# PARAMETRIZATION OF ALL REGULARLY IMPLEMENTING CONTROLLERS

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Abstract: In this short paper we deal with problems of controller parametrization in the context of behavioral systems. Given a full plant behavior, a subbehavior of the manifest plant behavior is called *regularly implementable* if it can be achieved as the controlled behavior resulting from the interconnection of the full plant behavior with a suitable controller behavior, in such a way that the number of outputs of the full controlled behavior is equal to the sum of the number of outputs of the full plant and the number of outputs of the controller. For the full interconnection case, we establish a parametrization of *all* controllers that regularly implement a given behavior. This result is used to obtain a parametrization of all stabilizing controllers. For the partial interconnection case we present a theorem that reduces the parametrization problem to the full interconnection case.


Keywords: behaviors, regular implementability, parametrization, control by interconnection.

## 1. INTRODUCTION

An important issue in the behavioral approach to control is that of *implementability*. The concept of implementability has been succesfully applied to resolve a number of important synthesis problems in the behavioral approach to control, in particular the synthesis of dissipative systems ([Willems and Trentelman(2002)]) and the behavioral versions of the problems of pole placement and stabilization ([Belur and Trentelman(2002)]). The concept was also studied in the paper [J.C. Willems and Trentelman(2003)], for nD behaviors in [Rocha and Wood(2001)], and for general behaviors in [van der Schaft(2003)]. A nice overview can also be found in [Belur(2003)]. Implementability deals with the the issue which system behaviors can be achieved ('implemented') by interconnecting a given system with a controller, and is thus concerned with the limits of performance of a given plant. In the behavioral framework this is made precise as follows. Given is a system behavior (plant) with two types of variables, the variable $w$ to be controlled, and the variable $c$ (the control variable) that we are allowed to put constraints on. A controller for our plant behavior is an additional system behavior, called controller behavior. Interconnecting the plant with the controller simply means requiring $c$ to be an element of the controller behavior. The space of all $w$ trajectories that remain possible after interconnecting the plant behavior with the controller behavior forms the so called *manifest controlled behavior*. A behavior is called implementable (with respect to the given plant behavior) if it can be achieved as manifest controlled behavior in this way. In the contexts of synthesis of dissipative systems, pole placement and stabilization, an important role is also played by *regular implementability.* A given behavior is called regularly implementable if it can be achieved by a controller behavior such that the number of outputs of the associated full controlled behavior is equal to the sum of the number of outputs of the plant and the number of outputs of the controller. In [Willems and Trentelman(2002)], for a given plant behavior a characterization was given of all implementable behaviors and in [Belur and Trentelman(2002)] a characterization was given of all regularly implementable behaviors. Once a given behavior is (regularly) implementable, it is important to know which controller behaviors implement it. In this paper we will, for the full interconnection case, give a parametrization of all controller behaviors that regularly implement a given behavior. This then will enable us to give a parametrization of all controllers that stabilize the system by full interconnection, thus generalizing the main result of [Kuijper(1995)]. We will also show that the parametrization problem for the general partial interconnection case can be reduced to the full interconnection case.

## 2. LINEAR DIFFERENTIAL SYSTEMS

In the behavioral approach to linear systems, a dynamical system is given by a triple $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^q$ is the signal space, and the behavior $\mathfrak{B}$ is a subset of $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ (the space of all locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^q$) consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. More precisely, there exists a real polynomial matrix $R$ with $q$ columns such that $\mathfrak{B} = \{w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\}$. Here $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is understood to hold in the distributional sense. Any such dynamical system $\Sigma$ is called a *linear differential system.* The set of all linear differential systems with $q$ variables is denoted by $\mathcal{L}^q$. Since the behavior $\mathfrak{B}$ of the system $\Sigma$ is the central item, we will mostly speak about the system $\mathfrak{B} \in \mathcal{L}^q$ (instead of $\Sigma \in \mathcal{L}^q$).

