CONTINUOUS TIME SYSTEM IDENTIFICATION OF NONPARAMETRIC MODELS WITH CONSTRAINTS

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Abstract: Although structural constraints such as model order and time delay have been incorporated in the continuous time system identification since its origin, the constraints on the estimated model parameters were rarely enforced. This paper proposes a continuous time system identification approach with constraints. It shows that by incorporating physical parameter information known *a priori* as hard constraints, the traditional parameter estimation schemes are modified to minimize a quadratic cost function with linear inequality constraints. Using the structure of Frequency Sampling Filters as the vehicle, the paper shows that the constraints can be readily imposed on continuous time frequency response estimation and step response estimation. In particular, *a priori* knowledge in both time-domain and frequency domain is utilized simultaneously as the constraints for the optimal parameter solution. A Monte-Carlo simulation study with 100 noise realization is used to demonstrate the improvement of the estimation results in terms of continuous time frequency response and continuous time step response.

Keywords: Continuous time system identification, nonparametric model, frequency response, step response, constraints.

1. INTRODUCTION

Although structural constraints such as model order and time delay have been incorporated in continuous time system identification since its origin, the constraints on the estimated parameters were rarely enforced. In order to pre-specify a set of constraints in the estimation, *a priori* knowledge about the continuous time system parameters is required. This *a priori* knowledge may come from a system that is partially known or from experiments that have been performed independently.

Typical examples include the situation where the gain and time delay of the system are known from a step response test, the critical frequency information is known from individual sinusoid test. Another kind of constraints may be associated with the general requirement of the estimation algorithm, such as the case where the stability of the estimated model in an iterative procedure is required to facilitate the convergence of the algorithm. The necessary condition to guarantee this is then translated into the constraints that enforce all estimated coefficients in the denominator to be positive. The initial results obtained by the authors have shown the potential for wide applications in the estimation of partially unknown systems. This paper will be restricted to a non-parametric approach but a subsequent paper will deal with both.

This paper is to examine the cases where introducing hard constraints on the estimated parameters may improve the quality of an estimated continuous time model. The primary objectives are to take advantage of *a priori* knowledge of a given physical system and to translate this knowledge into hard constraints that are embedded into the estimation algorithms. More specifically, the paper examines estimation of continuous time non-parametric model with pre-specified constraints on certain model parameters. In the estimation of nonparametric models, the frequency sampling filters structure (Bitmead and Anderson, 1981, Wang and Cluett, 2000) is used as the vehicle to obtain estimation of continuous time frequency response for $\omega \leq \frac{\pi}{\Delta t}$ (Δt is the sampling interval) as well as the estimation of continuous time step response. In this framework, a priori knowledge about time delay, steady state gain and critical frequency response is combined together as a set of equality (we know the parameters for sure) and inequality (we know the intervals of the parameters) constraints for the parameter estimation. The solutions for optimal parameters become a quadratic programming problem which can be solved readily by using an existing software package or by using a simplified algorithm as discussed in (Luenberger,1969). We also show that by using the frequency sampling filters structure, discrete noise model can be used effectively as part of the scheme without any compromising.

The contributions of this paper include

- Derive an algorithm for estimation of continuous time frequency response and step response with constraints. A priori knowledge of frequency response and step response information is simultaneously utilized as either linear equality or inequality constraints. This algorithm also includes discrete noise models in the estimation scheme.
- Monte-Carlo simulation studies on how the constraints affect the variance and bias of the continuous time parameter estimation schemes.

2. ESTIMATION OF NON-PARAMETRIC MODELS WITH CONSTRAINTS

The popular non-parametric representation in system identification consists of frequency response, step response and impulse response. It is called non-parametric (in contrast to parametric) in the sense that the process dynamics is captured by a set of response coefficients, instead of description of process poles and zeros. It is known that step response representation is invariant between system descriptions in continuous time and discrete time at the sampling instant, and continuous time frequency response representation can be closely approximated by the discrete frequency representation up to its Nyquist frequency. Therefore, unlike the parametric representation, the continuous time non-parametric presentations can be readily approximately by its discrete counter-parts, leading to the estimation from discrete data.

