INSTABILITY IN THE SIMPLEST CLASS OF CONTINUOUS SWITCHED LINEAR SYSTEMS WITH STABLE COMPONENTS [1]

Victoriano Carmona, Emilio Freire, Enrique Ponce, Francisco Torres

Escuela Superior de Ingenieros, Universidad de Sevilla,
Camino de los Descubrimientos, 41092-Sevilla, Spain,
e-mail: {vcarmona, efrem, eponcem, ftorres}@us.es

Abstract: For discontinuous switched linear systems, even when they are built by composing stable systems, examples of unstable systems are known. Here, three-dimensional homogeneous continuous piecewise linear systems composed by two linear systems sharing a boundary plane are considered. If the two linearization matrices are non-singular, then the only equilibrium point is at the origin, which is in the separation plane of the linear regions. For the important case where both matrices have complex eigenvalues and their whole spectra are in the open-left half of the complex plane, the possible counter-intuitive instability of the origin is analytically proved. Thus, in this paper, examples of unstable continuous switched linear system with just two stable components are shown.

Keywords: Piecewise linear systems, switched systems, stability

1. INTRODUCTION AND MAIN RESULTS

The stability issue is a problem of outstanding relevance in nonlinear control and has deserved a lot of attention in the past. Nevertheless, even for the seemingly simple case of continuous piecewise linear systems, several questions remain unsolved. In particular, for the class of switched linear systems with only two components, namely

$$\dot{\mathbf{x}} = A_\sigma \mathbf{x}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\sigma : [0, \infty) \to \{1, 2\}$ is a piecewise constant function of time, called a switching signal, several stability approaches have been recently proposed (Agrachev and Liberzon, 2001), (Boscain, 2002), (Liberzon and Morse, 1999), (Shorten *et al.*, 2003), (Shorten and Narendra, 2003), (King and Nathanson, 2004).

The value of the switching law σ at a given time s might just depend on s or $\mathbf{x}(s)$, or both, or may be generated using more sophisticated techniques. In this paper, particular attention is paid to the case of *continuous* switched systems, that is, switched systems with the additional constraint that the vector fields of the different subsystems coincide at the switching times, and furthermore the switching law is *autonomous*, only depending on the state $\mathbf{x}(\cdot)$. Thus, the resulting vector field is a piecewise linear system, changing continuously from one linear subsystem to the other when the state $\mathbf{x}(\cdot)$ hits certain boundary surface.

To be more specific, let us start with a rather elementary example. Consider the stability issue for the planar continuous piecewise linear (CPWL) system

$$\dot{\mathbf{x}} = \begin{cases} A^+\mathbf{x}, & \text{if } x \geqslant 0, \\ A^-\mathbf{x}, & \text{if } x < 0, \end{cases} \tag{2}$$

[1] Authors are partially supported by spanish Ministerio de Ciencia y Tecnología under Grant BFM2003-00336

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$, the dot denotes derivatives respect to the time s, and

$$A^+ = \begin{pmatrix} t^+ & -1 \\ d^+ & 0 \end{pmatrix}, \quad A^- = \begin{pmatrix} t^- & -1 \\ d^- & 0 \end{pmatrix}.$$

Consider for instance the case $t^+ = t^- = -1$, $d^+ = 1$, $d^- = 25$, and let us look for a Lyapunov function. A first approach is to investigate the existence of a common quadratic Lyapunov function (CQLF) for such system. Then, in order to apply the version of Shorten-Narendra theorem given in (King and Nathanson, 2004), a simple computation shows that

$$A^+ \cdot A^- = \begin{pmatrix} -24 & -1 \\ -1 & -1 \end{pmatrix},$$

and this matrix has real negative eigenvalues. Then it is concluded that the system has no CQLFs. So, for the given parameter values, in order to show the stability of system (2), more general Lyapunov functions should be used. See, for instance (Johansson and Rantzer, 1998).

