GLOBAL ASYMPTOTIC STABILIZATION BY USING THE CONTROL LYAPUNOV FUNCTION

Takayuki Tsuzuki * **Yuh Yamashita** *
Ryuji Enomoto **

* *Kita-ku N14W9, Sapporo, Hokkaido, 060-8028, Japan*
** *1-1 Ikegamicho, Toba, Mie prefecture, Japan*

Abstract: It is known that if state spaces of control systems are not contractible, the systems are not globally asymptotically stabilizable by using C^1 feedback laws. We set multiple singular points of a flow to solve the topological obstruction. Given Morse functions satisfying conditions of the control Lyapunov function except for the singular points, this paper propose a globally asymptotically stabilizing feedback laws of systems on manifolds.

Keywords: Lyapunov function; Stabilization; Global stability; Singular points

1. INTRODUCTION

This paper study the global asymptotic stabilization of nonlinear control systems on manifolds by using the idea of the control Lyapunov function and assigning multiple singular points.

For nonlinear control systems, the control Lyapunov function (CLF) is useful for establishing that controlled systems are asymptotically stable (Sontag, 1989; Malisoff and Sontag, 2000). A stabilizing feedback law that is nonsmooth at only the origin can be derived from the CLF for control affine systems.

However, it is known that if flows on a smooth manifolds have only one asymptotically stable point, the manifolds are diffeomorphic to $\mathbb{R}^n$ (Milnor, 1964), and if state spaces of control systems are not contractible, then the systems are not globally asymptotically stabilizable by using C^1 feedback law (Sontag, 1998). Hence the systems with noncontractible state spaces have no CLF.

The design method for gradient-like Morse-Smale controlled systems deals with the almost everywhere global asymptotic satbilization problem of control systems on manifolds(Enomoto and Shima, 1998*a*). The systems are said almost everywhere globally asymptotically stable to the origin, if all trajectories but a reduced set of zero Lebesgue measure converge asymptotically to the origin. In this design method, the topological obstructions are solved by the assignment of multiple singular points of gradient-like Morse-Smale flows to a global compact attractor. The gradient-like Morse-Smale flow is the flow such that the chain recurrent sets consist of finite number of hyperbolic singular points and each pair of stable and unstable manifolds of the singular points is transverse (Clark, 1999).

The assignment of singular points is determined by using the theory of the configuration space (Enomoto and Shima, 2003*c*; Enomoto and Shima, 2003*d*), the Poincaré-Hopf index formula (Enomoto and Shima, 2003*a*; Enomoto and Shima, 2003*b*) and the topological structures of flow that allow global asymptotic stability (Enomoto and Shima, 1998*b*; Enomoto *et al.*, 2004).

In this paper, given a singular point assignment, an extended CLF admitting the multiple singular points is defined. We show that the extended CLF can derive a globally asymptotically stabilizing feedback law.

2. PRELIMINARIES

2.1 Description of the system

Throughout this paper, control systems are described by the following equation:

$$\dot{x} = f(x,u) = f_0(x) + g(x)u, \tag{1}$$

where $x \in \mathbb{X}$ denotes the state and $u \in \mathbb{U}$ the scalar input. It is assumed that the vector fields f_0 and g are smooth, $\mathbb{X}$ is an n dimensional smooth manifold, $\mathbb{U}$ is a 1 dimensional manifold and $\mathbb{U} = \mathbb{R}$, where $\mathbb{R}$ represents the real numbers.

The state feedback law is considered as

$$u = k(x). \tag{2}$$

This feedback law is discontinuous on the union of the submanifolds:

$$S := S_1 \cup S_2 \cup \cdots \cup S_m, \tag{3}$$
$$S_i := \{x \in \mathbb{X} | \phi_i(x) = 0\}, \tag{4}$$

where $\phi_i(x)$ is a smooth function, $\nabla\phi_i(x) \neq 0$ for any $x \in S_i$ and S_i intersects transversely with S_j for each i, j. The subset S is called the switching surface. The switching surface divides $\mathbb{X}$ into two subsets:

$$\begin{aligned} G^+ &:= \{x \in \mathbb{X} | \phi(x) > 0\}, \\ G^- &:= \{x \in \mathbb{X} | \phi(x) < 0\}, \end{aligned} \tag{5}$$

where $\phi(x) := \phi_1(x) \cdots \phi_m(x)$. The feedback law (2) is smooth in each subset G^+ or G^-.

