NON AUTONOMOUS AFFINE SYSTEMS: CONTROL LYAPUNOV FUNCTION AND THE STABILIZATION PROBLEM [1]

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Abstract: Stabilization using the concept of control Lyapunov function is investigated for non linear non autonomous systems. First, necessary conditions for stabilization are given. Then, sufficient conditions ensure the existence of continuous stabilizing feedback for non autonomous affine systems. Lastly, a generalization of Sontag's formula is given. *Copyright ©2005 IFAC*

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1. INTRODUCTION

A standard problem in control theory is the stabilization of non linear systems. The problem of asymptotic controllability involving time varying control is completely solved for general non autonomous systems in Albertini and Sontag (1999) and Albertini and Sontag (1997). It is shown that asymptotic controllability is equivalent to the existence of a nonsmooth control Lyapunov function (CLF). The stabilization problem of non autonomous systems involving continuous dependent state control has not been addressed in Albertini and Sontag (1999) and Albertini and Sontag (1997). A seminal result is Artstein's theorem which proves, for autonomous affine systems, the existence of a smooth CLF is equivalent to the existence of a continuous feedback control. This result is a corollary of more general results involving relaxed control given in Artstein (1983). For affine systems Sontag gives in Sontag (1989) a general formula for the feedback law construction using a CLF. Contrary to what one can assume, the results on the stabilization of autonomous systems are not all valid for non autonomous ones. Hence, in this paper we extend the necessary and sufficient conditions for stabilization of Artstein's theorem and the Sontag's feedback to time-varying affine systems. On the other hand, there is no necessary and sufficient conditions for almost stabilization and the Sontag formula is not universal any more for non autonomous systems. The paper is organized as follows. Section 2 gives the formulation of the stabilization problem for non autonomous systems. Then, section 3 gives a necessary condition using the Kurzweil converse theorem (see the article by Kurzweil Kurzweil (1963) for the construction of the smooth CLF). Then it focuses on non autonomous affine systems: a sufficient condition is given and proved by using

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theorem Mickael (1956) on continuous selections. This two results leads to a necessary and sufficient condition fir stabilization. In section 4, a generalization of Sontag's formula is given. This formula is very interesting because with an additional condition related to the uniform property, it is an explicit and systematic construction, even if it is sometimes possible to find a more simple feedback control. This construction is illustrated through a system and its simulation.

2. PROBLEM FORMULATION

Through this paper $\mathcal{B}_\epsilon^n$ denotes the open ball in $\mathbb{R}^n$ centered at the origin of radius $\epsilon > 0$, $\|.\|_n$ the euclidian norm in $\mathbb{R}^n$ and $\mathcal{V}$ a neighborhood of the origin in $\mathbb{R}^n$. The paper aims at deriving an equivalent result to Artstein's one for systems in the following form

$$\dot{x} = f(t, x, u), \quad t \in \mathbb{R}, x \in \mathbb{R}^n \text{ and } u \in U, \quad (1)$$

where U is a non empty open set of $\mathbb{R}^m$ containing the origin and $f \in C^0(\mathbb{R} \times \mathbb{R}^n \times U, \mathbb{R}^n)$. In the following $x(t, t_0, x_0)$ will denote a solution starting from x_0 at t_0. First, let the origin be an equilibrium point of the system (1) with zero control. For example, assume that $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}$ and that the solution to the Cauchy Problem for initial condition near the origin has a unique solution. The question is under which conditions the stabilization problem for (1) may be solved. Thus, to start with, let us recall the meaning of this stabilization problem.

Definition 1. System (1) is *almost stabilizable* (respectively *almost C^k−stabilizable*), if there exists a feedback control law $u : \mathbb{R} \times \mathcal{V} \to U$ continuous (respectively C^k) on $\mathbb{R} \times \mathcal{V} \setminus \{0\}$ such that:

A1) $u(t, 0) = 0$ for all $t \in \mathbb{R}$,
A2) the origin is a uniformly asymptotically stable equilibrium of the closed-loop system:

$$\dot{x} = f(t, x, u(t, x)), \quad t \in \mathbb{R},\ x \in \mathcal{V}. \quad (2)$$

Moreover, if u is continuous (respectively C^k) on $\mathbb{R} \times \mathcal{V}$, then the system (1) is *stabilizable* (respectively *C^k−stabilizable*). If the system (1) is globally defined, it is *globally stabilizable* if there exists a continuous feedback control law: $u : \mathbb{R} \times \mathbb{R}^n \to U$ satisfying the two previous conditions A1) and A2) for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Let us recall that the origin of (2) is *uniformly asymptotically stable* if:

B1) the origin is uniformly stable for the system (2),
B2) the system is uniformly attractive: $\exists \delta > 0; \forall \epsilon > 0, \exists T(\epsilon) > 0;$

$$(x_0 \in \mathcal{B}_\delta^n) \Longrightarrow (x(t, t_0, x_0) \in \mathcal{B}_\epsilon^n)\ \forall t \geq t_0 + T(\epsilon).$$

A very important notion for stabilization is the CLF introduced by Artstein in Artstein (1983). In order to get rid of the time dependence of the stability motion, one needs to guarantee the uniform stability property. One adapts this notion for non autonomous systems and introduces an extension of the notion of decrescent Lyapunov function.