On the other hand, the use of a Lie-algebraic approach (Agrachev and Liberzon, 2001) cannot be successful. Effectively, when such approach assures the stability of a switched linear system, it does so under all the possible switching signals and it can be shown that there exists some switching signal for which the modified system (2) is unstable, namely

$$\dot{\mathbf{x}} = \begin{cases} A^+\mathbf{x}, & \text{if } xy(x^2 - y^2) \geqslant 0, \\ A^-\mathbf{x}, & \text{if } xy(x^2 - y^2) < 0. \end{cases} \tag{3}$$

Note that here the spirit of Example 2.1 in (Branicky, 1998) is followed and that system (3) is not continuous any longer.

However, by means of a thorough study of the flow, as done in (Freire *et al.*, 1998), it can be rigorously proved that planar CPWL systems with two zones, Hurwitz matrices and a equilibrium point in the separation line are always globally asymptotically stable, and so system (2) with the given parameter values is indeed stable.

Then, a quite natural question is whether higher dimensional CPWL systems with Hurwitz matrices are also globally asymptotically stable. As it will be seen, the question has already a negative answer for dimension three.

Thus, let us consider homogeneous CPWL systems in $\mathbb{R}^3$ with a separation plane, written in the form

$$\dot{\mathbf{x}} = \begin{cases} A^+\mathbf{x}, & \text{if } x \geqslant 0, \\ A^-\mathbf{x}, & \text{if } x < 0, \end{cases} \tag{4}$$

where $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$, the dot denotes derivatives respect to the time s, and

$$A^+ = \begin{pmatrix} t^+ & -1 & 0 \\ m^+ & 0 & -1 \\ d^+ & 0 & 0 \end{pmatrix}, \quad A^- = \begin{pmatrix} t^- & -1 & 0 \\ m^- & 0 & -1 \\ d^- & 0 & 0 \end{pmatrix}.$$

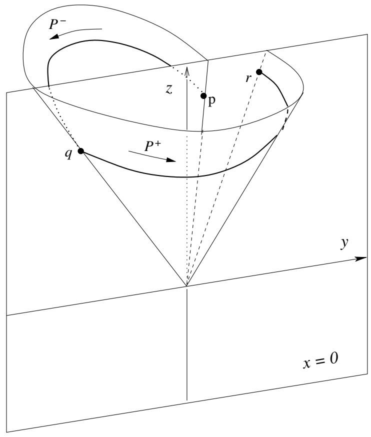

Fig. 1. The Poincaré half maps P^- and P^+ defined for the section $x = 0$.

In Proposition 16 of (Carmona *et al.*, 2002), it is shown that, under generic conditions, every CPWL system with two zones, can be written, by means of a linear change of variables, in the form (4). Such form, called generalized Liénard canonical form, is strongly related to the classical observability canonical form.

Note that the origin is always an equilibrium point for system (4), and it will be the only one when matrices A^+ and A^- are non-singular. Also, having in mind the previous planar example, it will not be useful to resort either to CQLF methods or to Lie-algebraic criteria to characterize its stability. Instead, a direct study of the flow behavior via Poincaré maps will be done.

In the following, our attention will be focussed on system (4) when the eigenvalues of its matrices are $\lambda^\pm$, $\alpha^\pm + i\beta^\pm$ and $\alpha^\pm - i\beta^\pm$ with $\beta^\pm > 0$, that is, both matrices have a pair of complex eigenvalues. In this case, the following two parameters

$$\gamma^+ = \frac{\alpha^+ - \lambda^+}{\beta^+} \quad \text{and} \quad \gamma^- = \frac{\alpha^- - \lambda^-}{\beta^-},$$

which are crucial for the analysis to be made in the sequel, will be assumed to be positive along the paper.

By means of the flow of system $\dot{\mathbf{x}} = A^-\mathbf{x}$ with $x \leqslant 0$, some points $\mathbf{p} = (0, y_0, z_0)$ with $y_0 > 0$ can be transformed into points $\mathbf{q} = (0, y_1, z_1)$ with $y_1 < 0$, so a left Poincaré half map P^- can be defined as $(y_1, z_1) = P^-(y_0, z_0)$. Analogously, a right Poincaré half map P^+ can be defined in some region of the half plane $x = 0$, $y < 0$, with range in the half plane $x = 0$, $y > 0$, so that the composed Poincaré map $P = P^+ \circ P^-$ can be defined from a subset of the half plane (y, z) with $y > 0$ into itself, see Fig. 1.