2.1 The Estimation Problem

In the sequel, we will show the unified framework of estimating continuous time non-parametric representations using discrete time systems approach. Assume that the continuous time system is stable with transfer function $G_c(s)$. The system is sampled uniformly with an interval Δt, and the system has a settling time T_s such that when $t \geq T_s$, the impulse response $h(t) \approx 0$. The corresponding discrete parameter to T_s is $N = \frac{T_s}{\Delta t}$. The discrete transfer function of the system can be represented in terms of the frequency response coefficients via the frequency sampling filters expression (Wang and Cluett, 2000):

$$G(z) = \sum_{l=-\frac{n-1}{2}}^{\frac{n-1}{2}} G(e^{jl\Omega}) H^l(z) \tag{1}$$

where n is an odd number to represent the number of frequencies included in the frequency sampling filters model; Ω is the fundamental sampling frequency defined by $\Omega = \frac{2\pi}{N}$. The lth frequency sampling filter is given as

$$\begin{aligned} H^l(z) &= \frac{1}{N} \frac{1 - z^{-N}}{1 - e^{jl\Omega} z^{-1}} \\ &= \frac{1}{N}(1 + e^{jl\Omega} z^{-1} + ... + e^{j(N-1)l\Omega} z^{-(N-1)}) \end{aligned}$$

At $z = e^{jl\Omega}$, $H^l(z) = 1$. Equation (1) can also be written in terms of real and imaginary parts of the discrete frequency response $G(e^{jl\Omega})$ (Bitmead and Anderson, 1981) as

$$\begin{aligned} G(z) &= \frac{1}{N}\frac{1-z^{-N}}{1-z^{-1}}G(e^{j0}) \\ &+ \sum_{l=1}^{\frac{n-1}{2}}[Re(G(e^{jl\Omega})F_R^l(z) \\ &+ Im(G(e^{jl\Omega})F_I^l(z)] \end{aligned} \tag{2}$$

where $F_R^l(z)$ and $F_I^l(z)$ are the lth second order filters given by

$$F_R^l(z) = \frac{1}{N}\frac{2(1-cos(l\Omega)z^{-1})(1-z^{-N})}{1-2cos(l\Omega)z^{-1}+z^{-2}}$$

$$F_I^l(z) = \frac{1}{N}\frac{2sin(l\Omega)z^{-1}(1-z^{-N})}{1-2cos(l\Omega)z^{-1}+z^{-2}}$$

- The frequency sampling filters model can be regarded as a hybrid structure between a continuous time system and a discrete time system when the sampling interval Δt is sufficiently small. For the continuous time frequency $\omega \leq \frac{\pi}{\Delta t}$, the continuous time frequency response $G_c(j\omega) \approx G(e^{j\omega\Delta t})$. Therefore, the coefficients of the discrete model are corresponding to continuous time frequency response at $\omega = 0, \frac{2\pi}{T_s}, \frac{4\pi}{T_s}, \ldots, \frac{\pi}{\Delta t}$.

Suppose that $u(k)$ is the process input, $y(k)$ is the process output and $v(k)$ is the disturbance signal. The output $y(k)$ can be expressed in a linear regression form by defining the parameter vector and the regressor vector as

$$\theta = \begin{bmatrix} G(e^{j0}) \\ Re(G(e^{j\Omega})) \\ Im(G(e^{-j\Omega})) \\ \vdots \\ Re(G(e^{j\Omega\frac{n-1}{2}})) \\ Im(G(e^{j\Omega\frac{n-1}{2}})) \end{bmatrix} \quad \phi(k) = \begin{bmatrix} f(k)^0 \\ f(k)_R^1 \\ f(k)_I^1 \\ \vdots \\ f(k)_R^{\frac{n-1}{2}} \\ f(k)_I^{\frac{n-1}{2}} \end{bmatrix}$$

