ON THE USE OF BACKSTEPPING TO ACHIEVE FINITE TIME STABILITY

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Abstract: Backstepping and a finite-time stability result of Garrard (Garrard, 1972) are combined to provide a recursive design algorithm for finite-time stability of a class of nonlinear systems.

Keywords: Nonlinear systems, control system design, feedback control, time-domain method, finite-time stability, backstepping.

1. INTRODUCTION

Finite-time stability [FTS] (sometimes also referred to as *short-time stability*) is a stability property concerned with the *quantitative* behaviour of the state of a system over a *finite, a priori fixed*, time interval. In particular, a system is *finite-time stable with levels* (α, β, T), [FTS(α, β, T)], if for all initial conditions with norm smaller than α, the norm of the state remains smaller than β on a time interval of length T. Compared with the concept of Lyapunov stability (involved with the *qualitative* behaviour of solution over an *infinite time interval*), the FTS concept is obviously more natural for many real systems, operating only for a finite time interval and subject to specific constraints on the state variables. It is worth remarking that FTS and Lyapunov stability are completely independent concepts (neither one implies the other).

The concept of FTS was first introduced in (Kamenkov, 1953), where conditions were given for nonlinear systems to be finite-time stable. A very complete study of FTS analysis for nonlinear systems appeared in (Weiss and Infante, 1965). However, design results for FTS of nonlinear systems were not available until (Garrard, 1972). Tractable LMI-based analysis and synthesis results for robust FTS of linear systems started appearing in the 1990's (Dorato *et al.*, 1997; Amato *et al.*, 2001) sparking renewed interest in FTS.

Meanwhile, significant developments in the field of stabilization of nonlinear systems appeared in the last two decades of the past century. In particular, some of the most impressive achievements in nonlinear control theory during the 1990's are essentially related to the discovery that globally asymptotically stabilizing control laws for special classes of nonlinear systems (having a suitable "triangular" or "interlaced" structure) can be obtained by fully constructive, recursive techniques ("backstepping" and "forwarding"); such a breakthrough

[1] Partially supported by MIUR (COFIN: "Robustness and optimization techniques for high performance control systems", FIRB: "Tiger") and ASI/ARS (I/R/277/02, I/R/125/02).
[2] Partially suported by the National Science Foundation under Information Technology Research (ITR) grants No. ANI-0312611, and NSF-0233205.

showed that the generally intractable problem of global asymptotic stabilization of nonlinear systems is easily solved if the system's structure can be exploited. A complete perspective on recursive designs is given in (Kristić *et al.*, 1995; Sepulchre *et al.*, 1997) and references therein.

The aim of this paper is to give constructive FTS design results for nonlinear systems in "strict-feedback" form (Kristić *et al.*, 1995). First, it is shown that any control law designed by standard backstepping [BS] achieves some level of FTS. Then, a synthesis result on FTS for nonlinear systems appeared in (Garrard, 1972) is combined with the idea of BS, providing a more effective modified BS design for FTS. Finally, a corollary of Garrard's theorem is stated whose application can further enhance the applicability of the modified BS design. A simple academic example is used to substantiate the theoretical results.

2. FTS IMPLICATIONS OF BACKSTEPPING

Consider the following system with state $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, in strict-feedback form:

$$\dot{x}_i = f_i(x_1, \ldots, x_i) + g_i(x_1, \ldots, x_i)x_{i+1}, \quad (1a)$$

for $i = 1, \ldots, n-1$, and

$$\dot{x}_n = f_n(x_1, \ldots, x_n) + g_n(x_1, \ldots, x_n)u, \quad (1b)$$

where each $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $u \in \mathbb{R}$ is the external input. The overall state is $\mathbf{x}' = [x_1 \ \cdots \ x_n]$, with $\mathbf{x}'$ denoting $\mathbf{x}$ transposed. The shorthand notations $\mathbf{x}_i' = [x_1 \ \cdots \ x_i]$ will be used throughout. Letting (with some abuse of notation) $f_i(\mathbf{x}_i) := f_i(x_1, \ldots, x_i)$, $g_i(\mathbf{x}_i) := g_i(x_1, \ldots, x_i)$, the following vector fields are defined:

$$\mathbf{f}_i(\mathbf{x}_i) := \begin{bmatrix} f_1(\mathbf{x}_1) + g_1(\mathbf{x}_1)x_2 \\ \vdots \\ f_{i-1}(\mathbf{x}_{i-1}) + g_{i-1}(\mathbf{x}_{i-1})x_i \\ f_i(\mathbf{x}_i) \end{bmatrix},$$

and $\mathbf{g}_i(\mathbf{x}_i) := \left[0 \ \cdots \ 0 \ g_i(\mathbf{x}_i)\right]'$, so that the subsystem of (1) with state $\mathbf{x}_i$ can be rewritten as

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) + \mathbf{g}_i(\mathbf{x}_i)v_i, \quad (2)$$

with *virtual control* $v_i = x_{i+1}$ for $i = 1, \ldots, n-1$, and $v_n = u$ (the actual control input). For simplicity, the functions $f_i(\cdot)$ and $g_i(\cdot)$ (as well as the $\phi_i(\cdot)$ appearing later) will be assumed to be smooth (such an assumption can be weakened to a sufficient regularity assumption).