Since if $\mathbf{x}(s)$ is a solution of system (4), then $\mu\mathbf{x}(s)$ for $\mu > 0$ is also a solution, it can be easily deduced that both Poincaré half maps transform half straight lines contained in the plane $x = 0$

and passing through the origin into half straight lines contained in the plane $x = 0$ that also pass through the origin, see Fig. 1 again.

A manifold $\mathcal{C}$ will be said a cone if for all $\mathbf{x} \in \mathcal{C}$ one has that $\mu\mathbf{x} \in \mathcal{C}$ for every $\mu \geqslant 0$. Thus, if there exists an invariant half straight line for the Poincaré map P, then it will be said that the system has a two-zonal invariant cone. Obviously, an invariant cone can degenerate into a plane.

The most interesting results of this work concern the dynamics of these systems in case of having one non-smooth invariant cone, which behaves in a similar way to the center manifold for non-hyperbolic differentiable systems. But it should be remarked that the determination of the dynamics on such manifold is here a non-local problem due to the lack of smoothness, which is a source of difficulties.

In Proposition 10 of (Carmona *et al.*, 2005*a*), a result about the stability of the origin in absence of invariant cones was given; namely, under the previous assumption of existence of complex eigenvalues for both matrices, if system (4) has no invariant cones, then the origin is an asymptotically stable equilibrium point if and only if $\lambda^+ < 0$ and $\lambda^- < 0$. Nevertheless, as far as the authors know, results about stability of the origin in presence of invariant cones have not been given yet. In this direction, the following theorem explains the difficulty in giving such kind of results.

Theorem 1. Assume that both matrices in system (4) are Hurwitz and have a pair of complex eigenvalues, in such way that $\gamma^+ > 0$ and $\gamma^- > 0$. The following statements hold.

(a) The origin has a one-dimensional stable manifold and a two-dimensional invariant manifold which is an attractive two-zonal cone. Generically, both manifolds are non-smooth.
(b) The dynamics on the invariant cone is either of stable focus type, or a center, or of unstable focus type, and there exist specific systems for the three cases.

Note that the above theorem assures that one can have a saddle-focus dynamics for the origin even when the vector field is composed by continuously matching two linear Hurwitz systems. It must be also remarked that without additional knowledge about parameter values it is not possible to discriminate the three cases of statement (b). The three possible situations described in the above theorem are sketched in Figure 2.

The mechanism which is responsible for the instability of the origin in the corresponding case of statement (*b*) of Theorem 1 can be elucidated by using Figure 3. In the figure, besides the plane $x = 0$, two half-planes $\Pi^{\pm}$ appear. They are the portions of focal planes of matrices $A^{\pm}$ in each half-space. In the figure, it also appears the cone $\mathcal{C}$ which is invariant for the flow and intersects the plane $x = 0$ at the lines containing the segments OA and OB. The orbit from the point A to point B, which is on the cone, uses the flow with $x < 0$ and it continues from B to C, also on the cone, using the flow with $x > 0$.

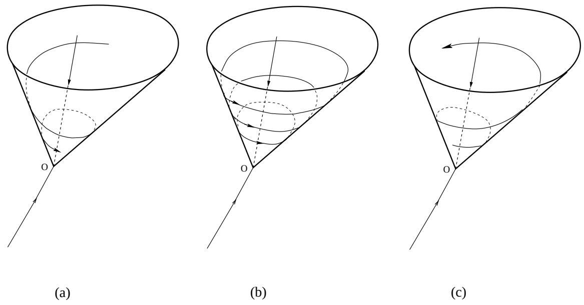

Fig. 2. The three situations predicted by Theorem 1. (a) The dynamics on the invariant cone is of stable focus type. (b) The center case. (c) Unstable focus dynamics on the invariant cone, what leads to a saddle-focus dynamics in a neighborhood of the origin.