In this paper, $\mathrm{cl}(A)$ denotes the closure of a set A, $\mathrm{int}(A)$ the interior of a set A, $\mathrm{co}(A)$ the convex hull of a set A, ∂A the boundary of a manifold A, and $A \backslash B$ the complement set of B with respect to A.

2.2 Definition of the solution

The controlled system with a discontinuous feedback law k is an ordinary differential equation with discontinuous right hand side:

$$\dot{x} = f(x, k(x)). \tag{6}$$

Filippov defined solutions of Eq. (6) as trajectories $x(t)$ satisfying the differential inclusion (Filippov, 1988) :

$$\dot{x} \in F(x) := \{f(x,u) | u \in K(x)\} \tag{7}$$

for almost every t, where K is the set-valued mapping

$$K(x) := \bigcap_{\varepsilon>0} \bigcap_{\mu(N)=0} \mathrm{cl}(\mathrm{co}(k(B_\varepsilon(x) \backslash N))). \tag{8}$$

Here $B_\varepsilon(x)$ is open ball of radius $\varepsilon > 0$ centered at x, N any subset, and $\mu(N)$ a measure of N.

Because Eq. (1) is linear with respect to the scalar input, $F(x)$ is nonempty, convex, bounded and closed set for all $x \in \mathbb{X}$. It is obvious also that F is upper semi-continuous because of the definition of $K(x)$. Then, solutions for Eq. (7) exist, though their uniqueness do not always hold.

For $\bar{x} \in S_i, x^+ \in \{x | \phi_i(x) > 0\}, x^- \in \{x | \phi_i(x) < 0\}$, the limits of the vector fields at $\bar{x}$ are

$$f_i^+(\bar{x}) := \lim_{x^+ \to \bar{x}} \nabla\phi_i(x^+) \cdot f(x^+, k(x^+)), \tag{9}$$
$$f_i^-(\bar{x}) := \lim_{x^- \to \bar{x}} \nabla\phi_i(x^-) \cdot f(x^-, k(x^-)). \tag{10}$$

If $f_i^+(\bar{x}) < 0$ and $f_i^-(\bar{x}) > 0$, the solutions converge to S_i in a neighborhood of $\bar{x}$. The solutions are called the sliding mode on S_i at $\bar{x}$. If $f_i^+(\bar{x}) > 0$ and $f_i^-(\bar{x}) < 0$, they are called the repelling mode on S_i at $\bar{x}$. The solutions starting going off S_i at $\bar{x}$ exist in the repelling mode on S_i.

Filippov's definition is inconvenient for our purpose, because it can contain stationary solutions in the repelling mode. In this paper, we propose a new definition of the solutions of Eq. (6) for the interval $[0, \infty)$.

Definition 1. If the repelling mode on S_i at $\bar{x} \in S_i$ occurs, the solutions starting from $\bar{x}$ of Eq. (6) for small positive time are defined by Filippov's except for solutions remaining in S_i. In the other cases, we use the Filippov's solutions.

If there exists no repelling mode at any $\bar{x} \in S$, the above definition is equal to Filippov's definition.

We define singular points including stationary points of flows.

Definition 2. We call a point x^* the singular point of the controlled system (6), if $0 \in F(x^*)$.

Let $x^* \in \cap_{j=1}^{\bar{m}} S_{i_j}$ be a singular point for $\bar{m} \leq m$. If the sliding mode on all S_{i_j} at x^* occurs, at least one solution starting from x^* is stationary. If the repelling mode on S_{i_j} for j happens, there is no stationary solution in x^*.