For given $\beta \geq \alpha > 0$, $T > 0$, it is desired to design a state feedback law $u = h(\mathbf{x})$ such that the closed loop system is *finite-time stable with respect to* (α, β, T) [FTS(α, β, T)], *i.e.* such that

$$\|\mathbf{x}(0)\| < \alpha \implies \|\mathbf{x}(t)\| < \beta, \ \forall t \in [0, T),$$

where $\|\mathbf{v}\|$ is the euclidean norm of vector $\mathbf{v}$. The reals α, β and T will be referred to as the "FTS levels"; when comparing two different control laws achieving FTS(α_1, β_1, T_1) and FTS(α_2, β_2, T_2), respectively, with $\beta_1 = \beta_2$, the first control law will be considered better than the second if $\alpha_1 \geq \alpha_2$, $T_1 \geq T_2$, and at least one inequality is strict. In this sense, for a given β, the "best" possible FTS levels are given by $\alpha = \beta$ and $T = +\infty$ (corresponding to forward invariance of the set $\{\mathbf{x} : \|\mathbf{x}\| \leq \beta\}$). Notice that FTS with levels $(\alpha, \beta, +\infty)$ corresponds to practical stability.

To start with, it is shown that standard BS designs automatically yield some level of FTS. Results in BS designs are usually given by first defining a stability property of interest, and then providing a "BS lemma" which shows that if a smooth virtual control law $v_i = \phi_i(\mathbf{x}_i)$ achieving that stability property for system (2) is known, then it is possible to find a smooth virtual control law $v_{i+1} = \phi_{i+1}(\mathbf{x}_{i+1})$ achieving the same stability property for system (2) with i replaced by $i+1$. Starting with $i = 1$ and recursively applying such a lemma $n-1$ times, a virtual control law $\phi_n(\mathbf{x}_n)$ (achieving the desired stability property) is found which can be directly implemented since $v_n = u$.

One interpretation of BS is to view it as a constructive, recursive procedure to determine a change of coordinates $\boldsymbol{\vartheta}_i(\cdot)$ such that in the new coordinates $\mathbf{z}_i = \boldsymbol{\vartheta}_i(\mathbf{x}_i)$ a control Lyapunov function [CLF] is simply given by $\|\mathbf{z}_i\|^2$, the square of the euclidean norm of $\mathbf{z}_i$. In particular, the $\mathbf{z}_i$ coordinates are given by $z_1 := x_1$, $z_i := x_i - \phi_{i-1}(\mathbf{x}_{i-1})$ for $i = 2, \ldots, n$, and $\mathbf{z}_i' = [z_1 \ \cdots \ z_i]$. Clearly, $\boldsymbol{\vartheta}_i(\cdot)$ is smooth and globally invertible provided that the $\phi_i(\cdot)$'s are smooth and globally defined. Moreover, $\boldsymbol{\vartheta}_i(\cdot)$ does not alter the triangular structure of (2), which in the new coordinates can be written as

$$\dot{\mathbf{z}}_i = \widetilde{\mathbf{f}}_i(\mathbf{z}_i) + \widetilde{\mathbf{g}}_i(\mathbf{z}_i)v_i. \quad (3)$$

In standard BS, the goal is global asymptotic stability [GAS] of the origin, and then the stability property considered in the "BS lemma" is

$$\mathbf{z}_i'\widetilde{\mathbf{f}}_i(\mathbf{z}_i) + \mathbf{z}_i'\widetilde{\mathbf{g}}_i(\mathbf{z}_i)\widetilde{\phi}_i(\mathbf{z}_i) < 0, \forall \mathbf{z}_i \in \mathbb{R}^i; \quad (4)$$

however, if the eventual goal is FTS, a weaker property is needed, requiring (4) only for states $\mathbf{z}_i$ in a subset of $\mathbb{R}^i$. In order to precisely state such a property, the following sets are defined for $i = 1, \ldots, n$:

$$\mathcal{S}_i^\beta := \{\mathbf{x}_i : 0 < \|\mathbf{x}_i\| \leq \beta\}, \quad (5a)$$

$$\widetilde{\mathcal{S}}_i^\beta := \{\mathbf{z}_i : 0 < \|\mathbf{z}_i\| \leq \beta\}. \quad (5b)$$

Let $\widetilde{\mathcal{S}}_{i,0}^\beta := \widetilde{\mathcal{S}}_i^\beta \cup \{0\}$ and $\partial\widetilde{\mathcal{S}}_{i,0}^\beta$ denote the boundary of $\widetilde{\mathcal{S}}_{i,0}^\beta$. Moreover, let $\widetilde{\phi}_i(\mathbf{z}_i) := \phi_i(\boldsymbol{\vartheta}_i^{-1}(\mathbf{z}_i))$.