The key point to understand the non intuitive behavior predicted by statement (b) is to see how the composition of two flows defined by Hurwitzian matrices can cause that the point C be farther from the origin than the starting point A. This can happen because each piece (AB and BC) of the orbit AC goes towards the origin by spiraling on the cone and approaching its focal plane, but the position of these planes are very different.

The rest of the paper is outlined as follows. In Section 2 Poincaré half maps and slope transition maps are given and its main properties are shown. In Section 3, the proof of Theorem 1 is sketched. From the ideas in the proof, it is immediate to build numerical examples corresponding to the two generic cases of Theorem 1 (obviously, the center case must be excluded). For more technical details, see (Carmona *et al.*, 2005*b*).

2. POINCARÉ HALF MAPS.

Since the flow of system (4) is made up by matching two linear flows, each Poincaré half map will be studied separately as a first step to compute the complete Poincaré map. In the following, the auxiliary function

$$\varphi_\gamma(\tau) = 1 - e^{\gamma\tau}(\cos\tau - \gamma\sin\tau),$$

introduced in (Andronov *et al.*, 1966) to study planar piecewise linear systems, will be extensively

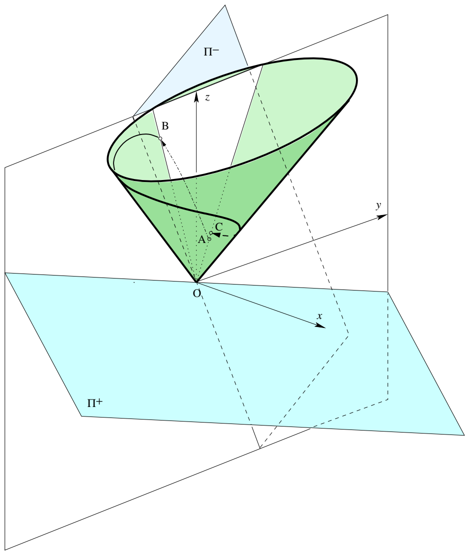

Fig. 3. Geometric sketch of an orbit on the invariant cone that, starting at the point A and spiraling towards the origin, approaches the half-plane Π^- up to the point B, and then goes into the region $x > 0$ approaching the half-plane Π^+ up to the point C. Counter-intuitively, the point C can be farther from the origin than point A.

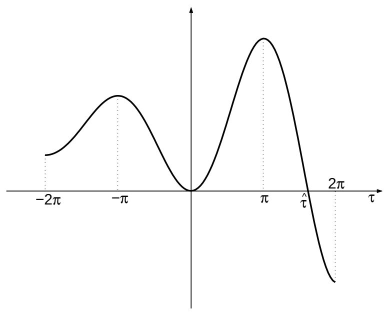

Fig. 4. The graph of the function $\varphi_\gamma(\tau)$ for $\gamma > 0$.

used, see Fig. 4. Note that this function satisfies $\varphi_{-\gamma}(\tau) = \varphi_\gamma(-\tau)$ and when $\gamma > 0$, a value $\hat{\tau} \in (\pi, 2\pi)$ exists such that $\varphi_\gamma(\hat{\tau}) = 0$.

To compute the left Poincaré half map P^-, the matrix A^- must be used. Starting from $(0, y_0, z_0)$ at $s = 0$, a direct integration of system (4) leads to the expression of $\mathbf{x}(s)$. Imposing that $\mathbf{x}(\tau/\beta^-) = (0, y_1, z_1)$, the initial and final coordinate ratios can be parameterized as follows,

$$u_0(\tau) = \frac{z_0}{y_0} = \lambda^- + \beta^- \left[(\gamma^-)^2 + 1\right] \frac{e^{\gamma^- \tau}\sin(\tau)}{\varphi_{\gamma^-}(\tau)}$$

$$u_1(\tau) = \frac{z_1}{y_1} = \lambda^- - \beta^- \left[(\gamma^-)^2 + 1\right] \frac{e^{-\gamma^- \tau}\sin(\tau)}{\varphi_{-\gamma^-}(\tau)}. \tag{5}$$

Furthermore,

$$\frac{y_1}{y_0} = -\frac{\varphi_{-\gamma^-}(\tau)}{\varphi_{\gamma^-}(\tau)} e^{(\gamma^- + \frac{\alpha^-}{\beta^-})\tau}. \tag{6}$$

In the above expressions, $y_0 > 0$, $y_1 < 0$, and $\tau \in (0, \hat{\tau}^-)$, where $\hat{\tau}^-$ is the first positive solution of $\varphi_{\gamma^-}(\tau) = 0$, see Fig.4. In the sequel, the parameter τ in (5) and (6) will be called the left phase.

