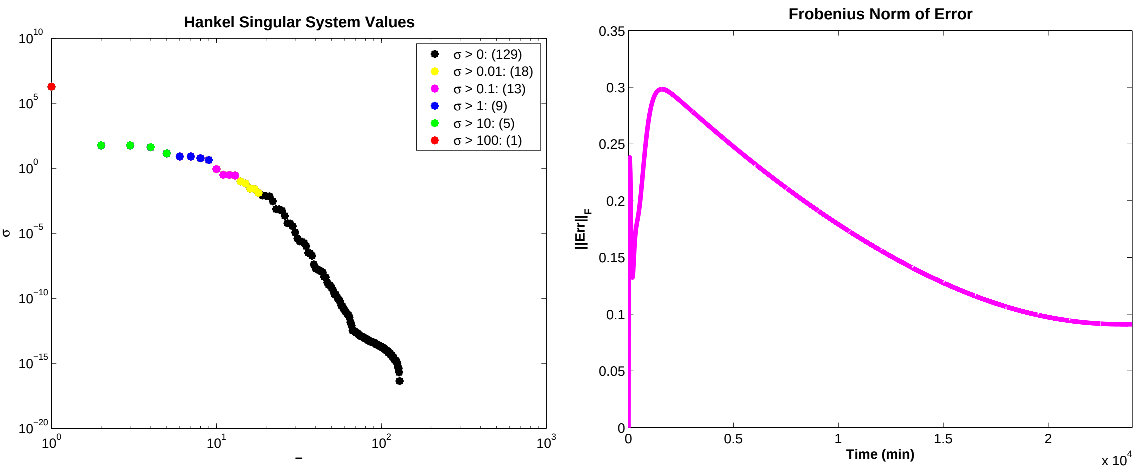

Figure 5. (a) Hankel singular values of the system; (b) Frobenius norm of the model output error

For the reduced order model, the MPC controller with the range control strategy respecting the constraints $0 \leq u \leq 5$ and $|\Delta u| \leq 1$ is designed. For the initial condition of temperature distribution as in Figure 6, the system in closed loop is controlled. The input time responses and the Frobenius norm response of control error are shown in Figure 7a. In Figure 7b, the temperature distribution, the system output with the set range reference, the control error and the reference temperature profile, respectively, are presented. Note that the system is simulated with the measurement noise σ_e.

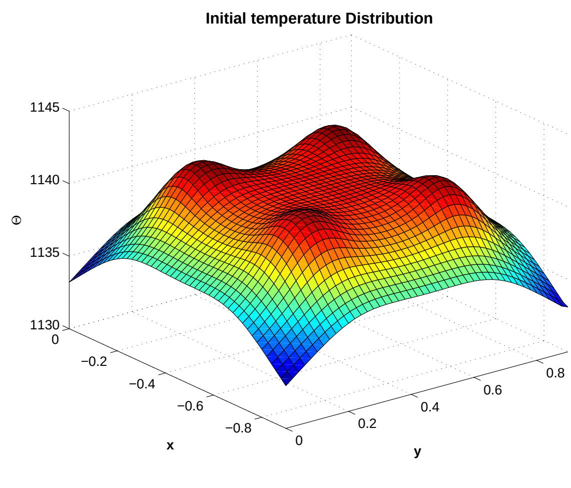

Figure 6. The initial temperature distribution

During the control simulation, the temperature on the left side of the area was changed at time 1000 minutes. The temperature was decreased by 10 K, which is represented the addition of the material for melting into the furnace.

From Figure 7a it follows that the input constraints are not violated and the Frobenius norm of the control error reaches a steady state value. The reference temperature profile is shown in Figure 7b. Note that the low and high limits of the set range reference are set to ± 2 K of this profile.

Figure 8 presents the simulation results as Figure 7 but in this case the predictive controller without the range control concept is used. If we compare Figures 7 and 8 we will observe the result which we expected. The input trajectories in Figure 7 are smoother than in Figure 8 because the manipulated inputs need not compensate the high-frequency component of the measurement noise.