We adopt the following definition of asymptotic stability of the singular point: If for a singular point any solution having any initial state in a neighborhood of the singular point is attracted to the singular point without excursion, the singular point is locally strongly asymptotically stable(Clarke, 1998).

3. INTRODUCTION OF THE SINGULAR POINT ASSIGNMENT

In this section, we study the problem of singular point assignment. The singular point assignment is useful in constituting Lyapunov functions on $\mathbb{X}$ in order to deal with the stabilization problem of the systems on manifolds.

Gradient-like Morse-Smale flows on C^∞ compact manifolds have Lyapunov-Morse functions (Smale, 1961; Franks, 1979). The Lyapunov-Morse function for the flow $\psi : \mathbb{X} \times \mathbb{R} \to \mathbb{X}$ is defined by Morse functions V satisfying

- for any $x \in \mathbb{X}$ except all singular points, t and $s > 0$,
$$V(\psi(x,t)) > V(\psi(x,t+s)), \qquad (11)$$
- the critical points of V coincide with the singular points of the flow.

Morse functions are the smooth functions with non-degenerate critical points(Milnor, 1963). Then, we consider the gradient-like Morse-Smale flows that are suitable for the control systems from two viewpoints.

The first is global topological structures of flows on $\mathbb{X}$. Assume that a gradient-like Morse-Smale flow on $\mathbb{X}$ has one stable point, the number of the singular points with the ν-dimensional unstable manifold is represented by l_ν, and M is a compact attractor that contains all singular points and is homotopy equivalent to $\mathbb{X}$. Then, the relative Poincaré-Hopf index formula holds (Pugh, 1968):

$$\chi(M, \partial M) = l_0 - l_1 + l_2 - \cdots + (-1)^n l_n, \quad (12)$$

where $\chi(M, \partial M)$ is the Euler characteristic of the pair of compact manifolds $(M, \partial M)$, $\chi(M, \partial M) = \chi(M) - \chi(\partial M)$, and $\chi(\circ)$ denotes the Euler characteristic of a compact manifold $\circ$.

The second is the existence of smooth feedback laws in a neighborhood of the singular point of the gradient-like Morse-Smale flow. For any singular point of the flow, there must exist feedback laws such that the point is a hyperbolic singular point with corresponding unstable manifold for the controlled system.

For gradient-like Morse-Smale flows with one stable singular point $p^{(0)} \in \mathbb{X}$ and the pair of singular points satisfying Eq. (12), the pair is called the specification of singular point assignment of the control system for the flows:

$$L = \{l_0, l_1, \ldots, l_n; p^{(0)}\}. \qquad (13)$$

There are Lyapunov-Morse functions for a gradient-like Morse-Smale flow such that all critical points correspond to the singular points of the flow. In the next section, we use the function to extend the definition of CLF.

4. GLOBAL ASYMPTOTIC STABILIZATION

Assume that a specification of singular point assignment and locations of the singular points of the flows are given for a gradient-like Morse-Smale flow. The points are called the desired singular points.

This section discusses the global asymptotic stabilization of the system by using an extended control Lyapunov function.

4.1 Definition of control Lyapunov-Morse function

The original CLF is applicable to global asymptotic stabilization, only if the controlled system has a single stable singular point. We extend CLF to global stabilization to the origin of the controlled system with multiple singular points by using a Lyapunov-Morse function the critical points of which are equal to the desired singular points.

Definition 3. Let a specification of singular point assignment and the locations of the singular points be determined. The control Lyapunov-Morse function (CLMF) is defined by the Morse function V satisfying the following:

- The function V is smooth, positive definite and proper. V is said to be proper, if the subsets
$$\{x \in \mathbb{X} | V(x) \le \alpha\} \qquad (14)$$
are bounded for every $\alpha \ge 0$.
- Except the desired singular points,
$$L_g V(x) = 0 \Rightarrow L_{f_0} V(x) < 0. \qquad (15)$$
- Any desired singular point p is identical with a critical point of V:
$$\frac{\partial V}{\partial x}(p) = 0. \qquad (16)$$
- For each critical point p on V, the number of the eigenvalues with negative real part of the Hessian of V at p is equal to the dimension of the unstable manifold of the desired singular point p.