Relations (5) allow to define in parametric form a map $u_1 = S^-(u_0)$, to be called as left slope transition map. It transforms the slope u_0 of the half straight line $x = 0$, $z = u_0 y$, with $y > 0$, which passes through the point (y_0, z_0) with $y_0 > 0$, into the slope u_1 of its corresponding image by means of map P^-. This image is defined by $x = 0$, $z = u_1 y$ with $y < 0$, and passes through the point $(y_1, z_1) = P^-(y_0, z_0)$ with $y_1 < 0$.

In the next lemma, whose proof is direct, some properties of the functions $u_0(\tau)$ and $u_1(\tau)$ are shown.

Lemma 2. For the functions $u_0(\tau)$ and $u_1(\tau)$ defined in (5), the following statements hold.

(a) The function $u_0(\tau)$ satisfies $u_0(\pi) = \lambda^-$,
$$\lim_{\tau \to \hat{\tau}^-} u_0(\tau) = \infty, \quad \text{and}$$
$$\frac{du_0}{d\tau} < 0, \text{ for all } \tau \in (0, \hat{\tau}^-).$$

(b) The function $u_1(\tau)$ satisfies $u_1(\pi) = \lambda^-$,
$$\lim_{\tau \to 0} u_1(\tau) = -\infty, \quad \text{and}$$
$$\frac{du_1}{d\tau} > 0 \text{ for all } \tau \in (0, \hat{\tau}^-).$$

(c) If $\gamma^- > 0$, then
$$\lim_{\tau \to \hat{\tau}^-} u_0(\tau) = -\infty,$$
$$\lim_{\tau \to \hat{\tau}^-} u_1(\tau) = u_H^- = \alpha^- - \beta^- \cot \hat{\tau}^-.$$

The left focal half plane Π^- of system (4) is given by the equation
$$(\lambda^-)^2 x - \lambda^- y + z = 0,$$
and is invariant for system $\dot{\mathbf{x}} = A^- \mathbf{x}$. Then, every point $(0, y_0, z_0)$ in the domain of P^- and located above the plane Π^-, i.e. $z_0 > \lambda^- y_0$, is transformed by the flow into the point $(0, y_1, z_1)$, which is also located above Π^-, i.e. $z_1 > \lambda^- y_1$. Analogously, when $z_0 < \lambda^- y_0$, one has $z_1 < \lambda^- y_1$.

Several properties of the map S^- that follow from Lemma 2 and from Proposition 15 in (Carmona *et al.*, 2005*a*) are gathered in the next result.

Proposition 3. For the slope transition map S^- the following statements are true, see Fig. 5.

(a) The map S^- is decreasing with $S^-(\lambda^-) = \lambda^-$, and has the straight line $u_1 = -u_0 + 2t^-/3$ as an oblique asymptote when $u_0 \to +\infty$.

(b) The interval $[\lambda^-, \infty)$ is mapped by S^- onto the interval $(-\infty, \lambda^-]$. The parametric representation (5) for these points is defined for $\tau \in (0, \pi]$.

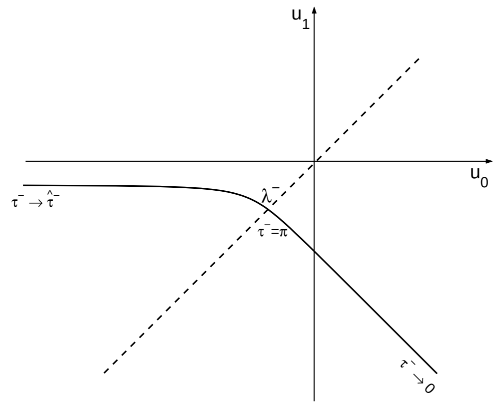

Fig. 5. The function $u_1 = S^-(u_0)$ for $\lambda^- < 0$, $\alpha^- < 0$, and $\gamma^- > 0$.