where

$$f(k)^0 = \frac{1}{N}\frac{1-z^{-N}}{1-z^{-1}}u(k)$$

$$f(k)_R^l = F_R^l(z)u(k); f(k)_I^l = F_I^l(z)u(k)$$

for $l = 1, 2, \ldots, \frac{n-1}{2}$. This allows us to write the linear regression with correlated residuals as

$$\begin{aligned} y(k) &= \phi(k)^T\theta + v(k) \\ v(k) &= \frac{\epsilon(k)}{D(z)} \end{aligned} \tag{3}$$

where $\epsilon(k)$ is a white noise sequence with zero means and standard deviation σ. Given a set of sampled finite amount of data

$$\{y(1), y(2), y(3), \ldots, y(M)\}$$
$$\{u(1), u(2), u(3), \ldots, u(M)\}$$

we can obtain an estimate of the frequency sampling filter model and an estimate of the noise model $\frac{1}{D(z)}$ using the generalized Least Squares method (Clarke, 1967, Soderstrom, 1974). More specifically, in the core estimation algorithm, we let

$$y_D(k) = \hat{D}(z)y(k); \phi_D(k) = \hat{D}(z)\phi(k)$$

The estimation of $\hat{\theta}$ is obtained by minimizing the quadratic performance index

$$\begin{aligned} J &= \sum_{k=1}^{M}[y_D(k) - \phi_D(k)\theta]^2 \\ &= \theta^T\sum_{k=1}^{M}[\phi_D(k)\phi_D(k)^T]\theta \\ &- 2\theta^T\sum_{k=1}^{M}[\phi_D(k)y_D(k)] + cons \end{aligned} \tag{4}$$

$\hat{D}(z)$ is estimated from the error sequence $e(k) = y(k) - \phi(k)^T\hat{\theta}$, $k = 1, 2, 3, \ldots, M$. The generalized Least Squares method is based on an iterative procedure and the iteration stops after the estimated parameters converge.

In order to obtain the estimated step response from the estimated frequency parameter vector θ, it can be easily verified (Wang and Cluett, 2000) that the step response of the system at the sample m is in a linear relation to the frequency parameter vector θ via

$$\hat{g}_m = Q(m)^T\hat{\theta} \tag{5}$$

where

$$Q(m) = \begin{bmatrix} \frac{m+1}{N} \\ 2Re(S(1,m)) \\ 2Im(S(1,m)) \\ \vdots \\ 2Re(S(\frac{n-1}{2}, m)) \\ 2Im(S(\frac{n-1}{2}, m)) \end{bmatrix}$$

$S(l, m) = \frac{1}{N}\frac{1-e^{jl\Omega(m+1)}}{1-e^{jl\Omega}}$, $l = 1, 2, \ldots, \frac{n-1}{2}$.

3. ESTIMATION WITH CONSTRAINTS

3.1 Imposing Constraints

Constraints on Frequency Parameters
Suppose that the continuous time system is known

at frequency $\gamma(< \frac{\pi}{\Delta t})$. By converting it to the discrete frequency $\gamma\Delta t$, from Equation (2) the frequency information can be expressed as

$$G(e^{j\gamma\Delta t}) = L(e^{j\gamma\Delta t})^T\theta \tag{6}$$

where

$$L(e^{j\gamma\Delta t}) = \begin{bmatrix} F(e^{j\gamma\Delta t})^0 \\ F(e^{j\gamma\Delta t})_R^1 \\ F(e^{j\gamma\Delta t})_I^1 \\ \dots \\ F(e^{j\gamma\Delta t})_R^{\frac{n-1}{2}} \\ F(e^{j\gamma\Delta t})_I^{\frac{n-1}{2}} \end{bmatrix}$$

This equation is then split into real and imaginary parts

$$Re(G(e^{j\gamma\Delta t})) = Re(L(e^{j\gamma\Delta t}))^T\theta \tag{7}$$
$$Im(G(e^{j\gamma\Delta t})) = Im(L(e^{j\gamma\Delta t}))^T\theta \tag{8}$$

