ON ε–INVARIANCE OF NONLINEAR SYSTEMS WITH FUNCTIONAL AND SIGNAL UNCERTAINTIES

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Abstract: Switching adaptive control scheme is developed, which provides global asymptotic stability property with respect to state vector of the plant with functional uncertainty. In the presence of disturbance the scheme ensures boundedness of all trajectories of the system. Under some mild conditions proposed solution ensures convergence of state vector of the plant to specified neighborhood of the origin for arbitrary initial conditions and bounded disturbances.

Keywords: adaptive control, disturbance rejection, robustness, nonlinear systems, switching.

1. INTRODUCTION

A considerable progress has been achieved during the last decade in solution of the problem of stabilization of nonlinear systems with known and unknown parameters under acting external disturbances. New design tools such as robust adaptive feedback linearization (Cambion, and Bastin, 1991; Kanellakopoulos, *et al*., 1991a; Kanellakopoulos, *et al*., 1991b; Sastry, and Isidori, 1989), robust adaptive backstepping (Krstić, *et al*., 1995; Voronov, *et al*., 2001), control Lyapunov functions (CLFs) (Sontag, 1989b; Freeman, and Kokotović, 1996; Liberzon, *et al*., 2001), passivation approach (Fradkov, *et al*., 1999) and switching adaptive control (Efimov, 2003; Kosmatopoulus and Ioannou, 2002) have been introduced. Using these new design tools, globally stabilizing controllers have been constructed for various classes of nonlinear systems, which typically provide boundedness of the system solution for bounded disturbances and asymptotic stability of the plant for vanishing disturbance. Despite the success of the aforementioned design tools to resolve a variety of adaptive control problems for nonlinear systems, the problem of adaptive control of nonlinear systems with functional uncertainty (Bobtsov, and Efimov, 2003; Haddad *et al*., 2001; Kosmatopoulus and Ioannou, 2002; Lin and Qian, 2001; Nikiforov, 1997) still has not advanced solution. For example, in the most of the cited before papers, which deal with functional uncertainty, were not analyzed the influence on stability properties of external disturbances appearance and, that is more, only practical stability is guaranteed for vanishing disturbances. Or, like in paper Kosmatopoulus and Ioannou (2002), authors supposed that all uncertainties are linearly included in equations of the plant. In recent work (Bobtsov, and Efimov, 2003) all this obstacles were vanquished, but with too restrictive assumptions imposed on the plant equations.

In this paper, we extend the result of (Bobtsov, and Efimov, 2003) for designing robust adaptive controllers for a large class of multi-input nonlinear systems with exogenous bounded input disturbances and functional uncertainty. Both can appear in nonlinear fashion in plant equations. The class of systems for which the proposed approach is applicable is characterized by the assumption that a robust CLF for the system is known and it admits some mild conditions. Additionally in the paper it is shown, that under suitable modification of result of (Bobtsov, and Efimov, 2003) an ε-invariance property (Bobtsov, 2003; Fomin, *et al*., 1981) can be assigned to the plant. In other words it does not matter how big is the amplitude of input disturbance and initial conditions of the system, all trajectories are attracted to predefined neighborhood of the origin (a ball of radios ε) uniformly with respect to functional uncertainty.

In the second section definitions and statements are presented. The main result is described and proven in the Section 3. Conclusion finishes the paper.

2. DEFINITIONS AND STATEMENTS

Let us consider nonlinear dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \boldsymbol{\omega}\left(\mathbf{x}, t\right)\right) + \mathbf{G}\left(\mathbf{x}\right)\mathbf{u}\,, \qquad (1)$$

where $\mathbf{x} \in R^n$ is state vector, $\mathbf{u} \in R^m$ is control; $\mathbf{f}(\mathbf{x}, \boldsymbol{\omega})$ and the columns of $\mathbf{G}(\mathbf{x})$ are continuous and locally Lipschitz vector fields on R^n, $\mathbf{f}(0, \boldsymbol{\omega}) = 0$ for any $\boldsymbol{\omega} \in R^p$; $\boldsymbol{\omega}(\mathbf{x}, t)$ is unknown vector function representing functional uncertainty of system. Let us introduce main restrictions on the system uncertainty.