Definition 2. A continuous function $V : \mathbb{R} \times \mathcal{V} \to \mathbb{R}_+$, with continuous partial derivatives is said to be a decrescent Lyapunov function in short *DCLF* for the system (1) if:

C1) $\forall t \in \mathbb{R}$, $V(t, 0) = 0$,
C2) V is positive definite and decrescent [2],
C3) $\forall t \in \mathbb{R}$, $\frac{\partial V}{\partial t}(t, 0) = 0$,
C4) there exists a positive definite function W, such that for all $t \in \mathbb{R}$, and for all $x \in \mathcal{V} \setminus \{0\}$:

$$\inf_{u \in U} \left[\frac{\partial V}{\partial t} + \left\langle \frac{\partial V}{\partial x}, f(t, x, u) \right\rangle \right] \leq -W(x) < 0.$$

Definition 3. A DCLF for the system (1) $V : \mathbb{R} \times \mathcal{V} \to \mathbb{R}_+$ satisfies the *small control property* if for each $\epsilon > 0$ there exists a $\delta > 0$, for all $t \in \mathbb{R}$ and all $x \in \mathcal{V} \setminus \{0\} \cap \mathcal{B}_\delta^n$, there exists at least $u \in U$ with:

D1) $u \in \mathcal{B}_\epsilon^m$,
D2) $\frac{\partial V}{\partial t} + \left\langle \frac{\partial V}{\partial x}, f(t, x, u) \right\rangle \leq -W(x) < 0$.

3. THEORETICAL RESULTS ON THE STABILIZATION PROBLEM

A necessary condition for stabilization is the following:

Proposition 4. If the system (1) is stabilizable, then there exists a smooth DCLF for the system (1) which satisfies the small control property.

Proof. Suppose that the system (1) is stabilizable by a given feedback control $u(t, x)$ which is defined and continuous on $\mathbb{R} \times \mathcal{V}$. Using the Kurzweil converse theorem in Kurzweil (1963), one knows that there exists a smooth decrescent Lyapunov function $V : \mathbb{R} \times \mathcal{V} \to \mathbb{R}_+$ for the closed-loop system (2). So, there exists a positive definite function W such that for all $t \in \mathbb{R}$, and for all $x \in \mathcal{V} \setminus \{0\}$:

$$\frac{\partial V}{\partial t} + \left\langle \frac{\partial V}{\partial x}, f(t, x, u(t, x)) \right\rangle \leq -W(x) < 0.$$

Moreover, as $f(t, 0, 0) = 0$ and

$$\frac{\partial V}{\partial t}(t, 0) + \left\langle \frac{\partial V}{\partial x}(t, 0), f(t, 0, 0) \right\rangle = 0,$$

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[2] A function $v : \mathbb{R} \times \mathcal{V} \to \mathbb{R}$ is *decrescent* if $\lim_{\|y\| \to 0} v(t, y) = 0$ uniformly in t.

one deduces that $\frac{\partial V}{\partial t}(t,0) = 0$. Thus, V is a smooth DCLF of (1). The continuity of $u(t,x)$ on $\mathbb{R}^n \times \{0\}$ is equivalent to $\lim_{\|x\|\to 0} u(t,x) = 0$ uniformly in t, which gives the small control property with $u = u(t,x)$. ■

Remark 5. Contrary to the result of Artstein, if the system (1) is almost stabilizable it is not possible to extend the closed-loop system to a continuous system in order to build a DCLF with the Kurzweil converse theorem. The assumption of stabilization under a continuous feedback at the origin is compulsory.