Condition 1. System (3) satisfies the inequality:

$$\mathbf{z}_i'\widetilde{\mathbf{f}}_i(\mathbf{z}_i) + \mathbf{z}_i'\widetilde{\mathbf{g}}_i(\mathbf{z}_i)\widetilde{\phi}_i(\mathbf{z}_i) < 0, \forall \mathbf{z}_i \in \widetilde{\mathcal{S}}_i^\beta,$$

under *virtual control* $v_i = \widetilde{\phi}_i(\mathbf{z}_i)$. △

$$\widetilde{\phi}_{j+1}(\mathbf{z}_{j+1}) := -\frac{1}{\widetilde{g}_{j+1}(\mathbf{z}_{j+1})}\left[\widetilde{f}_{j+1}(\mathbf{z}_{j+1}) + \widetilde{g}_j(\mathbf{z}_j)z_j + \gamma_{j+1}z_{j+1}\right] \quad (6)$$

Notice that $\frac{1}{2}\frac{d}{dt}\|\mathbf{z}_i\|^2 = \mathbf{z}_i'(\widetilde{\mathbf{f}}_i(\mathbf{z}_i) + \widetilde{\mathbf{g}}_i(\mathbf{z}_i)v_i)$, so that (in accordance with the above recalled interpretation of BS) Condition 1 essentially requires $\|\mathbf{z}_i\|^2$ to be a CLF for (3). Under the standard assumption that $\widetilde{g}_{j+1}(\mathbf{z}_{j+1}) \neq 0$, $\forall \mathbf{z}_{j+1} \in \widetilde{\mathcal{S}}_{j+1}^{\beta}$, a BS lemma (whose proof closely follows the reasoning in (Kristić *et al.*, 1995, Sec. 2.3.1)) for Condition 1 can be stated as follows.

Lemma 2. If Condition 1 holds for $i = j$, $1 \leq j < n$, then $\exists \widetilde{\phi}_{j+1}(\mathbf{z}_{j+1})$ such that Condition 1 holds for $i = j+1$. One such $\widetilde{\phi}_{j+1}(\mathbf{z}_{j+1})$ is given by (6) with $\gamma_{j+1} \in \mathbb{R}_{>0}$. △

Since BS makes $\|\mathbf{z}_i\|^2$ a CLF proving GAS on $\widetilde{\mathcal{S}}_i^{\beta}$ for system (3) under $v_i = \widetilde{\phi}_i(\mathbf{z}_i)$, and since the border of $\widetilde{\mathcal{S}}_i^{\beta}$ coincides with a level set of such a CLF, it follows that $\widetilde{\mathcal{S}}_i^{\beta}$ is a forward invariant set; in turn, forward invariance of $\widetilde{\mathcal{S}}_i^{\beta}$ implies that the BS control law is also a "best" FTS control law for (3) for any $i = 1, \ldots, n$, since it achieves FTS$(\beta, \beta, +\infty)$. Unfortunately, this very strong result does not automatically hold in the original $\mathbf{x}_i$ coordinates where the problem was formulated, since in general due to the change of coordinates the forward invariance of $\widetilde{\mathcal{S}}_i^{\beta}$ will not imply the forward invariance of $\mathcal{S}_i^{\beta}$. Anyway, some level of FTS is achieved for free by the BS control law, as shown in the following proposition.

Theorem 3. If Condition 1 holds and $\|\boldsymbol{\vartheta}_i(\mathbf{0})\| < \beta$, then (2) under *virtual control* $v_i = \phi_i(\mathbf{x}_i)$ is FTS$(\bar{\alpha}, \bar{\beta}, T)$, $\forall T > 0$, $\forall(\bar{\alpha}, \bar{\beta})$ satisfying

$$0 < \bar{\alpha} \leq \min_{\|\mathbf{z}_i\|=\beta} \|\mathbf{x}_i\| \leq \max_{\|\mathbf{z}_i\|=\beta} \|\mathbf{x}_i\| \leq \bar{\beta}, \quad (7)$$

where $\mathbf{z}_i = \boldsymbol{\vartheta}_i(\mathbf{x}_i)$. △

Proof: Since $\partial\widetilde{\mathcal{S}}_{i,0}^{\beta}$ is compact and $\boldsymbol{\vartheta}_i^{-1}(\cdot)$ is continuous, then also $\boldsymbol{\vartheta}_i^{-1}(\partial\widetilde{\mathcal{S}}_{i,0}^{\beta})$ is compact. By Weierstrass theorem, compactness of $\boldsymbol{\vartheta}_i^{-1}(\partial\widetilde{\mathcal{S}}_{i,0}^{\beta})$ and continuity of the norm function imply that both $\max_{\|\mathbf{z}_i\|=\beta}\|\mathbf{x}_i\|$ and $\min_{\|\mathbf{z}_i\|=\beta}\|\mathbf{x}_i\| \geq 0$ exist and are finite.

The existence of $\max_{\|\mathbf{z}_i\|=\beta}\|\mathbf{x}_i\|$ clearly implies the existence of $\bar{\beta} > 0$ such that $\boldsymbol{\vartheta}_i^{-1}(\widetilde{\mathcal{S}}_{i,0}^{\beta}) \subseteq \mathcal{S}_i^{\bar{\beta}}$.

In order to show that $\min_{\|\mathbf{z}_i\|=\beta}\|\mathbf{x}_i\| > 0$, so that $\bar{\alpha} > 0$ can be chosen, notice that since $\|\boldsymbol{\vartheta}_i(\mathbf{0})\| < \beta$, the point $\boldsymbol{\vartheta}_i(\mathbf{0})$ is an interior point of $\widetilde{\mathcal{S}}_{i,0}^{\beta}$, and then, by continuity of $\boldsymbol{\vartheta}_i(\cdot)$, $\mathbf{0}$ is an interior point of $\boldsymbol{\vartheta}_i^{-1}(\widetilde{\mathcal{S}}_{i,0}^{\beta})$; this, in turn, implies that the minimum distance of $\mathbf{0}$ from $\boldsymbol{\vartheta}_i^{-1}(\partial\widetilde{\mathcal{S}}_{i,0}^{\beta})$ is strictly positive, *i.e.* $\min_{\|\mathbf{z}_i\|=\beta}\|\mathbf{x}_i\| > 0$.