(c) If $\gamma^- = 0$, then $S^-(u_0) = -u_0 + 2t^-/3 = -u_0 + 2\lambda^-$ and the map has an affine graph.

(d) If $\gamma^- > 0$, then $S''(u_0) < 0$ for all u_0, and the image by S^- of the interval $(-\infty, \lambda^-]$ is the interval $\left(u_H^-, \lambda^-\right]$, where u_H^- is defined in Lemma 2. For these points, the parametric representation of S^- verifies $\tau \in [\pi, \hat{\tau}^-)$.

By considering now the system $\dot{\mathbf{x}} = A^+\mathbf{x}$ and integrating from $\mathbf{x}(0) = (0, y_1, z_1)$, after imposing that $\mathbf{x}(\tau/\beta^+) = (0, y_2, z_2)$, one obtains the parametric representation

$$u_1(\tau) = \frac{z_1}{y_1} = \lambda^+ + \beta^+ \left[(\gamma^+)^2 + 1\right] \frac{e^{\gamma^+\tau}\sin(\tau)}{\varphi_{\gamma^+}(\tau)}$$

$$u_2(\tau) = \frac{z_2}{y_2} = \lambda^+ - \beta^+ \left[(\gamma^+)^2 + 1\right] \frac{e^{-\gamma^+\tau}\sin(\tau)}{\varphi_{-\gamma^+}(\tau)} \tag{7}$$

and

$$\frac{y_2}{y_1} = -\frac{\varphi_{-\gamma^+}(\tau)}{\varphi_{\gamma^+}(\tau)} e^{(\gamma^+ + \frac{\alpha^+}{\beta^+})\tau} \tag{8}$$

where $y_1 < 0$, $y_2 > 0$, and now $\tau \in (0, \hat{\tau}^+)$, being $\hat{\tau}^+$ the first positive solution of $\varphi_{\gamma^+}(\tau) = 0$.

By changing the superscript $-$ to $+$, functions u_1 and u_2 defined in (7) satisfy the properties shown in Lemma 2 for u_0 and u_1 respectively. The parameter τ in (7) and (8) will be called as the right phase.

Equations (7) are a parametric representation of the right slope transition map $u_2 = S^+(u_1)$, defined in a similar way to map S^-, which also satisfies Proposition 3. Now, the image of points $(0, y_1, z_1)$ in the domain of P^+ located above (respectively, below) the right focal half plane Π^+ are above (respectively, below) Π^+.

Remark 4. If there exists $\bar{u}$ such that $S^+[S^-(\bar{u})] = \bar{u}$, then the existence of an invariant half straight line, or equivalently, of a two-zonal invariant cone is concluded. The existence of such fixed points in the composed map $S = S^+ \circ S^-$ corresponds with the existence of solutions for the equations

$$\begin{aligned} u_0(\tau^-) &= u_2(\tau^+), \\ u_1(\tau^-) &= u_1(\tau^+), \end{aligned}$$

or, what amounts the same, with the existence of solutions for $S^-(u) = (S^+)^{-1}(u)$. When $\lambda^+ = \lambda^- = \lambda$, both focal half planes are contained in the plane $\Pi \equiv \lambda^2 x - \lambda y + z = 0$, and so, the plane Π is a planar invariant cone. Then the above system of equations has trivially the solution $\tau^- = \tau^+ = \pi$.

Clearly, the ratio of distances from the origin for two consecutive impact points into the half plane $x = 0$, $y > 0$, satisfies

$$\frac{r_2^2}{r_0^2} = \frac{y_2^2 + z_2^2}{y_0^2 + z_0^2} = \frac{y_2^2}{y_0^2}\frac{1 + u_2^2}{1 + u_0^2}, \tag{9}$$

what in the case of being on an invariant cone ($u_2 = u_0$) reduces to

$$\frac{r_2}{r_0} = \frac{y_2}{y_0}. \tag{10}$$

3. PROOF OF THEOREM 1.

Let us begin this section by considering an auxiliary lemma related to the ratios given in (6) and (8), which will play a key role in determining the asymptotic behavior of the orbits on possible invariant cones.