Because of the definition of CLMF, the critical points are equal to the desired singular points.

The CLMF is identified as the original CLF, if the controlled system has a single stable singular point.

4.2 Designing a stabilizing feedback law

Given a CLMF V such that the set $S := \{x | L_g V(x) = 0\}$ is regarded as the union of submanifolds S_i defined by Eq. (4), the following feedback law is defined:

$$k(x) := k_S(x) + V(x)k_D(x), \tag{17}$$

where for a constant $\alpha > 0$, k_S and k_D is described below:

$$k_S := \begin{cases} -\dfrac{L_{f_0}V + \sqrt{(L_{f_0}V)^2 + (L_gV)^4}}{L_gV}, \\ \text{if } L_gV \neq 0 \\ 0, \quad \text{if } L_gV = 0, \end{cases} \tag{18}$$

$$k_D := -\alpha \mathrm{sign}(L_gV), \tag{19}$$

$$\mathrm{sign}(L_gV) := \begin{cases} 1, & L_gV > 0 \\ -1, & L_gV < 0 \end{cases}. \tag{20}$$

The first term k_S is a stabilizing formula (Sontag, 1989) and may be discontinuous at x satisfying $L_gV(x) = 0$. The second term k_D has discontinuity on the switching surface. Therefore, k is smooth on $\mathbb{X}\backslash S$.

Theorem 4. Assume that for any critical point p of V, $f_0(p) = 0$ and $g(p) \neq 0$. Then, the controlled system with k has the following properties:

(1) The critical points of V are equivalent to the singular points of the controlled system.
(2) V strictly decreases along the trajectories of the controlled system except the critical points.
(3) The origin is locally strongly asymptotically stable.

Before proving the theorem, we remark that the next lemma holds.

Lemma 5. Let V be a CLMF. If for any critical point p of V, $f_0(p) = 0$ and $g(p) \neq 0$. Then, V satisfies the small control property in a neighborhood of p.

PROOF. At first, the equivalence of the critical points and singular points is proven. If p is a critical point, then $L_{f_0}V(p) = L_gV(p) = 0$. The controlled system is

$$f(p, k(p)) = g(p)\{k_S(p) + V(p)k_D(p)\}. \tag{21}$$

Because of the small control property, k_S is continuous on $\mathbb{X}$ and equals zero on S. Then the right hand side of Eq. (7) includes zero:

$$\{f(p, u) | u \in [-\alpha V(p), \alpha V(p)]\} \ni 0. \tag{22}$$

By the definition, p is a singular point.

Let q be a singular point of the controlled system:

$$F(q) = \{f(q, u) | u \in \mathrm{cl}(\mathrm{co}(k(q)))\} \ni 0. \tag{23}$$

Then, the following equation holds:

$$\left\{ \frac{\partial V}{\partial x}(q) \cdot v \,\middle|\, v \in F(q) \right\} \ni 0. \tag{24}$$

For any point x such that $L_gV(x) \neq 0$, F is a single valued function and

$$\frac{\partial V}{\partial x}(x) \cdot F(x) < 0. \tag{25}$$

Hence, there is no singular point on $\mathbb{X}\backslash S$. For $x \in S$,

$$\frac{\partial V}{\partial x}(x) \cdot F(x) = -\sqrt{(L_{f_0}V(x))^2}. \tag{26}$$

Because for any singular point q Eq. (24) holds, $L_{f_0}V(q) = L_gV(q) = 0$ must be satisfied. It is clear that if q is a singular point, then q is the critical point by the definition of the CLMF. Therefore, the critical points are equivalent to the singular points.