If the frequency information is known quite accurately, then equality constraints based on equations (7) and (8) can be imposed in the solutions. This is particularly useful when the system has strong resonance, and the critical frequency information is used in the constraints to ensure good fitting. If the frequency information is known within certain bounds, then the inequality constraints can be imposed as

$$\begin{aligned} Re(G(e^{j\gamma\Delta t}))_{min} &\le real(L(e^{j\gamma\Delta t})\theta \\ &\le Re(G(e^{j\gamma\Delta t}))_{max} \\ Im(G(e^{j\gamma\Delta t}))_{min} &\le Im(L(e^{j\gamma\Delta t})\theta \\ &\le Im(G(e^{j\gamma\Delta t}))_{max} \end{aligned} \tag{9}$$

Constraints on Step Response Parameters Constraints on step response parameters will be based on equation (5). Given the *a priori* information about some step response coefficients g_m, $0 \le m \le N-1$, the equality constraint is formulated as

$$g_m = Q(m)^T\theta \tag{10}$$

where $Q(m)$ is defined by Equation (5). For inequality constraints, with specification of minimum and maximum of step responses, say $\underline{g}_m \le g_m \le \overline{g}_m$, then the inequality constraint on a step response coefficient g_m is formulated as

$$\underline{g}_m \le Q(m)^T\theta \le \overline{g}_m \tag{11}$$

3.2 Solution of the Estimation Problem with Constraints

The estimation problem with constraints is essentially to minimize the quadratic cost function

$$\begin{aligned} J = \theta^T &\sum_{k=1}^{M}[\phi_D(k)\phi_D(k)^T]\theta \\ &- 2\theta^T\sum_{k=1}^{M}[\phi_D(k)y_D(k)] + cons \end{aligned} \tag{12}$$

subject to equality constraints

$$M_1\theta = \gamma_1$$

and inequality constraints

$$M_2\theta \le \gamma_2$$

By defining $E = \sum_{k=1}^{M}[\phi_D(k)\phi_D(k)^T]$ and $F = -2\sum_{k=1}^{M}[\phi_D(k)y_D(k)]$, $M = [M_1^T \quad M_2^T]^T$, $\gamma = [\gamma_1^T \quad \gamma_2^T]^T$ the necessary conditions for this optimization problem (Kuhn-Tucker condition) are (Luenberger,1984)

$$\begin{aligned} E\Delta U + F + M^T\lambda &= 0 \\ M\Delta U - \gamma &\le 0 \\ \lambda^T(M\Delta U - \gamma) &= 0 \\ \lambda &\ge 0 \end{aligned} \tag{13}$$

where the vector λ contains the Lagrange multipliers. These conditions can be expressed in a simpler form in terms of the set of active constraints. Let S_{act} denote the index set of active constraints. Then the necessary conditions become

$$\begin{aligned} E\Delta U + F + \sum_{i\subset S_{act}}\lambda_i M_i^T &= 0 \\ M_i\Delta U - \gamma_i &= 0 \quad i \subset S_{act} \\ M_i\Delta U - \gamma_i &< 0 \quad i \not\subset S_{act} \\ \lambda_i &\ge 0 \quad i \subset S_{act} \\ \lambda_i &= 0 \quad i \not\subset S_{act} \end{aligned}$$

where M_i is the ith row of the M matrix. It is clear that if the active set were known, the original problem could be replaced by the corresponding problem having equality constraints only. Alternatively, suppose an active set is guessed and the corresponding equality constrained problem is solved. Then if the other constraints are satisfied and the Lagrange multipliers turn out to be nonnegative, that solution would be correct. In the case that only equality constraints are involved, the optimal solution has a closed-form as

$$\begin{bmatrix} E & M_1^T \\ M_1 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} -F \\ \gamma_1 \end{bmatrix} \tag{14}$$

Explicitly:

$$\lambda_1 = -(M_1E^{-1}M_1^T)^{-1}(\gamma_1 + M_1E^{-1}F) \tag{15}$$
$$\hat{\theta} = -E^{-1}(F + M_1^T\lambda_1) \tag{16}$$

When inequality constraints are required, an iterative algorithm is needed to solve the quadratic programming problem (Luenberger1984).