Lemma 5. Assume that $\lambda < \alpha < 0$, and let $\hat{\tau}$ be the first positive zero of $\varphi_\gamma(\tau) = 0$. Then, the continuous function

$$\Psi(\tau) = \begin{cases} \dfrac{\varphi_{-\gamma}(\tau)}{\varphi_\gamma(\tau)} e^{(\gamma + \frac{\alpha}{\beta})\tau}, & \text{if } 0 < \tau < \hat{\tau}, \\ 1, & \text{if } \tau = 0, \end{cases}$$

satisfies $e^{\frac{\lambda}{\beta}\pi} < \Psi(\tau) < 1$ for $0 < \tau < \pi$, $\Psi(\pi) = e^{\frac{\alpha}{\beta}\pi}$, and $\lim_{\tau \to \hat{\tau}} \Psi(\tau) = \infty$. Furthermore, there exists a unique value $\tau^* \in (\pi, \hat{\tau})$ such that $\Psi(\tau^*) = 1$, and $\Psi(\tau) > 1$ for $\tau^* < \tau < \hat{\tau}$.

Now, under the hypotheses of Theorem 1, it is immediate to show that the vectors

$$\left[1, \alpha^\pm, \left(\alpha^\pm\right)^2 + \left(\beta^\pm\right)^2\right]$$

are the eigenvectors associated to the real eigenvalues. Then, the one-dimensional stable manifold of the origin is the union of two half straight lines. Obviously, this manifold is non-smooth apart from the case $\alpha^+ = \alpha^-$, $\beta^+ = \beta^-$.

Since $\gamma^+ > 0$ and $\gamma^- > 0$, by using statement (d) of Proposition 3, it can be concluded that there exists only one value $\bar{u}$ such that $S^+[S^-(\bar{u})] = \bar{u}$, and so there is only one two-zonal invariant cone. In fact, it is easy to show that the derivatives of $S^\pm$ belong to the interval $(-1, 0)$, and then the attractiveness of the fixed point $\bar{u}$ follows.

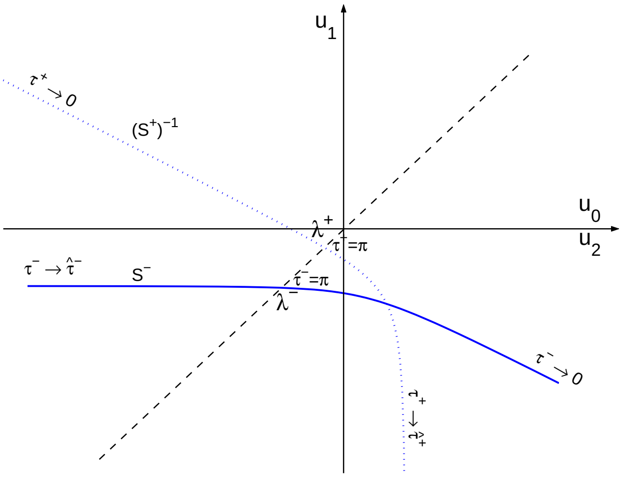

Fig. 6. Existence of one invariant cone when $\gamma^+ > 0$ and $\gamma^- > 0$. Note that $\tau^+ \in (\pi, \hat{\tau}^+)$ and $\tau^- \in (0, \pi)$.

Regarding statement (b) of the theorem, the three cases are related to the value of the quotient (10), so that there is a stable focus behavior if $y_2 < y_0$, there is a center when $y_2 = y_0$, and a unstable focus when $y_2 > y_0$. To verify that the three cases are possible, let us begin by considering the degenerate case $\lambda^+ = \lambda^-$. Here, from the last assertion of Remark 4, on the corresponding planar invariant cone it turns out that $\tau^- = \tau^+ = \pi$. Then, by using (6), (8), and (10)

$$\frac{r_2}{r_0} = \frac{y_2}{y_1}\frac{y_1}{y_0} = e^{\left(\frac{\alpha^+}{\beta^+} + \frac{\alpha^-}{\beta^-}\right)\pi} < 1,$$

where Lemma 5 has been taken into account. In this case, the dynamics on the planar invariant cone is of stable focus type. By perturbing this situation, it is immediate to get a similar dynamics on a non-planar invariant cone.