It is obvious that for any $x \in \mathbb{X}$

$$\max_{v \in F(x)} \left(\frac{\partial V}{\partial x} \cdot v \right) \leq 0. \tag{27}$$

Because the equation equals zero, only if $L_{f_0}V(x') = L_gV(x') = 0$, V strictly decreases along the trajectories except the singular points.

The origin is locally strongly asymptotically stable, since V is a local Lyapunov function on the controlled system in a neighborhood of the origin. □

4.3 Conditions for global asymptotic stability

If a CLMF satisfies the next condition, the repelling mode on S occurs at any singular point except the origin for Eq. (6), i.e. the origin is globally asymptotically stable.

Theorem 6. Let the assumption of theorem 4 hold. If $g^T(p)\frac{\partial^2 V}{\partial x^2}(p)g(p)$ is negative for any singular point p except the origin, then the origin is globally asymptotically stable.

PROOF. That the controlled system has the strict Lyapunov function V decreasing along the trajectory except the singular points has been proven.

We study the dynamics on S, because all singular points are on S. We regard L_gV as a new coordinate. The dynamics of L_gV is the following:

$$\frac{d}{dt}(L_gV(x)) = L_{f_0}L_gV(x) + L_g^2V(x)u. \tag{28}$$

The equivalent control u^{eq} represents the input such that Eq. (28) is equal to zero when $L_g^2V \neq 0$ (Utkin, 1977):

$$u^{eq}(x) := -\frac{L_{f_0}L_gV(x)}{L_g^2V(x)}. \tag{29}$$

Substituting the feedback law(17) into Eq. (28),

$$\frac{d}{dt}(L_g V) = L_{f_0} L_g V + L_g^2 V\{k_S + V k_D\}$$
$$= -L_g^2 V\{u^{eq} - k_S + \alpha V \mathrm{sign}(L_g V)\}. \tag{30}$$

The equation implies that the stability of $L_g V$ depends on $\alpha V - |u^{eq} - k_S|$ and $L_g^2 V$. Let $\alpha V - |u^{eq} - k_S|$ be positive. If $L_g^2 V$ is positive, the dynamics of $L_g V$ is asymptotically stable. If $L_g^2 V$ is negative, it is unstable.

The limits of vector field are

$$f_i^+(\bar{x}) = -L_g^2 V(\bar{x})\{u^{eq}(\bar{x}) - k_S(\bar{x}) + \alpha V(\bar{x})\}, \tag{31}$$

$$f_i^-(\bar{x}) = -L_g^2 V(\bar{x})\{u^{eq}(\bar{x}) - k_S(\bar{x}) - \alpha V(\bar{x})\}, \tag{32}$$

at $\bar{x} \in S_i$ satisfying $L_g^2 V(\bar{x}) \neq 0$.

For any singular point p

$$L_g^2 V(p) = g^T(p) \frac{\partial^2 V}{\partial x^2}(p) g(p). \tag{33}$$

The inequality $g^T(p)\frac{\partial^2 V}{\partial x^2}(p)g(p) < 0$ is sufficient to happen the repelling mode at any singular point p except the origin, because $u^{eq}(p) = k_S(p) = 0$ and $\alpha V(p) > 0$. Then, no existence of the solutions remaining in S follows from the definition of solutions. The origin is globally asymptotically stable. □

5. EXAMPLE

In this section, we study the problem of the global stabilization to the origin of the system;

$$\begin{aligned} \dot{\theta} &= \omega, \\ \dot{\omega} &= g \sin\theta - \zeta\omega + u, \end{aligned} \tag{34}$$

where $g = 10, \zeta = 0.1$.

If the number of rotations of θ is available, θ is an element of $\mathbb{R}$ and the exact linearization of Eq. (34) is solvable. However, if the number is not available, the state space is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Then, the system can not be linearized, and no original CLF exists.

The controlled system must satisfy the following properties from the solution of the problem of singular point assignment:

- For any singular point, $\omega = 0$.
- Because $\chi(M, \partial M) = 0$, the relative Poincaré-Hopf index formula derives

$$l_0 - l_1 + l_2 = 0. \tag{35}$$

From the above, $L = \{1, 1, 0, p_1\}$ can be chosen as the specification of singular point assignment. The stable singular point p_1 is assigned to the origin and the saddle singular point p_2 is assigned to $(\pi, 0)$.