As a consequence of the above reasoning, two positive scalars $\bar{\alpha}$ and $\bar{\beta}$ have been determined such that $\mathcal{S}_i^{\bar{\alpha}} \subseteq \boldsymbol{\vartheta}_i^{-1}(\widetilde{\mathcal{S}}_{i,0}^{\beta}) \subseteq \mathcal{S}_i^{\bar{\beta}}$. The proof is completed by considering that, since $\mathcal{S}_i^{\bar{\alpha}}$ is contained in the forward invariant set $\boldsymbol{\vartheta}_i^{-1}(\widetilde{\mathcal{S}}_{i,0}^{\beta})$, no motion starting from $\mathcal{S}_i^{\bar{\alpha}}$ can leave the set $\boldsymbol{\vartheta}_i^{-1}(\widetilde{\mathcal{S}}_{i,0}^{\beta})$, and since this set is contained in $\mathcal{S}_i^{\bar{\beta}}$, the considered system is FTS(α, β, T), for any $T > 0$. □

About the hypothesis that $\|\boldsymbol{\vartheta}_i(\mathbf{0})\| < \beta$, it is remarked that usually in applications $\mathbf{0}$ is an equilibrium of (1) and $\boldsymbol{\vartheta}_i(\mathbf{0}) = \mathbf{0}$.

3. A BACKSTEPPING LEMMA FOR FTS

Theorem 3 shows that the deterioration in the original $\mathbf{x}_i$ coordinates of the nice property enjoyed by the BS control law in the $\mathbf{z}_i$ coordinates is due to the deformation of the set $\mathbf{z}$ under the coordinate transformation (an example of such deformation can be seen in Fig. 1, where the locus $\|\mathbf{z}\| = 9$ in the is shown in the $\mathbf{x}$ coordinates as a solid line). Even when all the flexibility of BS is exploited (*e.g.* avoiding cancellations or adding compensating terms in order to limit such deformations), the FTS levels achievable by the above BS design are necessarily limited by two factors: first, the conditions imposed in the BS design above do not take into account that only a finite interval of time is considered in FTS; second, such conditions are required to hold even inside $\widetilde{\mathcal{S}}_i^{\alpha}$. Clearly, both factors are due to the fact that the above BS design is focused on GAS, and achieves FTS only as a byproduct. The second factor can be particularly detrimental considering that the above BS control law pushes the state towards the attractive manifolds $\mathbf{z}_i = 0$ (whose shapes are determined by the virtual control functions, and eventually by the dynamics that the virtual controls cancel, maybe unnecessarily), which can require certain states (acting as virtual controls) to become quite large in order to control some other states: clearly, if such phenomena take place inside $\widetilde{\mathcal{S}}_i^{\alpha}$ (*i.e.* in a region inside which FTS does not impose any constraint), the resulting deterioration appears to be quite unnecessary. In order to relax the shortcomings above, a modified BS algorithm is proposed, exploiting a result in (Garrard, 1972) and recalled next.

Theorem 4. [Garrard] System $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{t}) + \mathbf{B}\mathbf{u}$ with state feedback $\mathbf{u} = \mathbf{h}(\mathbf{x})$ is FTS(α, β, T) if

$$\mathbf{x}'\left(\mathbf{f}(\mathbf{x}, \mathbf{t}) + \mathbf{B}\mathbf{h}(\mathbf{x})\right) - \delta\|\mathbf{x}\|^{\mathbf{2}} \leq \mathbf{0}, \quad \forall \mathbf{x} \in \mathcal{R},$$

where $\delta := \frac{1}{T}\ln\left(\frac{\beta}{\alpha}\right)$, $\mathcal{R} := \{\mathbf{x} : \alpha \leq \|\mathbf{x}\| \leq \beta\}$. △

$$\widetilde{\phi}_{j+1}(\mathbf{z}_{j+1}) := -\frac{1}{\widetilde{g}_{j+1}(\mathbf{z}_{j+1})}\left[\widetilde{f}_{j+1}(\mathbf{z}_{j+1}) + \widetilde{g}_j(\mathbf{z}_j)z_j + \gamma_{j+1}(z_{j+1})\mathbf{z}_j'\left(\widetilde{\mathbf{f}}_j(\mathbf{z}_j) + \widetilde{\mathbf{g}}_j(\mathbf{z}_j)\widetilde{\phi}_j(\mathbf{z}_j)\right)\right] \quad (8)$$

For general nonlinear systems, the application of Garrard's result, in order to design the control law u, is far from trivial. However, at least for systems in strict-feedback form, a synergic use of BS and Garrard's result allows to systematically design control laws achieving FTS, possibly with better FTS levels than those obtained in Section 2. The key idea consists in noticing that Garrard's result is based on bounding the derivative of a function of the form $\ln\left(\frac{\|\mathbf{x}\|^2}{\alpha^2}\right)$, which, apart from the logarithm, is a quadratic function, like the CLFs obtained by BS. In order to simplify subsequent statements, in analogy to (5) the following sets are defined for $i = 1, \ldots, n$:

$$\mathcal{R}_i^{\alpha,\beta} := \{\mathbf{x}_i : \alpha \leq \|\mathbf{x}_i\| \leq \beta\}, \quad (9a)$$
$$\widetilde{\mathcal{R}}_i^{\alpha,\beta} := \{\mathbf{z}_i : \alpha \leq \|\mathbf{z}_i\| \leq \beta\}. \quad (9b)$$