The behavioral approach makes a distinction between the behavior as the space of all solutions to a set of (differential) equations, and the set of equations itself. A set of equations in terms of which the behavior is defined, is called a *representation* of the behavior. If a behavior $\mathfrak{B}$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ then we call this a kernel representation of $\mathfrak{B}$. Suppose $R$ has $p$ rows. Then the kernel representation is said to be *minimal* if every other kernel representation of $\mathfrak{B}$ has at least $p$ rows. A given kernel representation, $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is minimal if and only if the polynomial matrix $R$ has full row rank. The number of rows in any minimal kernel representation of $\mathfrak{B}$ is denoted by $\mathtt{p}(\mathfrak{B})$. This number is called the *output cardinality* of $\mathfrak{B}$. It corresponds to the number of outputs in any input/output representation of $\mathfrak{B}$ (see [Polderman and Willems(1997)]). If the behavior $\mathfrak{B}$ has the property that $\mathtt{p}(\mathfrak{B}) = q$, the number of variables (so all variables are output), then we call $\mathfrak{B}$ *autonomous.*

For autonomous behaviors we have the notion of stability: if $\mathfrak{B}$ is autonomous then it is called stable if for all $w \in \mathfrak{B}$ we have $\lim_{t\to\infty} w(t) = 0$. If $\mathfrak{B}$ is autonomous then there exists a $q \times q$ polynomial matrix $R$ with $\det(R) \neq 0$ such that $\mathfrak{B}$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. $\mathfrak{B}$ is stable if and only if $R$ is Hurwitz.

## 3. IMPLEMENTABILITY

We will now review the basic issues of control in a behavioral framework. Let $\Sigma_{\mathrm{p}} = (\mathbb{R}, \mathbb{R}^q, \mathcal{P})$ (the plant) have plant behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{q}}$. A controller is a second system $\Sigma_{\mathrm{c}} = (\mathbb{R}, \mathbb{R}^q, \mathcal{C})$ with controller behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{q}}$. The *full interconnection* of $\Sigma_{\mathrm{p}}$ and $\Sigma_{\mathrm{c}}$ is defined as the system which has the intersection $\mathcal{P} \cap \mathcal{C}$ as its behavior. This *controlled behavior* is again an element of $\mathfrak{L}^{\mathtt{q}}$. The full interconnection is called *regular* if $\mathtt{p}(\mathcal{P} \cap \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C})$. We now discus the issue of *implementability.* Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{q}}$ be a given behavior, which should be interpreted as a 'desired' behavior. A fundamental question is whether this $\mathcal{K}$ can be achieved as controlled behavior, i.e. whether there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{q}}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$. If such $\mathcal{C}$ exists, then we call $\mathcal{K}$ *implementable by full interconnection* (with respect to $\mathcal{P}$). If $\mathcal{K}$ can be achieved by regular interconnection, i.e. if there exists $\mathcal{C}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$ and $\mathtt{p}(\mathcal{P} \cap \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C})$, then we call $\mathcal{K}$ *regularly implementable by full interconnection.*

If, instead of all system variables, we only allow to interconnect the plant to a controller through a specific subset of the system variables, we speak about *partial interconnection.* In that case we have a plant with two kinds of variables, $w$ and $c$. The variables $w$ are interpreted as the variables to be controlled, the variables $c$ are those through which we can interconnect the plant to a controller (the *control variables*). More specific, assume we have a linear differential plant behavior $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{q+k}$, with system variable $(w, c)$, were $w$ takes its values in $\mathbb{R}^q$ and $c$ in $\mathbb{R}^k$. Let $\mathcal{C} \in \mathfrak{L}^k$ be a controller behavior, with variable $c$. The interconnection of $\mathcal{P}_{\mathrm{full}}$ and $\mathcal{C}$ through $c$ is defined as the system behavior $\mathcal{K}_{\mathrm{full}}(\mathcal{C}) \in \mathfrak{L}^{q+k}$, defined as

$$\mathcal{K}_{\mathrm{full}}(\mathcal{C}) = \{(w, c) \mid (w, c) \in \mathcal{P}_{\mathrm{full}} \text{ and } c \in \mathcal{C}\},$$