A s s u m p t i o n 1. *There are an unknown constant*

$m\in[0,+\infty)$, *unknown Lebesgue measurable and essentially bounded signal* $\mathbf{w}:R_{\geq 0}\to R_{\geq 0}$ *and a known function* $r:R^n\to R_{\geq 0}$, *such, that for all* $t\geq 0$

$$|\boldsymbol{\omega}(\mathbf{x},t)|\leq m\ r(\mathbf{x})+\mathbf{w}(t),\ \mathbf{x}\in R^n,\ r(0)=0.\ \square$$

Assumption 1 does not imply boundedness of function $\boldsymbol{\omega}(\mathbf{x},t)$ with respect to variable $\mathbf{x}$. The presence of unknown parameter m leads to necessity of adaptive controller construction. Opposite to classical adaptive control theory (Fomin, *et al.*, 1981; Fradkov, *et al.*, 1999; Krstic, *et al.*, 1995) here there is no assumption on compactness of admissible values set for unknown parameter m. Such complication allows to take into account the presence of unmodeled dynamics, nonlinear parameterization of the plant equations by vector of unknown parameters, and even case with unknown function $r(\mathbf{x})$ under suitable approximation (see paper (Bobtsov, and Efimov, 2003) for detailed explanation of these connections).

Further, we suppose, that for system (1) some differentiable Lyapunov function $V:R^n\to R_{\geq 0}$ is given:

$$\dot V=L_{\mathbf{f}(\mathbf{x},\mathbf{d})}V(\mathbf{x})+L_{\mathbf{G}}V(\mathbf{x})\mathbf{u},$$

where $L_{\mathbf{f}}V(\mathbf{x})=\partial V(\mathbf{x})/\partial\mathbf{x}\ \mathbf{f}(\mathbf{x},\mathbf{d})$,

$$L_{\mathbf{G}}V(\mathbf{x})=\partial V(\mathbf{x})/\partial\mathbf{x}\ \mathbf{G}(\mathbf{x}).$$

According to Lemma 2.1 in (Lin and Qian, 2001), Lemma 9 in (Sontag,1998) or discussion in section 4 of paper (Liberzon, *et al.*, 2001), the first term of above expression can be majorized as follows

$$L_{\mathbf{f}(\mathbf{x},\mathbf{d})}V(\mathbf{x})\leq a(\mathbf{x})+\chi(|\mathbf{d}|),$$

where a is some continuous function, $a(0)=0$, $\chi\in\mathcal{K}_\infty$ (definitions of classes $\mathcal{K}$ and $\mathcal{K}_\infty$ are standard (Sontag, 1989b)). Utilizing the same arguments and substituting upper bound of function ω from Assumption 1 in function χ it is possible to introduce new designations:

$$\chi(|\boldsymbol{\omega}(\mathbf{x},t)|)\leq\mu\ \rho(\mathbf{x})+\delta(|\mathbf{w}(t)|),$$

where μ is a new unknown constant and ρ is some new known continuous function ($\rho(0)=0$), which is dependent on functions r and χ; $\delta\in\mathcal{K}$. Hence:

$$\dot V\leq a(\mathbf{x})+\mu\ \rho(\mathbf{x})+L_{\mathbf{G}}V(\mathbf{x})\mathbf{u}+\delta(|\mathbf{w}(t)|).\quad (2)$$

It is necessary to note, that under property

$$|L_{\mathbf{G}}V(\mathbf{x})|\equiv 0\ \Rightarrow\ a(\mathbf{x})+\mu\rho(\mathbf{x})<0$$

this function V becomes input-to-state stable (ISS) CLF (see (Sontag and Wang, 1995a; Liberzon, *et al.*, 2001; Efimov, 2002a) for these terms introductions) for the system (1). This fact is equivalent to ISS stabilization of system (1) by state feedback, that will be exploited further in the paper. So, let us define the main requirement to properties of system (1).

A s s u m p t i o n 2. *There exists a differentiable Lyapunov function* $V(\mathbf{x})$, *such, that*

$$\alpha_1(|\mathbf{x}|)\leq V(\mathbf{x})\leq\alpha_2(|\mathbf{x}|)$$

for some functions $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, *and inequality (2) holds for all* $\mathbf{x}\neq 0$ *and any* $\mu\geq 0$ *with properties:*

1. $|L_{\mathbf{G}}V(\mathbf{x})|\equiv 0\ \Rightarrow\ a(\mathbf{x})+\mu\rho(\mathbf{x})<-\alpha(|\mathbf{x}|)$, $\alpha\in\mathcal{K}_\infty$;
2. $\lim\limits_{|\mathbf{x}|\to 0}\sup\ [a(\mathbf{x})+\mu\rho(\mathbf{x})]/|L_{\mathbf{G}}V(\mathbf{x})|\leq 0$. $\square$

It is worth to note, that the first property of Assumption 2 supposes, that on the set where control can not affect on the dynamics of system (1) (i.e. on the set, where $|L_{\mathbf{G}}V(\mathbf{x})|\equiv 0$) this system is asymptotically stable for *any* admissible values of μ. In fact, if there exists some another adaptive controller which solves posed problem with some Lyapunov function V, then again we should meet this constrain: on subset $|L_{\mathbf{G}}V(\mathbf{x})|\equiv 0$ controller can not affect on sign of time derivative of function V.