Condition 5. System (3) satisfies the inequality:
$\mathbf{z}_i'\widetilde{\mathbf{f}}_i(\mathbf{z}_i) + \mathbf{z}_i'\widetilde{\mathbf{g}}_i(\mathbf{z}_i)\widetilde{\phi}_i(\mathbf{z}_i) - \delta\|\mathbf{z}_i\|^2 < 0, \forall \mathbf{z}_i \in \widetilde{\mathcal{R}}_i^{\alpha,\beta}$, under *virtual control* $v_i = \widetilde{\phi}_i(\mathbf{z}_i)$. △

In order to compactly state the subsequent Lemma 7, the following class of functions Γ_ε is defined for $\varepsilon \in \mathbb{R}_{>0}$. A function $\gamma(\cdot)$ belongs to Γ_ε if it is smooth, $\gamma(\cdot) : \mathbb{R} \to [0, 1]$, and $\exists \bar{\varepsilon} \in (0, \varepsilon)$:

$$\gamma(s) = \begin{cases} 0 & \text{if } |s| \leq \bar{\varepsilon}/2, \\ 1 & \text{if } |s| \geq \bar{\varepsilon}. \end{cases}$$

As a consequence of the strict inequality in Condition 5, it is possible to choose $\varepsilon \in \mathbb{R}_{>0}$ such that the following property holds.

Condition 6. The inequality in Condition 5 is satisfied $\forall \mathbf{z}_i \in \widetilde{\mathcal{R}}_i^{\underline{\alpha},\beta}$, where $\underline{\alpha} := \sqrt{\alpha^2 - \varepsilon^2}$. △

Under the standard assumption that $\widetilde{g}_{j+1}(\mathbf{z}_{j+1}) \neq 0, \forall \mathbf{z}_{j+1} \in \widetilde{\mathcal{R}}_{j+1}^{\alpha,\beta}$, a BS lemma for Condition 5 can be stated as follows.

Lemma 7. If Condition 5 holds for $i = j$, $1 \leq j < n$, then $\exists \widetilde{\phi}_{j+1}(\mathbf{z}_{j+1})$ such that Condition 5 holds for $i = j+1$. One such $\widetilde{\phi}_{j+1}(\mathbf{z}_{j+1})$ is given by (8) with $\gamma_{j+1}(\cdot) \in \Gamma_\varepsilon$, and ε such that Condition 6 holds. △

Proof: It is enough to show that the virtual control function (8) implies Condition 5 for $i = j+1$. Let $\Delta_i := \mathbf{z}_i'\widetilde{\mathbf{f}}_i(\mathbf{z}_i) + \mathbf{z}_i'\widetilde{\mathbf{g}}_i(\mathbf{z}_i)\widetilde{\phi}_i(\mathbf{z}_i)$. Substituting $v_{j+1} = \phi_{j+1}(\mathbf{x}_{j+1})$ and writing the left hand side of the inequality in Condition 5 for $i = j+1$ yields:

$$\Delta_{j+1} - \delta\|\mathbf{z}_{j+1}\|^2 = (1 - \gamma_{j+1}(z_{j+1}))\Delta_j - \delta\|\mathbf{z}_{j+1}\|^2$$

In order to show that $\Delta_{j+1} - \delta\|\mathbf{z}_{j+1}\|^2 < 0$, $\forall \mathbf{z}_{j+1} \in \widetilde{\mathcal{R}}_{j+1}^{\alpha,\beta}$, consider the following cases:

- If $|z_{j+1}| \geq \varepsilon$, then $\gamma_{j+1}(z_{j+1}) = 1$, so that
$$\Delta_{j+1} - \delta\|\mathbf{z}_{j+1}\|^2 = -\delta\|\mathbf{z}_{j+1}\|^2 < 0$$
since $\mathbf{z}_{j+1} \in \widetilde{\mathcal{R}}_{j+1}^{\alpha,\beta}$ implies $\|\mathbf{z}_{j+1}\| \neq 0$;
- If $|z_{j+1}| < \varepsilon$, then $\|\mathbf{z}_j\| \geq \underline{\alpha}$ since
$$\begin{aligned}\|\mathbf{z}_j\|^2 &= \|\mathbf{z}_{j+1}\|^2 - z_{j+1}^2 \\ &\geq \alpha^2 - z_{j+1}^2 \\ &\geq \alpha^2 - \varepsilon^2 =: \underline{\alpha}^2\end{aligned}$$

and then $\mathbf{z}_j \in \widetilde{\mathcal{R}}_j^{\underline{\alpha},\beta}$, as $\|\mathbf{z}_j\| \leq \|\mathbf{z}_{j+1}\| \leq \beta$. Since ε has been chosen so that Property 6 is satisfied, $\Delta_j - \delta\|\mathbf{z}_j\|^2 < 0$, $\forall \mathbf{z}_j \in \widetilde{\mathcal{R}}_j^{\underline{\alpha},\beta}$. Since $\gamma_{j+1}(\cdot) \in \Gamma_\varepsilon$, then also $(1 - \gamma_{j+1}(z_{j+1})) \in [0, 1]$, and $(1 - \gamma_{j+1}(z_{j+1}))\Delta_j - \delta\|\mathbf{z}_j\|^2 < 0$. As a consequence, taking into account that $\|\mathbf{z}_j\| \leq \|\mathbf{z}_{j+1}\|$, so that $\Delta_{j+1} - \delta\|\mathbf{z}_{j+1}\|^2 \leq \Delta_{j+1} - \delta\|\mathbf{z}_j\|^2$, it follows that $\Delta_{j+1} - \delta\|\mathbf{z}_{j+1}\|^2 < 0$. □