which is called *the full controlled behavior.* The interconnection of $\mathcal{P}_{\rm full}$ and $\mathcal{C}$ through $c$ is called regular if $\mathtt{p}(\mathcal{K}_{\rm full}(\mathcal{C})) = \mathtt{p}(\mathcal{P}_{\rm full})+\mathtt{p}(\mathcal{C})$. In addition to $\mathcal{K}_{\rm full}$, we have the *manifest controlled behavior* $\mathcal{K}(\mathcal{C})$ that is obtained by eliminating $c$ from $\mathcal{K}_{\rm full}(\mathcal{C})$:

$$\mathcal{K}(\mathcal{C}) = \{w \mid \exists\ c \text{ such that } (w,c) \in \mathcal{K}_{\rm full}\}^{\rm closure} \ .$$

Here, like elsewhere in this paper, the closure should be understood to be taken in the topology of $\mathcal{L}_1^{\rm loc}$, see [Polderman and Willems(1997)]. Again, let $\mathcal{K} \in \mathfrak{L}^{\mathtt{q}}$ be a given behavior, which should be interpreted as a 'desired' behavior. As in the full interconnection case, the question is whether this $\mathcal{K}$ can be achieved as controlled behavior, i.e. whether there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{q}}$ such that $\mathcal{K} = \mathcal{K}(\mathcal{C})$. If such $\mathcal{C}$ exists, then we call $\mathcal{K}$ *implementable by partial interconnection (through c, with respect to* $\mathcal{P}_{\rm full}$*).* If $\mathcal{K}$ can be achieved by regular partial interconnection, i.e. if there exists $\mathcal{C}$ such that $\mathcal{K} = \mathcal{K}(\mathcal{C})$ and $\mathtt{p}(\mathcal{K}_{\rm full}(\mathcal{C})) = \mathtt{p}(\mathcal{P}_{\rm full})+\mathtt{p}(\mathcal{C})$, then we call $\mathcal{K}$ *regularly imlementable by partial interconnection.*

Necessary and sufficient conditions for a given $\mathcal{K} \in \mathfrak{L}^{\mathtt{q}}$ to be (regularly) implementable by partial interconnection have been obtained in [Willems and Trentelman(2002)] as well as in [Belur and Trentelman(2002)]. These conditions are given in terms of the *manifest plant behavior* and *hidden behavior* associated with the full plant behavior $\mathcal{P}_{\rm full}$, which are defined as follows. The manifest plant behavior is obtained from $\mathcal{P}_{\rm full}$ by eliminating $c$:

$$\mathcal{P} = \{w \mid \exists\ c \text{ such that } (w,c) \in \mathcal{P}_{\rm full}\}^{\rm closure}$$

The hidden behavior consists of those $w$ trajectories that appear in $\mathcal{P}_{\rm full}$ with $c$ equal to zero:

$$\mathcal{N} = \{w \mid (w,0) \in \mathcal{P}_{\rm full}\}.$$

According to [Willems and Trentelman(2002)], a given $\mathcal{K} \in \mathfrak{L}^{\mathtt{q}}$ is implementable by partial interconnection through $c$ if and only if $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$. Conditions for regular implementability were given in [Belur and Trentelman(2002)]:

*Proposition 1.* **:** Let $\mathcal{P}_{\rm full} \in \mathfrak{L}^{q+k}$. $\mathcal{K} \in \mathfrak{L}^{\mathtt{q}}$ is regularly implementable by partial interconnection through $c$ if and only if $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$ and $\mathcal{K} + \mathcal{P}_{\rm cont} = \mathcal{P}$. Here, $\mathcal{P}_{\rm cont}$ denotes the controllable part of the manifest plant behavior $\mathcal{P}$ defined above.