An example of class of systems which possess conditions of this assumption is the following one with *input* functional uncertainty:

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})+\mathbf{G}(\mathbf{x})[\boldsymbol{\omega}(\mathbf{x},t)+\mathbf{u}+\mathbf{w}_1(t)].\quad (3)$$

For system (3) property 1 of Assumption 2 takes form

$$|L_{\mathbf{G}}V(\mathbf{x})|\equiv 0\ \Rightarrow\ L_{\mathbf{f}}V(\mathbf{x})=a(\mathbf{x})<0\ \text{ for all }\ \mathbf{x}\neq 0,$$

the last fact simply means that system (3) can be asymptotically stabilised by (continuous) state feedback if functional uncertainty $\boldsymbol{\omega}$ and external disturbance $\mathbf{w}$ are missing (for details see Sontag (1989b), another condition for system (3) to be asymptotically stabilised by state feedback was presented in (Efimov, 2002b)). The second property of Assumption 2 is called small control property, which is included to base continuity of control law at point $\mathbf{x}=0$ (Liberzon, *et al.*, 2001; Efimov, 2002a).

Function ρ by construction is a continuous one, so there exists function $\rho_1\in\mathcal{K}_\infty$, such, that for all $\mathbf{x}\in R^n$ inequality $\rho(\mathbf{x})\leq\rho_1(|\mathbf{x}|)$ holds. To use function ρ_1 instead of ρ in control algorithms the following property will be utilized.

A s s u m p t i o n 3. $\rho(\mathbf{x})=\rho_1(|\mathbf{x}|)$, $\rho_1\in\mathcal{K}_\infty$. $\square$

According to these statements, the solving problem consists in development of adaptive controller, which provides for any initial condition $\mathbf{x}_0\in R^n$ and constant $m\geq 0$:

- asymptotic stability of (1) then $\mathbf{w}(t)\equiv 0$, $t\geq 0$;
- boundedness trajectories of overall system for any L_∞ bounded $\mathbf{w}(t)$.
- for arbitrary $\varepsilon>0$, for any finite initial conditions of the overall system and for any essentially bounded input $\mathbf{w}(t)$ should exist $T>0$, such, that $|\mathbf{x}(t)|\leq\varepsilon$ for $t\geq T$ (ε-invariance property for the plant).

3. MAIN RESULTS

For this purpose we will use the theory of input-to-state stable systems (Sontag, 1989a; Sontag, 1995). It is worth to stress, that ISS system has global asymptotic stability property for vanishing input and for any bounded inputs trajectories stay asymptotically bounded by L_∞ norm of the input. So, main characteristics of ISS property coincide with requirements formulated in control goal. Hence, it is possible to base solution of the task on ISS theory using.

3.1. Non adaptive control

At first, for the sake of simplicity, let us assume that constant m from Assumption 1 is known. It is required to design a control law ensuring a input-to-state stability of a system (1) for any function $\boldsymbol{\omega}$ that satisfies Assumption 1. It is well known "universal" control (Liberzon, *et al.*, 2001; Wang, 1996), which provides for any function ρ global asymptotic stability of system (1) (robust stability with specified stability margin ρ):

$$\mathbf{u}=-\kappa_1(\psi(\mathbf{x}),\beta(\mathbf{x}))L_{\mathbf{G}}V(\mathbf{x})^T, \qquad (4)$$

where $\psi(\mathbf{x})=a(\mathbf{x})+\mu\rho(\mathbf{x})$, $\beta(\mathbf{x})=|L_{\mathbf{G}}V(\mathbf{x})|$,

$$\kappa_1(r,s)=\begin{cases}\dfrac{r+\sqrt{r^2+s^4}}{s^2}, if\ s\neq 0;\\ 0, if\ s=0.\end{cases}$$

For the task of global asymptotic stabilisation of system (1) while disturbance $\mathbf{w}$ is missing such controls were formulated in papers (Sontag, 1989b ; Wang, 1996). The extension of these results on the problem IOS stabilisation was proposed in (Efimov, 2002a).