4. MONTE-CARLO SIMULATION STUDY

This section is to illustrate the strength of the proposed approach in the estimation of non-parametric models through a Monte-Carlo simulation study. The relay experiment proposed by Astrom and Hagglund (Astrom and Hagglund, 1984) is particularly suitable for continuous time identification as the sampling interval in the experiment can be chosen as small as desired. Wang et al. (1999) extended this experiment to include identification of more general class of models other than simple frequency response points. The same set of design parameters as in Wang and Gawthrop (2000) is used here to generate the input excitation signal. In the Monte-Carlo simulation study, a white noise sequence with standard deviation of 0.8 is used to generate $e(k)$ and the disturbance $\xi(k) = \frac{0.1}{1-0.9z^{-1}}$. 100 realizations of the white noise sequence are generated by changing the seed of the generator from 1 to 100.

The system used for simulation is given by the transfer function

$$G(s) = \frac{e^{-3s}}{(s^2+0.4s+1)(s+1)^3} \tag{17}$$

The sampling interval for this system is $\Delta t = 0.1$ second. The settling time T_s is estimated as 40 seconds, hence the number of samples to steady state $N = \frac{T_s}{\Delta t} = 400$. By using frequency sampling filters to parametrize this system, the number of frequency required is 63, yielding the number of parameters in the FSF model as $n = 125$. Figures 1- 3 show the magnitude and phase of the frequency responses when estimated without constraints. From the distribution of the responses, it is seen that the estimation of the non-parametric models is unbiased. However, the variances are large both for the frequency response and step response. To introduce equality constraints on the estimation, *a priori* knowledge about the system is required. The *a priori* knowledge for this system is assumed as time delay being approximately 1 seconds, the gain being 1, and the first pair of frequency response $G_c(\frac{2\pi}{T_s}) = 0.5305 - j0.8317$. The *a priori* knowledge about the steady state gain of the system is translated into the constraint on the first parameter of the FSF model while the *a priori* knowledge about the frequency response information is translated into two equality constraints on the second and third parameters of the FSF model. Note that a frequency information at an arbitrary frequency can be translated into

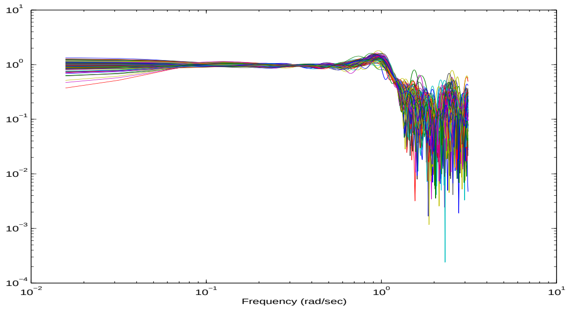

Fig. 1. Monte-Carlo simulation results without constraints: estimated frequency amplitude plot.

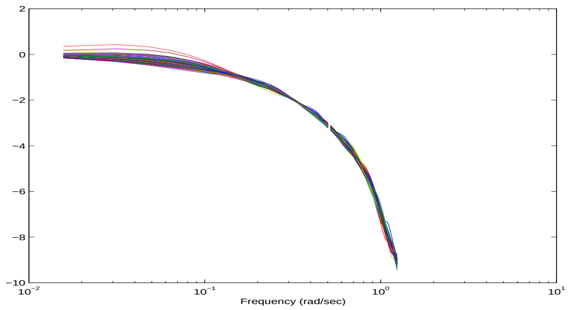

Fig. 2. Monte-Carlo simulation results without constraints: estimated frequency phase plot.