The controllable part of a given behavior is defined as its largest controllable subbehavior. For details we refer to [Polderman and Willems(1997)]. In addition to characterizing all subbehaviors $\mathcal{K}$ of the manifest plant behavior $\mathcal{P}$ that are (regularly) implementable, we consider the problem of characterizing all ('desired') subbehaviors $\mathcal{K}_{\rm full}$ of the *full* plant behavior $\mathcal{P}_{\rm full}$ that are (regularly) implementable. To be precise, let $\mathcal{P}_{\rm full} \in \mathfrak{L}^{q+k}$ be a full plant behavior. Then $\mathcal{K}_{\rm full} \in \mathfrak{L}^{q+k}$ is called implementable through $c$ if there exists $\mathcal{C} \in \mathfrak{L}^{k}$ such that $\mathcal{K}_{\rm full}(\mathcal{C}) = \mathcal{K}_{\rm full}$. It is called regularly implementable through $c$ if there exists such $\mathcal{C}$ with, in addition, $\mathtt{p}(\mathcal{K}_{\rm full}(\mathcal{C})) = \mathtt{p}(\mathcal{P}_{\rm full}) + \mathtt{p}(\mathcal{C})$. We have the following theorem:

*Theorem 2.* **:** Let $\mathcal{P}_{\rm full} \in \mathfrak{L}^{q+k}$, with system variable $(w,c)$. Then $\mathcal{K}_{\rm full} \in \mathfrak{L}^{q+k}$ is implementable through $c$ if and only if

$$\mathcal{N} \times \{0\} \subseteq \mathcal{K}_{\rm full} \subseteq \mathcal{P}_{\rm full}. \tag{1}$$

$\mathcal{K}_{\rm full}$ is regularly implementable through $c$ if and only if (1) holds and

$$\mathcal{K}_{\rm full} + (\mathcal{P}_{\rm full})_{\rm cont} = \mathcal{P}_{\rm full}. \tag{2}$$

Here, $(\mathcal{P}_{\rm full})_{\rm cont}$ denotes the controllable part of $\mathcal{P}_{\rm full}$.

## 4. ALL CONTROLLERS THAT REGULARLY IMPLEMENT A GIVEN BEHAVIOR: THE FULL INTERCONNECTION CASE

Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{q}}$ be a given plant behavior. It was shown in [Belur and Trentelman(2002)] that a given $\mathcal{K} \in \mathfrak{L}^{\mathtt{q}}$ is regularly implementable by full interconnection if and only if $\mathcal{K} \subseteq \mathcal{P}$ and $\mathcal{K}+\mathcal{P}_{\rm cont} = \mathcal{P}$. where, again, $\mathcal{P}_{\rm cont}$ denotes the controllable part of $\mathcal{P}$. If this condition holds, then the question arises how to compute *all* controllers $\mathcal{C} \in \mathfrak{L}^{\mathtt{q}}$ that regularly implement the given behavior $\mathcal{K}$. In this section, we will give a parametrization of all such controllers $\mathcal{C}$. This problem was considered before in [Kuijper(1995)] for the case that the plant behavior $\mathcal{P}$ is controllable, and the given subbehavior $\mathcal{K}$ is autonomous. The approach in [Kuijper(1995)] is heavily based on the use of image representations for $\mathcal{P}$. Here, we will establish a parametrization for arbitrary $\mathcal{P}$ and arbitrary (regularly implementable) $\mathcal{K}$.

In the following, let $\mathcal{K} \in \mathfrak{L}^{\mathtt{q}}$. Let $K$ be a real polynomial matrix such that $K(\frac{\rm d}{{\rm d}t})w = 0$ is a (not necessarily minimal) kernel representation of $\mathcal{K}$. Our first goal is to find a condition on the polynomial matrix $K$ that is necessary and sufficient for $\mathcal{K}$ to be regularly implementable. In order to be able to formulate such condition, let $R(\frac{\rm d}{{\rm d}t})w = 0$ be a minimal kernel representation of $\mathcal{P}$, and let $R_1(\frac{\rm d}{{\rm d}t})w = 0$ be a minimal kernel representation of the controllable part $\mathcal{P}_{\rm cont}$ of $\mathcal{P}$.