The ISS property uniformly with respect to uncertain function $\boldsymbol{\omega}$ for system (1), (4) follows by ISS-Lyapunov function candidate V analysis. Substitute in (2) control (4). According to Assumption 2, for $|L_{\mathbf{G}}V(\mathbf{x})|=0$ term $a(\mathbf{x})+\mu\rho(\mathbf{x})$ is negatively definite and radially unbounded. So, let $|L_{\mathbf{G}}V(\mathbf{x})|\neq 0$, then inequality (2) takes form:

$$\dot{V}\le -\sqrt{[a(\mathbf{x})+\mu\rho(\mathbf{x})]^2+|L_{\mathbf{G}}V|^4}+\delta(|\mathbf{w}|).$$

The square root function is concave one, and, hence, inequality $\sqrt{a+b}\ge\sqrt{0.5a}+\sqrt{0.5b}$ is satisfied, thus

$$\dot{V}\le -\sqrt{a(\mathbf{x})^2+|L_{\mathbf{G}}V|^4}-\mu\rho(\mathbf{x})+\delta(|\mathbf{w}|).$$

Due to Assumption 2 (the first property):

$$\sqrt{a(\mathbf{x})^2+|L_{\mathbf{G}}V|^4}\ge\alpha(|\mathbf{x}|)$$

for some $\alpha\in\mathcal{K}_\infty$ and finally we obtain

$$\dot{V}\le -\alpha(|\mathbf{x}|)-\mu\rho(\mathbf{x})+\delta(|\mathbf{w}|).$$

That is sufficient to conclude that V is an ISS-Lyapunov function and system (1), (4) is ISS (Sontag and Wang, 1995b). If Assumption 3 holds, then

$$\dot{V}\le -\alpha(|\mathbf{x}|)-\mu\rho_1(|\mathbf{x}|)+\delta(|\mathbf{w}|)$$

and ISS system obtains robust stability margin dependent on $\mu\rho_1(\cdot)$.

3.2. Adaptive and robust adaptive controls

Suppose, that constant m from Assumption 1 is unknown and, hence, constant μ is unknown too. Then for system (1) control laws (4) should be modified as follows:

$$\mathbf{u}=-\kappa_1(\psi(\mathbf{x}),\beta(\mathbf{x}))L_{\mathbf{G}}V(\mathbf{x})^T, \qquad (5)$$

where $\psi(\mathbf{x})=a(\mathbf{x})+\hat{\mu}\rho(\mathbf{x})$; $\hat{\mu}$ is adjustable parameter, estimation of unknown constant μ. Updating algorithm for parameter $\hat{\mu}$ is selected in the following way:

$$\dot{\hat{\mu}}=\gamma\rho(\mathbf{x}), \qquad (6)$$

where $\gamma>0$ is a design parameter.

L e m m a 1 . L*et Assumptions 1 and 2 be true. Then system (1), (5), (6) possesses the following properties uniformly with respect to functional uncertainty* $\boldsymbol{\omega}$ *satisfied Assumption 1:*

1. *Forward completeness for any essentially bounded and Lebesgue measurable disturbance* $\mathbf{w}$.
2. *Asymptotic gain property with respect to* $\mathbf{x}$ *:*

$$\sup\lim_{t\to+\infty}|\mathbf{x}(t)|\le\gamma(\|\mathbf{w}\|),$$

where function $\gamma(s)=\alpha_1^{-1}\circ\alpha_2\circ\alpha^{-1}\circ\delta(s)$ *belongs to class* $\mathcal{K}$, $\|\mathbf{w}\|=\operatorname{ess\,sup}_{t\ge 0}\{|\mathbf{w}(t)|\}$.

3. *If Assumption 3 is true, then for any* $\varepsilon>0$, *for any initial conditions* $\mathbf{x}(0)\in R^n$, $\hat{\mu}(0)\in R_{\ge 0}$ *and any* $\|\mathbf{w}\|<+\infty$ *there exists*

$$T=T(\varepsilon,\mathbf{x}(0),\hat{\mu}(0),\|\mathbf{w}\|)>0$$

such, that $|\mathbf{x}(t)|\le\varepsilon$ *for all* $t\ge T$.