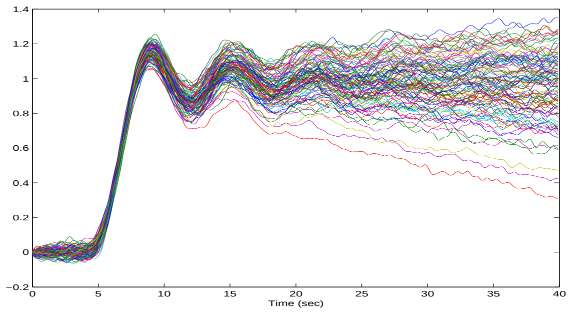

Fig. 3. Monte-Carlo simulation results without constraints: estimated step response plot.

constraints in a linear combination of the parameters of the FSF model. Similarly the *a priori* information about time delay is translated into a set of linear equality constraints in terms of the parameters of the FSF model. Four constraints have been put on the time delay at the sampling instant $k = 0, 3, 6, 9$. The reason for not using every sampling instant is because the solution is ill-conditioned when constraint is imposed on every sampling instant. As it is seen from the Monte-Carlo simulation study, this approach is adequate for this purpose. With the equality constraints imposed on the estimated parameters, the Generalized Least Squares method is modified to have the constraints on the system parameters, but not on the noise model parameters. Figures 4-6 show the magnitude and phase of the frequency responses when estimated with constraints. In comparison with the estimation results obtained

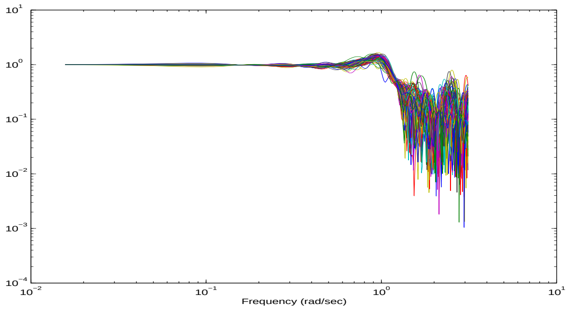

Fig. 4. Monte-Carlo simulation results with constraints: estimated frequency amplitude plot

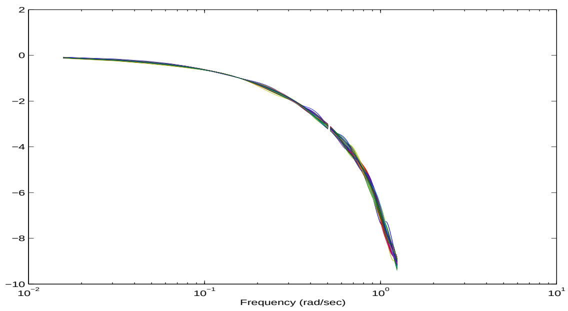

Fig. 5. Monte-Carlo simulation results with constraints: estimated frequency phase plot

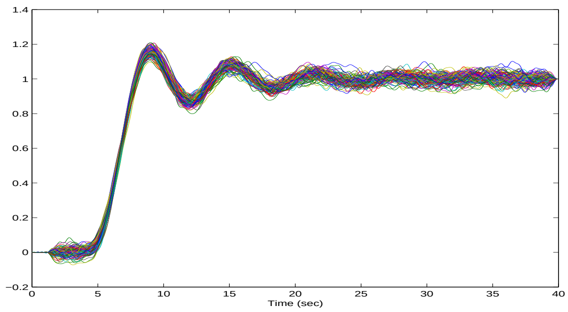

Fig. 6. Monte-Carlo simulation results with constraints: estimated step response plot.

without constraints, it is seen that the results are improved.

5. CONCLUSIONS

This paper discussed estimation of continuous time nonparametric models with constraints, where a priori knowledge is incorporated to improve the estimation results. A Monte-Carlo simulation study is used to demonstrate show the improvement of the estimation in a noise environment. The authors are currently working on analysis of the bias and variances when constraints are introduced in the estimation algorithm and the extension of this estimation to continuous time transfer function models. With recent work in unstable systems using FSF model (Gawthrop and Wang, 2004), the authors envisage the extension of this work to unstable systems.

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