As before (see the comments after Lemma 2), by Theorem 4 if $v_i = \widetilde{\phi}_i(\mathbf{z}_i)$ were the actual control for (3), then Condition 5 would imply FTS(α, β, T) in the $\mathbf{z}_i$ coordinates; however, clearly FTS(α, β, T) would not automatically hold in the original $\mathbf{x}_i$ coordinates where the problem was formulated. The FTS levels achieved by the proposed BS control law can be quantified as shown in the following theorem.

Theorem 8. If Condition 5 holds and $\|\boldsymbol{\vartheta}_i(\mathbf{0})\| < \alpha$, then (2) under *virtual control* $v_i = \phi_i(\mathbf{x}_i)$ is FTS$(\bar{\alpha}, \bar{\beta}, T)$, for the same T, $\forall(\bar{\alpha}, \bar{\beta})$ satisfying

$$0 < \bar{\alpha} \leq \min_{\|\mathbf{z}_i\|=\alpha} \|\mathbf{x}_i\| \leq \max_{\|\mathbf{z}_i\|=\beta} \|\mathbf{x}_i\| \leq \bar{\beta}, \quad (10)$$

where $\mathbf{z}_i = \boldsymbol{\vartheta}_i(\mathbf{x}_i)$. △

Proof: Consider the same notations of the proof of Theorem 3. FTS(α, β, T) of (3) under the considered virtual control implies that, under the same control, motions of (2) starting inside $\boldsymbol{\vartheta}_i^{-1}(\widetilde{\mathcal{S}}_{i,0}^\alpha)$ never leave $\boldsymbol{\vartheta}_i^{-1}(\widetilde{\mathcal{S}}_{i,0}^\beta)$ on time intervals shorter than T. By reasonings wholly similar to those used in the proof of Theorem 3, the proof is then completed by showing that $\boldsymbol{\vartheta}_i^{-1}(\partial\widetilde{\mathcal{R}}_i^{\alpha,\beta})$ is the union of the compact sets $\boldsymbol{\vartheta}_i^{-1}(\partial\widetilde{\mathcal{S}}_{i,0}^\alpha)$ and $\boldsymbol{\vartheta}_i^{-1}(\partial\widetilde{\mathcal{S}}_{i,0}^\beta)$ (so that, by Weierstrass theorem, the required minimum and maximum exist and are finite), and that the hypothesis $\|\boldsymbol{\vartheta}_i(\mathbf{0})\| < \alpha$ implies that $\mathbf{0}$ is an interior point of $\boldsymbol{\vartheta}_i^{-1}(\widetilde{\mathcal{S}}_{i,0}^\alpha)$ (so that the required minimum is strictly greater than zero).□

About the hypothesis that $\|\boldsymbol{\vartheta}_i(\mathbf{0})\| < \alpha$, it is remarked that when FTS is the goal (instead of GAS on $\widetilde{\mathcal{S}}_i^\beta$), the presence of a region $\widetilde{\mathcal{S}}_{i,0}^\alpha$ where no specific requirement is imposed makes it simpler to guarantee even the stronger condition $\boldsymbol{\vartheta}_i(\mathbf{0}) = \mathbf{0}$, even if $\mathbf{0}$ is not an equilibrium of (1).

Though the deterioration of the FTS levels in the original $\mathbf{x}_i$ coordinates with respect to the FTS levels guaranteed by the BS control law in the $\mathbf{z}_i$ coordinates inevitably affects the result in Theorem 8, the advantages offered by Theorem 8 with respect to Theorem 3 are essentially due to a larger flexibility in the choice of the virtual control functions: in fact, virtual control laws satisfying Condition 1 also satisfy Condition 5 for any choice of $\alpha \in (0, \beta)$ and $T > 0$, but the converse is not necessarily true. Such greater flexibility is obtained by explicitly accounting for the interest in a finite interval of time and by avoiding to impose constraint on the virtual control for states inside $\widetilde{\mathcal{S}}_i^\alpha$ (compare with the comments following Theorem 3). A subject of current research is how the enhanced flexibility due to the use of Condition 5 should be exploited in order to design a control law achieving the "best possible" FTS levels in the $\mathbf{x}_i$ coordinates. Clearly, the deformations involved in the change of coordinates are null when the change of coordinates is the identity, and then intuition would suggest that at any step the virtual control law should be chosen in such a way to introduce as little deformation as possible (*i.e.* to be as close as possible to 0); however, there is no formal proof that such a step-by-step "optimization" actually leads to the "flattest" possible transformation as yet.

As remarked in (Kristić *et al.*, 1995), the full exploitation of BS is achieved when unnecessary cancellations of useful nonlinearities are avoided. Along the same lines, the enhanced flexibility due to the use of Theorem 5 can be exploited by using the following corollary of Theorem 4.