P r o o f . The proof of the Lemma can be found in paper (Bobtsov, and Efimov, 2003) with minimum modifications. ■

If function ρ is separated from zero, then variable $\hat{\mu}$ can infinitely increase. Therefore, the control (5) improves its robust stabilization ability. This increasing is the cost of disturbance attenuation in adaptive system. To compensate this shortage in paper (Bobtsov, and Efimov, 2003) an additional negative feedback in parameter updating algorithm (6) was introduced:

$$\dot{\hat{\mu}}=\gamma\rho(\mathbf{x})-\gamma k\hat{\mu}, \qquad (9)$$

where $k>0$. Unfortunately algorithm (9) leads to a steady state error in the system response provided by constant μ, even then disturbance is vanishing.

R e m a r k 1 . Value of constant μ does not belong to some known compact set. Therefore, to add robust properties in the system we can not borrow from (Pomet, and Praly, 1992) projection modification of algorithm (6) instead of (9). □

According to the result of Lemma above, starting from some time instant $T>0$ further increasing of $\hat{\mu}$ does not add something to system properties. Indeed, in common case if $\hat{\mu}(t)\ge\mu$ or with Assumption 3 $\hat{\mu}(t)\ge\mu_\varepsilon$ for $t\ge T$, then prospective stabilization goal is reached and further growth of control amplitude is not desirable (μ_ε is value of $\hat{\mu}$ for which plant state converges to ε vicinity of the origin). Thus stopping of $\hat{\mu}$ increasing after $t=T$ can ensure overall boundedness of system trajectories. In paper (Bobtsov, and Efimov, 2003) a switching scheme was developed, which utilized described above ideas. In supervision algorithm in work (Bobtsov, and Efimov, 2003) the information about function V time derivative is required, that seriously complicate application of that result. Here we will improve switching adaptive algorithm from (Bobtsov, and Efimov, 2003) to provide some new results and overcome this obstacle.

3.3. Switching adaptive control

In Lemma 1 it was established, that there exists a moment of time $T \geq 0$, after which $\hat{\mu}(t) \geq \mu$ and algorithm of adaptation (6) can be switched off (or $\hat{\mu}(t) \geq \mu_\varepsilon$ for case of Assumption 3 satisfying). So, new algorithm of adaptation can be formalised as follows for $\gamma > 0$:

$$\dot{\hat{\mu}} = F_i(\mathbf{x}),\ i = 1,2\ ;\ F_1(\mathbf{x}) = \gamma\rho(\mathbf{x})\ ;\ F_2(\mathbf{x}) = 0\ . \quad (10)$$

Control system with adaptation algorithm (10) becomes a switching one, where signal $i(t) \in \{1,2\}$ describes a current dynamics of variable $\hat{\mu}(t)$. While $i(t) = 1$ dynamics of system (1), (5), (10) possesses properties, which were established in Lemma 1: forward completeness, global asymptotic stability of variable $\mathbf{x}$ for vanishing disturbance $\mathbf{w}$. For $i(t) = 2$ this system becomes equivalent to non adaptive system (1), (5) with some frozen value of variable $\hat{\mu}(t)$. If it happens, that $i(t) = 2$ but still $\hat{\mu}(t) < \mu$, then the behaviour of the system is unknown and should be investigated. Then $i(t) = 2$ and $\hat{\mu}(t) \geq \mu$, inequality (8) would be true and, like in non adaptive case, system recovers ISS property.

While $\hat{\mu}(t) \geq \mu$ or $\hat{\mu}(t) \geq \mu_\varepsilon$, inequality (8) is satisfied and plant state space vector is bounded and the following property is true

$$\mathbf{x}(t) \in \Theta\ ,\ \Theta = \{\mathbf{x} : |\mathbf{x}| \leq \varepsilon\}, \quad (11)$$

where $\varepsilon = \alpha_1^{-1} \circ \alpha_2 \circ \alpha^{-1} \circ \delta(W_{\max})$ and $W_{\max}$ is an upper bound of external disturbance $\mathbf{w}$, i.e. $|\mathbf{w}(t)| \leq W_{\max}$ for almost all $t \geq 0$. Such constant $W_{\max}$ possibly unknown always exists according to suppositions posed on signal $\mathbf{w}(t)$. If Assumption 3 is satisfied, then $\varepsilon > 0$ in (11) can be picked up arbitrary. Property (11) helps to design a supervisor. In this work so-called *dwell time* technique (Morse, 1995) will be used to prevent *chattering regime* arising. So, supervision algorithm can be described as follows:

$$i(t) = \begin{cases} i(t_k) \text{ if } \tau < \tau_D\ ; \\ \begin{cases} 1, & \text{if } \mathbf{x}(t) \notin \Theta; \\ 2, & \text{if } \mathbf{x}(t) \in \Theta; \end{cases} \text{ if } \tau \geq \tau_D\ ; \end{cases} \quad (12)$$

$$\dot{\tau} = 1,\ \tau(t_k) = 0,$$

where auxiliary variable τ represents internal supervisor timer dynamics, $\tau_D > 0$ is dwell time constant and t_k, $k = 0,1,2,...$ are instants of switching (instants then signal $i(t)$ changes its value), k is number of current switching. The operating of algorithm (12) can be explaining in the following way: after each switching internal timer τ is initialised to zero. While $\tau < \tau_D$ signal $i(t)$ does not change its value. Dwell time presence in algorithm (12) help us to prevent fast switching arising in the system (1), (5), (10), (12). After dwell time signal $i(t)$ can be set up to 1, if property (11) does not satisfy and, consequently, variable $\mathbf{x}$ is not bounded; signal $i(t)$ would be set up to 2, if variable $\mathbf{x}$ is bounded. The knowledge of constant $W_{\max}$ is supposed in algorithm (12) for case when Assumption 3 is not satisfied. Properties of proposed switched system are summarised in the following theorem.

T h e o r e m 1. *Let Assumptions 1 and 2 be true. Then system (1), (5), (10), (12) has*
a) *asymptotically bounded solution* $\tilde{\mathbf{x}}$ *: there exists* $T_1 > 0$*, such, that*

$$\mathbf{x}(t) \in \Theta,\ \hat{\mu}(t) = const\ ,\ t \geq T_1\ ;$$

b) *if, additionally, for each fixed* $\hat{\mu} < \mu$ *system (1), (5) possesses unbounded solution, i.e. for each* $\varepsilon_0 > 0$ *there exists an* $T_\varepsilon > 0$*, such, that* $|\mathbf{x}(t)| > \varepsilon_0$ *for* $t > T_\varepsilon$*, then asymptotic gain property with respect to variable* $\mathbf{x}(t)$ *holds for the system :*

$$\sup \lim_{t \to +\infty} |\mathbf{x}(t)| \leq \gamma(\|\mathbf{w}\|),\ \gamma \in \mathcal{K}\,.$$

If also Assumption 3 holds, then constant $\varepsilon > 0$ *in definition of the set* Θ *(11) can be chosen arbitrary.*

P r o o f. Presence of dwell time in algorithm (12) bounds number of switchings $N_{[t_A, t_B]}$ on any time interval $[t_A, t_B]$ in the following obvious way:

$$N_{[t_A, t_B]} \leq \frac{t_B - t_A}{\tau_D} + 1\,.$$

Therefore, the solution of the system is well defined at the least locally and absolutely continuous (Filippov, 1988; Morse, 1995).

The advance of switched system consists in possibility of system dynamics analysing without supervisor system behaviour consideration, i.e. at first we can investigate properties of switched system (1), (5), (10) for frozen values of signal $i(t)$ and after that analyse the influence of algorithm (12). Let us consider three situations or, better to say, three sets of time instants Λ_i, $i = \overline{1,3}$. Where $\Lambda_1 = \{t : i(t) = 1\}$ and if $t \in \Lambda_1$, then system (1), (5), (10) admits all properties claimed in Lemma 1. Let $\Lambda_2 = \{t : i(t) = 2 \text{ and } \hat{\mu}(t) < \mu^*\}$ where

$$\mu^* = \max\{\mu, \mu_\varepsilon\}\,.$$

Then solution of system (1), (5) is bounded for $t \in \Lambda_2$. Indeed, by construction in this case $\mathbf{x}(t) \in \Theta$ and $\hat{\mu}(t) \leq \mu^*$ for $t \in \Lambda_2$. Note also, that case $\mu^* = \mu_\varepsilon$ may be included into consideration only if Assumption 3 holds. Finally let $t \in \Lambda_3$, where $\Lambda_3 = \{t : i(t) = 2 \text{ and } \hat{\mu}(t) \geq \mu^*\}$. Then system (1), (5) is ISS. Note, that if $i(t) = 2$, then it is possible to analyse system (1), (5) with some fixed value of $\hat{\mu}(t)$ instead of whole system (1), (5), (10). In this situation time derivative of Lyapunov function candidate (7) can be rewritten as follows:

$$\dot{U} \leq a(\mathbf{x}) + \mu\,\rho(\mathbf{x}) + L_{\mathbf{G}}V\,\kappa_1(\psi(\mathbf{x}), \beta(\mathbf{x}))L_{\mathbf{G}}V^T + \\ + \delta(|\mathbf{w}(t)|) \leq -\alpha(|\mathbf{x}|) + (\mu - \hat{\mu})\rho(\mathbf{x}) + \delta(|\mathbf{w}|)\,.$$