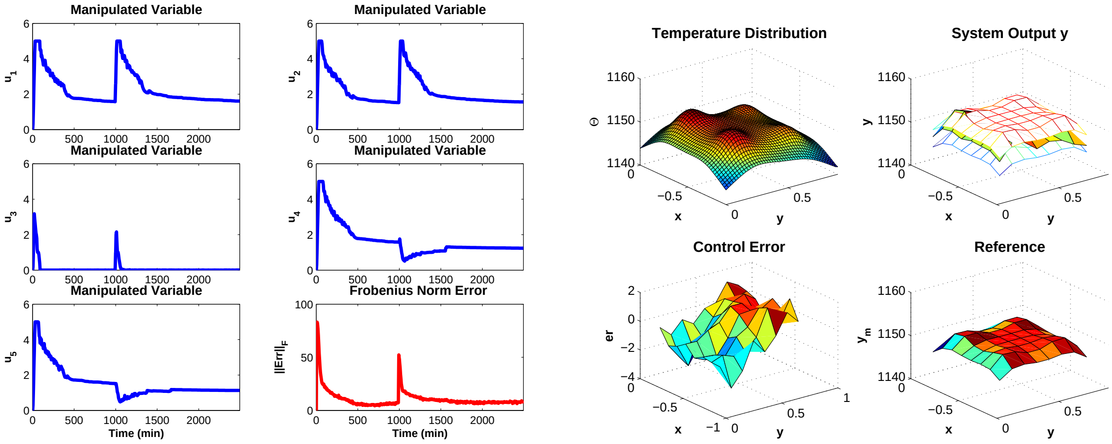

Figure 7. Range control predictive control approach: (a) The inputs time responses and Frobenius norm of control error time response; (b) Steady state temperature distribution with its reference

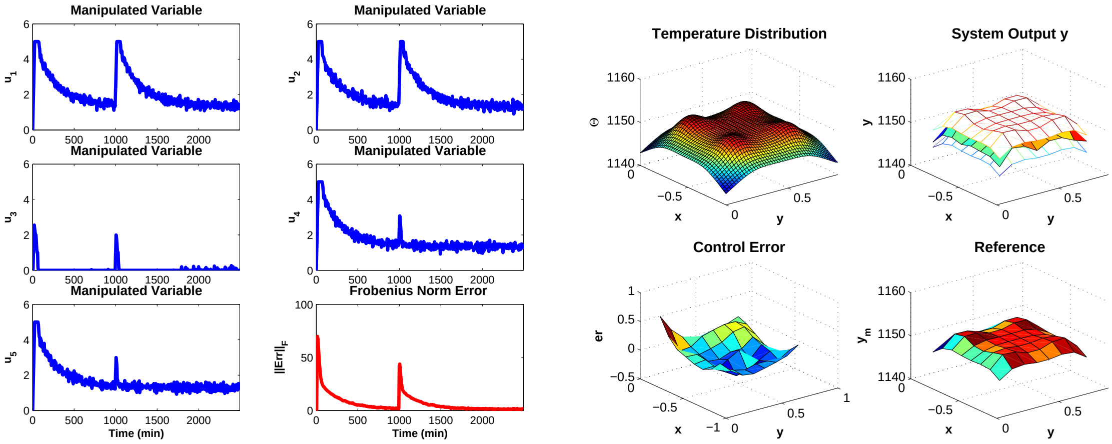

Figure 8. Classical predictive control approach: (a) The inputs time responses and Frobenius norm of control error time response; (b) Steady state temperature distribution with its reference

6. CONCLUSION

The state space model of the distributed parameters system which is described by the linear two-dimensional parabolic partial differential equation and the model reduction by the balanced truncation method are described.

The range control methodology of the model predictive control is introduced and is applied to the distributed parameter model which can describe a heat transfer process. Because of a large dimension of the model, the balanced truncation reduction method was used.

The predictive controller with the range control strategy was compared with the classical predictive control approach. The expected results was obtained. In the case of the range control approach the manipulated variables are smoother than in the classical predictive control approach because the manipulated variables do not compensate the high-frequency component of the measurement noise or inaccuracy of the model at high frequencies. The predictive controller with the range control strategy leads to more "calm control".

REFERENCES

Babuška, I., Práger, M. and Vitásek, E. (1966). *Numerical Processes in Differential Equations*. London - New York - Sydney, Interscience Publishers.

Havlena, V. and Findejs, J. (2005). Application of Model Predictive Control to Advanced Combustion Control. *Control Engineering Practice* **13** (No.6), 671–680.

Horn, R. A. and Johnson, Ch. R. (1985). *Matrix Analysis*. Cambridge University Press, United Kingdom.

Long, C. A. (1999). *Essential Heat Transfer*. Longman Group, England.

Maciejowski, J. M. (2002). *Predictive Control with Constraints*. Prentice Hall, Harlow, England.

Roubal, J., Trnka, P. and Havlena, V. (2004). Simulation of Two-Dimensional Distributed Parameter Systems. In: *Matlab Conference*. Prague, Czech Republic. ⟨http://www.humusoft.cz/matlab04/⟩.

Zhou, K., Doyle, J. C. and Glover, K. (1996). *Robust and Optimal Control*. Prentice Hall, New Jersey.