Corollary 9. System $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{t}) + \mathbf{B}\mathbf{u}$ under the state feedback $\mathbf{u} = \mathbf{h}(\mathbf{x})$ is FTS(α, β, T) if $\exists(\breve{\alpha}, \breve{\beta})$ such that $\alpha \leq \breve{\alpha} < \breve{\beta} \leq \beta$ and

$$\mathbf{x}' \left(\mathbf{f}(\mathbf{x}, \mathbf{t}) + \mathbf{B}\mathbf{h}(\mathbf{x})\right) - \breve{\delta} \left\|\mathbf{x}\right\|^2 \leq \mathbf{0}, \quad \forall \mathbf{x} \in \breve{\mathcal{R}},$$

with $\breve{\delta} := \frac{1}{T} \ln\left(\frac{\breve{\beta}}{\breve{\alpha}}\right)$, $\breve{\mathcal{R}} := \{\mathbf{x} : \breve{\alpha} \leq \|\mathbf{x}\| \leq \breve{\beta}\}$. △

Proof: by the very definition of FTS, any system which is FTS$(\breve{\alpha}, \breve{\beta}, T)$ will also be FTS(α, β, T). By Theorem 4, the above inequality is a sufficient condition for FTS$(\breve{\alpha}, \breve{\beta}, T)$ to hold. □

The interest in the above corollary could appear quite limited, since in order to show FTS(α, β, T), it requires the stronger property of FTS$(\breve{\alpha}, \breve{\beta}, T)$. However, doing so relaxes the condition of Theorem 4, requiring it to hold only on a smaller set. Clearly, in view of Corollary 9, the result in Theorem 8 can be strengthened to

$$0 < \bar{\alpha} < \min_{\|\mathbf{z}_i\| = \breve{\alpha}} \|\mathbf{x}_i\| < \max_{\|\mathbf{z}_i\| = \breve{\beta}} \|\mathbf{x}_i\| < \bar{\beta}.$$

4. EXAMPLES

The use of the finite-time stabilization results in Sections 2 and 3 will be considered for system

$$\dot{x}_1 = q(x_1, x_2) + x_2, \tag{11a}$$

$$\dot{x}_2 = (2 - 0.5x_1^2)x_2 - x_1 + u. \tag{11b}$$

Since $n = 2$, once a virtual control law $v_1 = \phi_1(x_1)$ satisfying Condition 1 [Condition 5] is available, a single application of Lemma 2 [Lemma 7] yields a change of variables $z_1 = x_1$, $z_2 = x_2 - \phi_1(x_1)$, which makes straightforward the choice of $u = \phi_2(\mathbf{x})$ making (11) FTS$(\beta, \beta, +\infty)$ [FTS(α, β, T)] in the $\mathbf{z}$ coordinates; moreover, if $\phi_1(\cdot) \equiv 0$ then $\mathbf{x} \equiv \mathbf{z}$ and the same FTS levels are achieved in the original coordinates, too.

Several choices for $q(x_1, x_2)$ are considered to highlight different issues. **Case 1** (with $q(\cdot, \cdot) \equiv 0$) can be compared to (Garrard, 1972, III.B), which considers the same system; **Case 2** and **Case 3** (with $q(x_1, x_2) = 2x_1^3$) show the advantages (in term of FTS levels achieved in the $\mathbf{x}$ coordinates) of Lemma 7 vs. Lemma 2; **Case 4** (with $q(x_1, x_2)$ non-smooth and depending on x_2) shows the usefulness of Corollary 9, and the possibility to deal with systems which are not in strict-feedback form and/or non-smooth.

In order to show the fact that (6) and (8) are only possible choices, and that usually the flexibility of backstepping is better exploited by direct inspection of Condition 1 and Condition 5, the control laws in the examples are possibly derived by the latter approach, without using (6) and (8).

Case 1 [Van der Pol's oscillator]. When $q(x_1, x_2) \equiv 0$, it was shown in (Garrard, 1972) that $u = \gamma x_2$, $\gamma \in [-2, -1.39]$ makes (11) FTS$(1, 2, 2)$ (but not asymptotically stable). Since Condition 5 for $i = 1$ reduces to

$$x_1\phi_1(x_1) - \delta x_1^2 < 0, \quad \forall x_1 : \alpha \leq |x_1| \leq \beta,$$

the choice $\phi_1(\cdot) \equiv 0$ is feasible for any $\delta > 0$, *i.e.* for any $\alpha \in (0, \beta)$ (possibly arbitrarily close to β), and for any (possibly arbitrarily large) $T \in (0, \infty)$; moreover, it yields $\mathbf{x} = \mathbf{z}$. Lemma 7 implies the existence of a $\phi_2(\cdot)$ ensuring Condition 5 for $i = 2$, *i.e.* FTS(α, β, T) for (11). Without resorting to (8), the choice

$$u = \phi_2(x_1, x_2) = -(2 - 0.5x_1^2)x_2 \tag{12}$$

is immediately suggested by inspection of the inequality in Condition 5 for $i = 2$:

$$x_1x_2 + x_2[(2 - 0.5x_1^2)x_2 - x_1 + u] - \delta \left\|\mathbf{x}\right\|^2 < 0.$$

which clearly holds for all $\mathbf{x}$ with $0 < \alpha \leq \|\mathbf{x}\| \leq \beta$ when the control law (12) is used.