For $t \in \Lambda_3$ the last inequality takes form: $\dot{U} \le -\alpha(|\mathbf{x}|) + (\mu - \hat{\mu})\rho(\mathbf{x}) + \delta(|\mathbf{w}|) \le -\alpha(|\mathbf{x}|) + \delta(|\mathbf{w}|)$, that is equivalent to ISS property (Sontag and Wang, 1995b). Note, that interval of overall system solution definition $[0, T) = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ and on each time instants subset Λ_i, $i = \overline{1,3}$ solution can not escape to infinity. So, system is forward complete and solution is defined for all $t \ge 0$.

Now let us suppose that number of switchings are finite, but in this case the control goal is realized. Indeed, let after the last switching $i(t) = 1$, but it is not possible, due to result of Lemma 1: in this case always would exist a time instant $T > 0$, such, that $\mathbf{x}(T) \in \Theta$. Therefore case $i(t) = 1$ can not correspond to the last switching. If after last switching $i(t) = 2$, then desired control goal is reached. Suppose that switchings are infinite. It means, that there is infinite sequence of intervals $\{[0,t_1),[t_1,t_2),....,[t_k,t_{k+1}),...\}$, where, we can suppose it without losing generality, on each even interval $[t_{2k}, t_{2k+1})$, $k = 1,2,...$ equality $i(t) = 1$ holds, and on each odd interval $i(t) = 2$.

By construction of algorithm (12) $t_{k+1} - t_k \ge \tau_D$ for all $k \ge 0$. Further, we know, that if $\hat{\mu}(T_2) \ge \mu^*$ for some $T_2 \ge 0$, then also $\hat{\mu}(t) \ge \mu^*$ for all $t \ge T_2$ and inequality (8) is satisfied for all such $t \ge T_2$. So, to prove the main result of the Theorem we should base, that always exists such T_2. If a time T_2 exists, then obviously exists a time T_1. Now as a contradiction, we assume, that neither time T_2 neither time T_1 does not exist. It means that for all $t \ge 0$, $\hat{\mu}(t) < \mu^*$, or equivalently:

$$\hat{\mu}(t) = \hat{\mu}(0) + \sum_k \int_{t_{2k}}^{t_{2k+1}} F_1(\mathbf{x}(t))dt =$$
$$= \hat{\mu}(0) + \gamma \sum_k \int_{t_{2k}}^{t_{2k+1}} \rho(\mathbf{x}(t))dt < \mu^* .$$

Due to continuity of function ρ it is possible to use mean value theorem to rewrite sum of integrals as follows:

$$\gamma \sum_k \int_{t2k}^{t2k+1} \rho(\mathbf{x}(t))dt = \gamma \sum_k (t_{2k+1} - t_{2k})\rho(\mathbf{x}(\mathcal{T}_k)),$$

where time $\mathcal{T}_k \in [t_{2k}, t_{2k+1})$. Finally we obtain:

$$\gamma \tau_D \sum_k \rho(\mathbf{x}(\mathcal{T}_k)) \le \gamma \sum_k (t_{2k+1} - t_{2k})\rho(\mathbf{x}(\mathcal{T}_k)) <$$
$$< \mu^* - \mu(0) ,$$