Then, since $\mathcal{P}_{\text{cont}} \subseteq \mathcal{P}$, there exists a nonsingular polynomial matrix $D$ such that

$$R = DR_1.$$

Next, let $C_0$ be a polynomial matrix such that

$$\begin{pmatrix} R_1 \\ C_0 \end{pmatrix}$$

is a unimodular polynomial matrix (i.e. its determinant is a nonzero constant). Furthermore, let $N$ and $M$ be polynomial matrices such that

$$\begin{pmatrix} R_1 \\ C_0 \end{pmatrix} \begin{pmatrix} N & M \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & I_{q-p} \end{pmatrix},$$

where the integer $p$ is equal to the number of rows of $R$. Note that $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is then an observable image representation of the controllable part $\mathcal{P}_{\text{cont}}$ of $\mathcal{P}$. Now consider the polynomial matrix $KM$, and let $Q$ be a polynomial matrix with the following three properties:

(1) $QKM = 0$
(2) $Q$ has $\operatorname{rowdim}(KM) - \operatorname{rank}(KM)$ rows
(3) $Q(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$.

(For a given matrix 'rowdim' denotes the number of rows of the matrix.) The above three conditions together are equivalent with: $Q(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ is a minimal kernel representation of the system with image representation $v = K(\frac{\mathrm{d}}{\mathrm{d}t})M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$. Finally, let $W$ be a polynomial matrix such that

$$\begin{pmatrix} Q \\ W \end{pmatrix} \tag{3}$$

is unimodular. Note that $W$ must have its numbers of rows equal to $\operatorname{rank}(KM)$. Then the following lemma indeed gives necessary and sufficient conditions, in terms of the representing polynomial matrix $K$, for regular implementability of $\mathcal{K}$:

*Lemma 1.* **:** $\mathcal{K}$ is regularly implementable by full interconnection if and only if

$$\begin{pmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ W(\frac{\mathrm{d}}{\mathrm{d}t})K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} w = 0 \tag{4}$$

is a minimal kernel representation of $\mathcal{K}$.

We will now apply lemma 1 to establish the main result of this section. It gives, for a given regularly implementable subbehavior $\mathcal{K}$ of $\mathcal{P}$, a parametrization of all controllers that regularly implement $\mathcal{K}$.

*Theorem 3.* **:** Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{q}}$, with minimal kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{q}}$ be regularly implementable by full interconnection and let $K$ be a polynomial matrix such that $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a kernel representation of $\mathcal{K}$. Let $W$ be such that (3) is unimodular. Let $C$ be a polynomial matrix and define $\mathcal{C} \in \mathfrak{L}^{\mathtt{q}}$ as the behavior represented by $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Then the following statements are equivalent:

(1) the controller $\mathcal{C}$ has minimal kernel representation $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and regularly implements $\mathcal{K}$,
(2) there exists a polynomial matrix $F$ and a unimodular polynomial matrix $U$ such that

$$C = FR + UWK. \tag{5}$$

**Proof :** (2. $\Rightarrow$ 1.) First note that since $\mathcal{K}$ is regularly implementable, by lemma 1 the polynomial matrix $\operatorname{col}(R, WK)$ has full row rank (for given matrices $A$ and $B$ we denote by $\operatorname{col}(A, B)$ the matrix obtained by stacking $A$ over $B$). Since

$$\begin{pmatrix} I_p & 0 \\ F & U \end{pmatrix} \begin{pmatrix} R \\ WK \end{pmatrix} = \begin{pmatrix} R \\ FR + UWK \end{pmatrix}, \tag{6}$$

this implies that also $C = FR + UWK$ has full row rank, so $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a minimal representation of $\mathcal{C}$. It also follows from (6) that $\mathcal{C}$ implements $\mathcal{K}$. Clearly, the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is regular.