Case 2 [Standard Backstepping]. Consider $q(x_1, x_2) = 2/(1 + 5x_1^2)$, and the objective of achieving $\text{FTS}(\alpha, 5, 10)$ with α as close as possible to $\beta = 5$. Choosing the virtual control law $\phi_1(x_1) = -q(x_1, x_2) - \gamma_1 x_1$, with $\gamma_1 \in \mathbb{R}_{>0}$, Condition 1 for $i = 1$ is satisfied, and such a choice results in $z_2 = x_2 + 2/(1+5x_1^2) + \gamma_1 x_1$. The actual control obtained applying (6) with $\gamma_2 \in \mathbb{R}_{>0}$ is:

$$u = -[x_2(2 - 0.5x_1^2 - 10x_1/(1 + 5x_1^2)^2 + \gamma_1 + \gamma_2) + (2/(1 + 5x_1^2))(\gamma_1 + \gamma_2) + \gamma_1\gamma_2 x_1]$$

The above controller makes (11) FTS(β,β,∞), for any choice of β, in the $\mathbf{z}$ coordinates; however, in the original $\mathbf{x}$ coordinates, the FTS levels deteriorate. For $\gamma_1 = 0.1$, according to (7) the level curve $\|\mathbf{z}\| = 9$ shows that the FTS levels $(1, 5, \infty)$ are achieved (Fig. 1).

Case 3 [Backstepping lemma for FTS]. Consider the same $q(x_1, x_2)$ and objective of **Case 2**. For the given choice of FTS levels, the virtual control $\phi_1(\cdot) \equiv 0$ satisfies Condition 5 for $i = 1$ and yields $z_2 = x_2$. By the same reasoning in **Case 1**, the control law (12) can be shown to achieve the desired FTS levels $(4, 5, T)$ in the $\mathbf{x}$ coordinates (coinciding with the $\mathbf{z}$ coordinates). The comparison with **Case 2** shows that the backstepping design for FTS, being tailored to exploit the finite time horizon and the state limitations of interest in FTS, can deliver better controllers (in the sense specified in Section 2) than standard backstepping.

Case 4 [A non-strict feedback system]. Let $q(x_1, x_2) := x_2^2 \vartheta(x_1)$, where

$$\vartheta(x_1) := \begin{cases} \cos^2\left(\frac{\pi}{2}\frac{x_1}{A}\right) & \text{if } |x_1| \leq A, \\ 0 & \text{otherwise.} \end{cases}$$

The design objective is to achieve a FTS(1/2, 2, 1) closed loop system. Notice that system (11) is not in the strict-feedback form of (1), due to the term x_2^2 in (11a); moreover, even if (11a) were replaced by $\dot{x}_1 = \vartheta(x_1) + x_2$, trying to cancel the term $\vartheta(x_1)$ for $1/2 \leq |x_1| \leq 2$ could lead to a choice $\phi_1(\cdot) \not\equiv 0$, introducing distortion in the change of coordinates and deterioration of the FTS levels; finally, $\vartheta(x_1)$ is not even differentiable (hence, not smooth) when $|x_1| = 1$.

However, since $\vartheta(x_1) = 0$ on $\mathcal{R}_1^{1,2}$, Condition 5 for $i = 1$ is satisfied on $\mathcal{R}_1^{1,2}$ for $\phi_1(\cdot) \equiv 0$, which yields $\mathbf{x} = \mathbf{z}$. Lemma 7 implies the existence of a $\phi_2(\cdot)$ ensuring Condition 5 for $i = 2$, *i.e.* ensuring that (11) is FTS(1, 2, 1), and then *a fortiori* FTS(1/2, 2, 1) (as in the proof of Corollary 9). Once again, the same reasoning used in **Case 1** shows that (12) is a suitable choice for $\phi_2(\cdot)$.

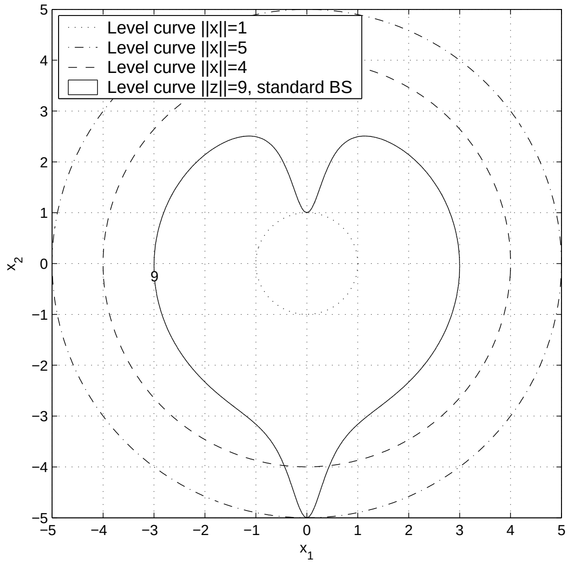

Fig. 1. FTS levels obtained by using standard and modified backstepping.

5. CONCLUSIONS

A constructive recursive design of control laws achieving finite-time stability for a class of nonlinear systems having a triangular (feedback) structure has been proposed, exploiting a result in (Garrard, 1972) and the idea of backstepping.

Extensions to the present work are being developed, considering discrete time systems, robust finite-time stabilization, and recursive designs for nonlinear systems having more general structures.

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