it means that sum of infinite series $\rho(\mathbf{x}(\mathcal{T}_k))$, $k = 0,1,...$ converges to value smaller than $(\mu^* - \hat{\mu}(0))\gamma^{-1}\tau_D^{-1}$. The last fact ensures that there exists an infinite sequence of time instants $\mathcal{T}_{i_0}, \mathcal{T}_{i_1}, \mathcal{T}_{i_2}, ...$ such, that $\rho(\mathbf{x}(\mathcal{T}_{i_k})) \to 0$ while index k grows to infinity. We should exclude from consideration case $\rho(\mathbf{x}(\mathcal{T}_{k^*})) = 0$ for some finite k^*, as since in such situation, according to positive semidefiniteness of function ρ, equality $\rho(\mathbf{x}(t)) = 0$ for all $t \in [t_{2k^*}, t_{2k^*+1})$ immediately follows. By continuity of function ρ also $\rho(\mathbf{x}(t_{2k^*})) = 0$ and $\rho(\mathbf{x}(t_{2k^*+1})) = 0$, but in such situation $\mathbf{x}(t) \in \Theta$ on ends of the interval and algorithm (12) should not change value of i at time t_{2k^*}. By the same arguments, for any $\varepsilon_0 > 0$ in this case there exists some index l_ε, such, that inequality $\rho(\mathbf{x}(t)) \le \varepsilon_0$ holds for all $t \in [t_{2l}, t_{2l+1})$, $l \ge l_\varepsilon$. This conclusion is satisfied for $\varepsilon_0 = \mu^{-1}\tilde{\varepsilon}$ with any $\tilde{\varepsilon} > 0$, hence for all $t \in [t_{2l}, t_{2l+1})$, $l \ge l_\varepsilon$ time derivative of function V possesses the following inequality:

$$\dot{V}(t) \le -\alpha(|\mathbf{x}(t)|) + \delta(|\mathbf{w}(t)|) + \tilde{\varepsilon} .$$

Due to arbitrary choice of $\tilde{\varepsilon}$ there exists some index $l > 0$, such, that $\mathbf{x}(t_{2l}) \in \Theta$ and algorithm (12) should not change further value of i, we receive a contradiction. Thus, there exists a time $T_2 \ge 0$, such, that $\hat{\mu}(t) \ge \mu^*$ for all $t \ge T_2$ and inequality (8) is satisfied, then there exists a time $T_1 \ge 0$, such, that $\mathbf{x}(t) \in \Theta$ for all $t \ge T_1$. So, the point (a) of the Theorem 1 is proven.

Note, that time T_2 always exists only with supposition that there is no time T_1. In general case it is possible a situation, when time T_1 exists, but time T_2 is not. A condition provided that additionally for system there exists time T_2 is formulated in point (b) of Theorem 1. Indeed, suppose that system (1), (5) possesses unbounded solution $\mathbf{x}(t)$ for any $\hat{\mu} < \mu$ and let $t \ge T_1$. Then trajectory $\mathbf{x}(t)$ should leave set Θ in finite time, but it contradicts definition of time T_1, hence, in such situation time T_2 should exist and $T_2 \le T_1$. Then for all $t \ge T_2$ inequality (8) is satisfied, from which asymptotic gain property holds with γ. If Assumption 3 holds, then for arbitrary $\varepsilon > 0$ it is possible to calculate constant $\mu_\varepsilon > 0$ as a solution of inequality

$$\alpha_1^{-1} \circ \alpha_2 \circ \rho_1^{-1} \circ \delta(\mu_\varepsilon^{-1} \| \mathbf{w} \|) \le \varepsilon . \qquad \blacksquare$$

Let us additionally emphasize and explain result of Theorem 1. If uncertainty of system (1) possesses requirements of Assumption 1 and system (1) is ISS stabilizable by continuous feedback with respect to input $\boldsymbol{\omega}$ (with known ISS-Lyapunov function, this requirement is fixed in Assumption 2), then control (5) with adaptation algorithm (10) and supervisor (12) provides boundedness of overall system solution and attractiveness of variable $\mathbf{x}$ to some neighbourhood of the origin. If additionally Assumption 3 holds, then this neighbourhood can be chosen arbitrary. If signal uncertainty $\mathbf{w}$, which reflects influence of exogenous disturbances on system dynamics, is vanishing, then global asymptotic stability property of the system with respect to variable $\mathbf{x}$ is proven. As it was mentioned before, for system (3) with input appearance of uncertainty $\boldsymbol{\omega}$ ISS stabilizability can be weakened to simple global asymptotic stabilizability with known Lyapunov function.

4. CONCLUSION

In this work a switching adaptive controller is designed, which for uncertain system (1) provides boundedness of overall system solution and attractiveness of variable $\mathbf{x}$ to some neighbourhood of the origin. Applicability conditions of proposed solution assume, that functional uncertainty admits requirements of Assumption 1 and system (1) is ISS stabilizable by continuous feedback (Assumption 2). If additionally Assumption 3 holds, then radios of attracting neighbourhood can be chosen arbitrary. If disturbance signal is absent, then global asymptotic stability property of the system with respect to variable $\mathbf{x}$ is proven. Additionally, there is no restriction on compactness of admissible values set for system uncertain parameters.

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