To show the existence of unstable dynamics on the invariant cone, let us choose $-\beta^- = \lambda^- < \alpha^- < 0$, and $\lambda^- < \lambda^+ < \alpha^+ < 0$. Then from Proposition 3, the phases of Poincaré half maps S^- and S^+ in the fixed point $\bar{u}$ satisfy $0 < \tau^- < \pi$, and $\pi < \tau^+ < \hat{\tau}^+$; see Fig. 6.

Keeping $\alpha^+, \lambda^+, \beta^+$ and α^- fixed, and putting $\lambda^- = -\beta^- = \lambda^+ - k$, by increasing k, the fixed point $\bar{u}$ moves towards u_H^+ while $S^-(\bar{u})$ tends to $-\infty$. Then, the right phase τ^+ goes towards $\hat{\tau}^+$, so that from Lemma 5 it turns out that $|y_2/y_1| \to \infty$, while the left phase τ^- remains in the interval $0 < \tau^- < \pi$. Thus, again from Lemma 5, one obtains

$$\left|\frac{y_1}{y_0}\right| > e^{\frac{\lambda^- \pi}{\beta^-}} = e^{-\pi}.$$

Hence, for a value of k sufficiently large, from (10) it follows that

$$\frac{r_2}{r_0} = \frac{y_2}{y_1}\frac{y_1}{y_0} > 1.$$

Therefore, the orbits contained in the invariant cone move away from the origin like in an unstable focus.

Finally, by considering the case $k = 0$, which corresponds with the already analyzed case $\lambda^+ = \lambda^-$, and using the continuity, there also exists a certain value k^* such that $r_2 = r_0$, and then the dynamics on the cone is of center type.

REFERENCES

Agrachev, A. and D. Liberzon (2001). Lie-algebraic stability criteria for switched systems. *SIAM J. Control Optim.* **40**, 253–269.

Andronov, A.A., A. Vitt and S. Khaikin (1966). *Theory of oscillations*. Pergamon Press. Oxford.

Boscain, U. (2002). Stability of planar switched systems: the linear single input case. *SIAM J. Control Optim.* **41**, 89–112.

Branicky, M. (1998). Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Automat. Contr.* **43**, 475–482.

Carmona, V., E. Freire, E. Ponce and F. Torres (2002). On simplifying and classifying piecewise linear systems. *IEEE Trans. Circuits Systems I: Fund. Theory Appl.* **49**, 609–620.

Carmona, V., E. Freire, E. Ponce and F. Torres (2005*a*). Bifurcation of invariant cones in piecewise linear homogeneous systems. *To appear in Int. J. of Bifurcation and Chaos.*

Carmona, V., E. Freire, E. Ponce and F. Torres (2005*b*). The continuous matching of two stable linear systems can be unstable. *Preprint.*

Freire, E., E. Ponce, F. Rodrigo and F. Torres (1998). Bifurcation sets of continuous piecewise linear systems with two zones. *Int. J. of Bifurcation and Chaos* **8**, 2073–2097.

Johansson, M. and A. Rantzer (1998). Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Trans. on Automatic Control* **43**, 555–559.

King, C. and M. Nathanson (2004). On the existence of a common quadratic Lyapunov function for a rank one difference. *arXiv:math* **OC/0403467**, 1–18.

Liberzon, D. and A.S. Morse (1999). Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine* pp. 59–70.

Shorten, R. and K.S. Narendra (2003). On common quadratic Lyapunov functions for pairs of stable lti systems whose system matrices are in companion form. *IEEE Trans. on Automatic Control* **48**, 618–621.

Shorten, R., K.S. Narendra and O. Mason (2003). A result on common quadratic Lyapunov functions. *IEEE Trans. on Automatic Control* **48**, 110–112.