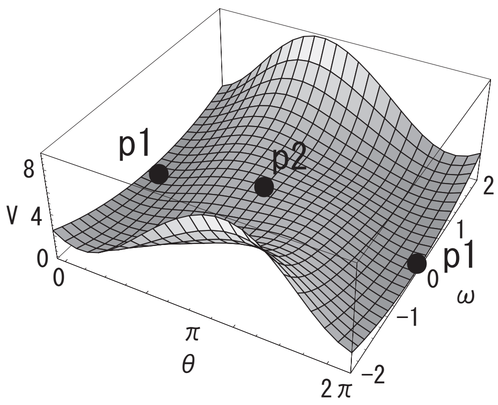

Fig. 1. V on $\mathbb{X}$

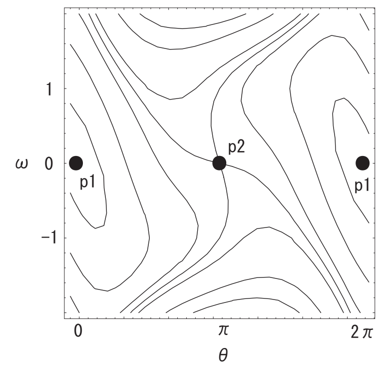

Fig. 2. Levelset of V on $\mathbb{X}$

5.1 Stabilization by using a CLMF

The following function V is a candidate of CLMF:

$$V(x) = 1-\cos\theta+\frac{1}{4}(\omega^2-1)^2(1-\cos\theta)+\frac{1}{2}(\omega+\sin\theta)^2. \tag{36}$$

Figure 1 illustrates V on $\mathbb{X}$, and Fig.2 shows the levelset of V. V is smooth, positive definite and proper.

$L_g V$ is expressed by

$$L_g V(x) = \omega(\omega^2 - 1)(1 - \cos\theta) + \omega + \sin\theta. \tag{37}$$

Figure 3 illustrates $L_{f_0} V$ on S. The solutions of $L_{f_0} V = 0$ are only $(0, 0)$ and $(\pi, 0)$. It follows from Fig.3 that except for the desired singular points, the following holds:

$$L_g V(x) = 0 \Rightarrow L_{f_0} V(x) < 0. \tag{38}$$

Because the critical points of V are $(0, 0)$ and $(\pi, 0)$, the desired singular points are identical with the critical points.

The eigenvalues of the Hessian of V at p_1 are $\{2.80425, 0.445752\}$ and those at p_2 are $\{-1.693, 0.443\}$. The dimension of the unstable manifold of each desired singular point is equivalent to the number of the eigenvalues with negative real part.

V satisfies the conditions of the CLMF.