(1. $\Rightarrow$ 2.) Assume that $C$ has full row rank, and $\mathcal{C}$ regularly implements $\mathcal{K}$. Then

$$\begin{pmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} w = 0 \text{ and } \begin{pmatrix} R(\frac{\mathrm{d}}{\mathrm{d}t}) \\ W(\frac{\mathrm{d}}{\mathrm{d}t})K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} w = 0$$

are both a minimal representation of $\mathcal{K}$. Consequently, there exists a unimodular polynomial matrix

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

such that

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} R \\ WK \end{pmatrix} \begin{pmatrix} N & M \end{pmatrix} = \begin{pmatrix} R \\ C \end{pmatrix} \begin{pmatrix} N & M \end{pmatrix}.$$

As a consequence we obtain

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} I_p & 0 \\ WKN & WKM \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & I_{q-p} \end{pmatrix}.$$

This implies $V_{12}WKM = 0$. Since $WKM$ has full row rank, we get $V_{12} = 0$. By the above this implies that $V_{11} = I_p$, so

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ V_{21} & V_{22} \end{pmatrix}.$$

It follows that $V_{22}$ is unimodular. We also have

$$\begin{pmatrix} I_p & 0 \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} R \\ WK \end{pmatrix} = \begin{pmatrix} R \\ C \end{pmatrix},$$

from which $C = V_{21}R + V_{22}WK$. This completes the proof of the theorem $\square$

Summarizing the above, for a given plant $\mathcal{P}$ with kernel representation $R(\frac{d}{dt})w = 0$ and a given regularly implementable subbehavior $\mathcal{K} \subseteq \mathcal{P}$ with kernel representation $K(\frac{d}{dt})w = 0$, a parametrization of all controllers that regularly implement $\mathcal{K}$ is obtained in the following steps:

(1) find a polynomial matrix $M$ such that $w = M(\frac{d}{dt})\ell$ is an observable image representation of the controllable part $\mathcal{P}_{\text{cont}}$ of $\mathcal{P}$,
(2) find a polynomial matrix $Q$ such that
$$Q(\tfrac{d}{dt})v = 0$$
is a minimal kernel representation of the auxiliary system with image representation $v = K(\frac{d}{dt})M(\frac{d}{dt})\ell$,
(3) find a polynomial matrix $W$ such that
$$\begin{pmatrix} Q \\ W \end{pmatrix}$$
is unimodular,
(4) the controllers $\mathcal{C} \in \mathfrak{L}^{\mathrm{q}}$ that regularly implement $\mathcal{K}$ are parametrized by
$$\mathcal{C} = \{w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathrm{q}}) \mid (FR + UWK)(\tfrac{d}{dt})w = 0\}$$
with $F$ ranging over all polynomial matrices with $p$ columns and $r$ rows, and $U$ ranging over all unimodular $r \times r$ polynomial matrices. Here, $r := \text{rank}(KM)$.

## 5. ALL STABILIZING CONTROLLERS: THE FULL INTERCONNECTION CASE

The problem of stabilization by regular full interconnection is formulated as follows. Let $\mathcal{P} \in \mathfrak{L}^{\mathrm{q}}$ be a given plant behavior. Find a controller behavior $\mathcal{C}$ such that the controlled behavior $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$ is autonomous and stable, and the interconnection is regular. It was proven in [Willems(1997)] that there exists such stabilizing controller $\mathcal{C}$ if and only if the plant behavior $\mathcal{P}$ is stabilizable. In this section we will solve the problem of parametrizing *all* stabilizing controller behaviors for $\mathcal{P}$. This problem was considered before in [Kuijper(1995)] under the assumption that $\mathcal{P}$ is controllable.

Assume that $\mathcal{P}$ is represented by the minimal kernel representation $R(\frac{d}{dt})w = 0$, with $R(\xi)$ a real polynomial matrix. Assume that $\mathcal{P}$ is stabilizable. This is equivalent to the condition that $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}^+$ (the closed right half complex plane). The following corollary yields a parametrization of all stabilizing controllers for the stabilizable plant $\mathcal{P}$:

*Corollary 2.* **:** Let $\mathcal{P} \in \mathfrak{L}^{\mathrm{q}}$ be stabilizable. Let $R_1(\frac{d}{dt})w = 0$ be a minimal kernel representation of the controllable part $\mathcal{P}_{\text{cont}}$. Let $C_0$ be such that
$$\begin{pmatrix} R_1 \\ C_0 \end{pmatrix}$$
is unimodular. Then for any $\mathcal{C} \in \mathfrak{L}^{\mathrm{q}}$ represented by the kernel representation $C(\frac{d}{dt})w = 0$ the following statements are equivalent:

(1) $\mathcal{P} \cap \mathcal{C}$ is autonomous and stable, the interconnection is regular and the kernel representation $C(\frac{d}{dt})w = 0$ is minimal,
(2) there exist a polynomial matrix $F$, a unimodular polynomial matrix $U$, and a Hurwitz polynomial matrix $D$ such that
$$C = FR_1 + UDC_0. \tag{7}$$

**Proof :** The proof is an application of theorem 3. Because of space limitations we will omit it here. $\square$

If, in the above, we assume that $\mathcal{P}$ is controllable, then we can take $R = R_1$, and we recover the parametrization of all stabilizing controllers that was obtained in [Kuijper(1995)].

## 6. ALL CONTROLLERS THAT REGULARLY IMPLEMENT A GIVEN BEHAVIOR: THE PARTIAL INTERCONNECTION CASE

In this section we will study the partial interconnection case. Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$, with system variable $(w, c)$, were $w$ takes its values in $\mathbb{R}^q$ and $c$ in $\mathbb{R}^k$. From section 3, for a given controller behavior $\mathcal{C} \in \mathfrak{L}^k$ recall the definition of the full controlled behavior $\mathcal{K}_{\text{full}}(\mathcal{C}) \in \mathfrak{L}^{q+k}$. Recall also the conditions on a given $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{q+k}$ to be regularly implementable (see theorem 2). In this section we will consider the problem of parametrizing, for a given regularly implementable $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{q+k}$, all controllers $\mathcal{C} \in \mathfrak{L}^k$ that regularly implement $\mathcal{K}_{\text{full}}$.

Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$, and let $\mathcal{K}_{\text{full}}$ be implementable through $c$. We will first consider the problem of finding a controller $\mathcal{C} \in \mathfrak{L}^k$ that implements $\mathcal{K}_{\text{full}}$. We will derive a representation of one such controller in terms of representations of $\mathcal{P}_{\text{full}}$ and $\mathcal{K}_{\text{full}}$. Let $R_1(\frac{d}{dt})w + R_2(\frac{d}{dt})c = 0$ and $K_1(\frac{d}{dt})w + K_2(\frac{d}{dt})c = 0$ be kernel representations of $\mathcal{P}_{\text{full}}$ and $\mathcal{K}_{\text{full}}$, respectively. Clearly, $R_1(\frac{d}{dt})w = 0$ is a representation of $\mathcal{N}$. Since $\mathcal{N} \times \{0\} \subseteq \mathcal{K}_{\text{full}}$ we have that, for all $w$, $R_1(\frac{d}{dt})w = 0$ implies

$K_1(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. As a consequence, there exists a polynomial matrix $F_1$ such that $K_1 = F_1 R_1$. Now define a controller behavior $\mathcal{C}^* \in \mathfrak{L}^k$ by

$$\mathcal{C}^* := \{c \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k) | (K_2 - F_1 R_2)(\tfrac{\mathrm{d}}{\mathrm{d}t})c = 0\} \quad (8)$$

This controller behavior indeed implements $\mathcal{K}_{\mathrm{full}}$ through $c$:

*Lemma 3.* **:** $\mathcal{K}_{\mathrm{full}} = \mathcal{K}_{\mathrm{full}}(\mathcal{C}^*)$.

Given $\mathcal{K}_{\mathrm{full}} \in \mathfrak{L}^{q+k}$, we denote by $\mathcal{K}_c$ the behavior obtained by elimimating $w$. Likewise, $\mathcal{P}_c$ denotes the behavior obtained from $\mathcal{P}_{\mathrm{full}}$ by eliminating $w$:

$$\mathcal{K}_c = \{c \mid \exists\ w \text{ such that } (w, c) \in \mathcal{K}_{\mathrm{full}}\}^{\mathrm{closure}} \quad .$$

$$\mathcal{P}_c = \{c \mid \exists\ w \text{ such that } (w, c) \in \mathcal{P}_{\mathrm{full}}\}^{\mathrm{closure}} \quad .$$

The following theorem reduces the problem of parametrizing all controllers that regularly implement $\mathcal{K}_{\mathrm{full}}$ via *partial* interconnection to that of parametrizing all controllers that regularly implement $\mathcal{K}_c$ via *full* interconnection. Due to space limitations, the proof is omitted.

*Theorem 4.* **:** Let $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{q+k}$. Let $\mathcal{K}_{\mathrm{full}} \in \mathfrak{L}^{q+k}$ be regularly implementable. Let $\mathcal{C} \in \mathfrak{L}^k$. Then $\mathcal{C}$ regularly implements $\mathcal{K}_{\mathrm{full}}$ via interconnection through $c$ if and only if $\mathcal{C}$ regularly implements $\mathcal{K}_c$ via full interconnection with $\mathcal{P}_c$. In other words, $\mathcal{K}_{\mathrm{full}}(\mathcal{C}) = \mathcal{K}_{\mathrm{full}}$ and $\mathtt{p}(\mathcal{K}_{\mathrm{full}}(\mathcal{C})) = \mathtt{p}(\mathcal{C}) + \mathtt{p}(\mathcal{P}_{\mathrm{full}})$ if and only if $K(\mathcal{C}) = \mathcal{K}_c$ and $\mathtt{p}(\mathcal{K}(\mathcal{C})) = \mathtt{p}(\mathcal{C}) + \mathtt{p}(\mathcal{P}_c)$.

In this way, the problem of parametrizing all controllers that regularly implement $\mathcal{K}_{\mathrm{full}}$ via interconnection through $c$ w.r.t. $\mathcal{P}_{\mathrm{full}}$ is reduced to the problem of parametrizing all controllers that regularly implement $\mathcal{K}_c$ by full interconnection w.r.t. $\mathcal{P}_c$. The latter problem was already solved in section 4. In the following it will be outlined how such parametrization can be obtained in terms of the polynomial matrices representing $\mathcal{P}_{\mathrm{full}}$ and $\mathcal{K}_{\mathrm{full}}$.

Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ and $K_1(\frac{\mathrm{d}}{\mathrm{d}t})w + K_2(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be kernel representations of $\mathcal{P}_{\mathrm{full}}$ and $\mathcal{K}_{\mathrm{full}}$, respectively. In order to be able to apply theorem 4, we compute representations of $\mathcal{P}_c$ and $\mathcal{K}_c$. One way to do this is to first eliminate $w$ from $\mathcal{P}_{\mathrm{full}}$ as follows: let $U$ be a unimodular matrix such that

$$UR_1 = \begin{pmatrix} R_{11} \\ 0 \end{pmatrix}$$

and such that $R_{11}$ has full row rank. Likewise, partition

$$UR_2 = \begin{pmatrix} R_{12} \\ R_{22} \end{pmatrix}.$$

It is then easily seen that $\mathcal{P}_c$ has kernel representation

$$R_{22}(\tfrac{\mathrm{d}}{\mathrm{d}t})c = 0. \quad (9)$$

By lemma 3, $\mathcal{K}_{\mathrm{full}}$ is represented by

$$\begin{pmatrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t}) & R_2(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & (K_2 - F_1 R_2)(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} \begin{pmatrix} w \\ c \end{pmatrix} = 0,$$

where $F_1$ is such that $K_1 = F_1 R_1$. Hence, a kernel representation of $\mathcal{K}_c$ is given by

$$\begin{pmatrix} R_{22}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ (K_2 - F_1 R_2)(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} c = 0. \quad (10)$$

In order to parametrize all controllers that regularly implement $\mathcal{K}_c$ by full interconnection w.r.t. $\mathcal{P}_c$ one can now apply theorem 3 to $\mathcal{P}_c$ and $\mathcal{K}_c$ represented respectively by (9) and (10). For details we refer to future work.

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