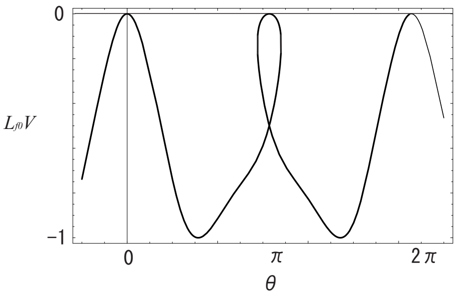

Fig. 3. $L_f V$ on S

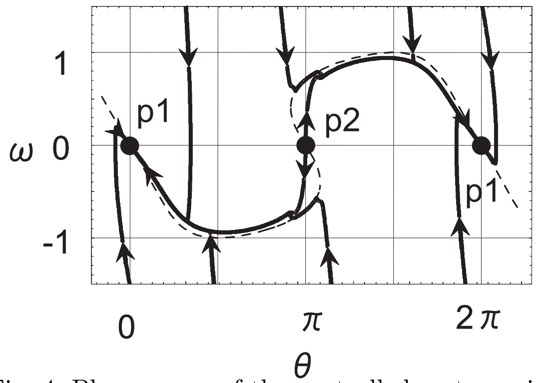

Fig. 4. Phase space of the controlled system with $u = k(x)$ for $V(x)$ and S

The repelling mode on S occurs at p_2 and the sliding mode on S occurs at p_1, because $g^T(p_1)\frac{\partial^2 V}{\partial x^2}(p_1)g(p_1) = 1$, $g^T(p_2)\frac{\partial^2 V}{\partial x^2}(p_2)g(p_2) = -1$. Therefore, the origin of the controlled system with the feedback law (17) is globally asymptotically stable.

Figure 4 indicates the simulation result for the feedback law (17) for $\alpha = 1$. The repelling mode on S happens at p_2, and the sliding mode at p_1.

6. CONCLUSIONS

We have defined the control Lyapunov-Morse function for a specification of singular point assignment and the location of the singular points.

Given the control Lyapunov-Morse function, a feedback law (17) was defined. We have shown that if for each singular point except the origin $g^T \frac{\partial^2 V}{\partial x^2} g$ is negative, the controlled system with the feedback law is globally asymptotically stable to the origin.

REFERENCES

Clark, R. (1999). *Dynamical systems.* CRC Press. Boca Raton.

Clarke, F.H. (1998). Asymptotic stability and smooth Lyapunov functions. *Journal of Differential Equations* **149**, 69–114.

Enomoto, R. and M. Shima (1998*a*). Control system design for global asymptotic stabilization via the singular point assignment (in Japanese). *Trans. of SICE* **34**, 126–134.

Enomoto, R. and M. Shima (1998*b*). Singular point assignment via the Conley index theory (in Japanese). *Trans. of SICE* **34**, 584–592.

Enomoto, R. and M. Shima (2003*a*). Boundary tangency manifolds for state feedback systems and index formula (in Japanese). *Trans. of SICE* **39**, 646–653.

Enomoto, R. and M. Shima (2003*b*). Intersection theory of boundary tangency manifolds for state feedback systems (in Japanese). *Trans. of SICE* **39**, 654–661.

Enomoto, R. and M. Shima (2003*c*). Intersection theory on configuration spaces and its applications I (in Japanese). *Trans. of ISICE* **16**, 1–10.

Enomoto, R. and M. Shima (2003*d*). Intersection theory on configuration spaces and its applications II (in Japanese). *Trans. of ISICE* **16**, 11–19.

Enomoto, R., T. Tsuzuki and M. Shima (2004). Feedback control design using the transition of topological structure of flows (in Japanese). *Trans. of SICE* **40**, 210–219.

Filippov, A.F. (1988). *Differential equations with discontinuous righthand sides.* Kluwer Academic Publishers. Dordrecht.

Franks, J.M. (1979). Morse-Smale flows and homotopy theory. *Topology* **18**, 199–215.

Malisoff, M. and E. D. Sontag (2000). Universal formulas for feedback stabilization with respect to Minkowski ball. *Systems & Control Letters* **40**, 247–260.

Milnor, J.W. (1963). *Morse Theory.* Princeton Univ. Press. Princeton.

Milnor, J.W. (1964). Differential topology. In: *Lectures on Modern Mathematics* (T.L. Saaty, Ed.). Vol. II. John Wiley & Sons, Inc.. London.

Pugh, C.C. (1968). A generalized Poincaré index formula. *Topology* **7**, 217–226.

Smale, S. (1961). On gradient dynamical systems. *Annals of Mathematics* **74**, 199–206.

Sontag, E.D. (1989). A 'universal' construction of Artstein's theorem on nonlinear stabilization. *Systems & Control Letters* **13**, 117–123.

Sontag, E.D. (1998). *Mathematical Control Theory; Determinisitic Finite Dimensional Systems Second Edition.* Springer. New York.

Utkin, V.I. (1977). Variable structure systems with sliding modes: a survey. *IEEE Trans. on Automatic Control* **22**, 212–222.