This part deals with special classes of systems described by Artstein (1983). These are non autonomous affine systems of the form

$$\dot{x} = f_0(t,x) + \sum_{i=1}^{m} f_i(t,x)u_i, t \in \mathbb{R}, x \in \mathbb{R}^n, u \in U \tag{3}$$

where $f_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are continuous. One denotes by $f_u(t,x) = f_0(t,x) + \sum_{i=1}^{m} f_i(t,x)u_i$. With no loss of generality, let us assume that $f_0(t,0) = 0$ for all $t \in \mathbb{R}$. If the system (3) is stabilizable by the control $u(t,x)$ then the closed-loop system is given by

$$\dot{x} = f_{u(t,x)}(t,x), \quad t \in \mathbb{R}, x \in \mathbb{R}^n. \tag{4}$$

Let V be a DCLF for the system (3). For all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, let us define the following terms:

$$\begin{aligned} a(t,x) &= \frac{\partial V}{\partial t}(t,x) + \left\langle \frac{\partial V}{\partial x}(t,x), f_0(t,x) \right\rangle \\ b_i(t,x) &= \left\langle \frac{\partial V}{\partial x}(t,x), f_i(t,x) \right\rangle \text{ for } 1 \le i \le m \\ B(t,x) &= (b_1(t,x), ..., b_m(t,x)) \\ b(t,x) &= \sum_{i=1}^{m} b_i(t,x)^2 = \|B(t,x)\|^2 \end{aligned}$$

The proposition (6) is a generalization of a result due to Artstein in (Artstein, 1983, Theorem 5.1) to non autonomous systems. Our proof is far from different of the Artstein's proof. We use an the Mickael theorem on continuous selection which allow to give a simple proof.

Proposition 6. If V is a DCLF for the system (3) then the system (3) is almost stabilizable. If in addition, V satisfies the small control property, the system (3) is stabilizable. Moreover, if the assumptions hold globally and if V is radially unbounded [3], then the system (3) is almost globally stabilizable.

To prove this result, we need some set valued map concepts that can be found in Aubin and Cellina (1984). A *set valued function* F on the vector space $\mathcal{X}$ to the vector space $\mathcal{Y}$ is a function that associates with any $x \in \mathcal{X}$ a subset $F(x)$ of $\mathcal{Y}$. F is *lower semi-continuous* if $\{x \in \mathcal{X} : F(x) \cap O \neq \emptyset\}$ is open in $\mathcal{X}$ for every open $O \subset \mathcal{Y}$.

Proof. There exists a neighborhood $\mathcal{V}$ of the origin and a positive function W such that for all $x \in \mathcal{V}\backslash\{0\}$ and $t \in \mathbb{R}$, one defines the set valued function $\Phi : \mathbb{R} \times \mathcal{V}\backslash\{0\} \to 2^U$, $(t,x) \mapsto \Phi(t,x)$ where 2^U will denote the family of non-empty subsets of U and:

$$\Phi(t,x) = \{v \in U : a(t,x) + \langle B(t,x), v\rangle \le -W(x)\}.$$

$\Phi(t,x)$ is a non-empty, closed, convex set for all $(t,x) \in \mathbb{R} \times \mathcal{V}\backslash\{0\}$. As a and B are continuous, W can be chosen continuous with no loss of generality and thus Φ is lower semi- continuous on $\mathbb{R} \times \mathcal{V}\backslash\{0\}$. One may apply the Mickael theorem (see Mickael (1956) or Aubin and Cellina (1984)) to find a selection $u : \mathbb{R} \times \mathcal{V}\backslash\{0\} \to U$ (that is a continuous function u on $\mathbb{R} \times \mathcal{V}\backslash\{0\}$ such that $u(t,x) \in \Phi(t,x)$, extended by $u(t,0) = 0$ for all $t \in \mathbb{R}$. Then, V is a Lyapunov function for the closed loop system (4). As V is decrescent, the theorem of Lyapunov (see Slotine and Weiping (1991)), implies that the system (3) is almost stabilizable. If V satisfies the small control property, one may extend Φ on $\mathbb{R} \times \{0\}$ by $\Phi(t,0) = \{0\}$ for all $t \in \mathbb{R}$. Φ is now lower semi-continuous on $\mathbb{R} \times \mathcal{V}$, so there is a selection $u : \mathbb{R} \times \mathcal{V} \to U$ which stabilizes the system (3). Finally, the region of attraction of V contains the set

$$\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : V(t,x) < \min_{x \in \partial\mathcal{V}} V(t,x)\}$$

where $\partial\mathcal{V}$ is the boundary of $\mathcal{V}$. So, if V is radially unbounded $\partial\mathcal{V}$ is infinite and the region of attraction contains the whole space. ■

The two propositions 4 and 6 allows us to give a necessary and sufficient condition for the stabilization problem.

Corollary 7. The system (3) is stabilizable if and only if there exists a DCLF for the system (1) which satisfies the small control property.

It is important to notice that contrary to autonomous systems, there is no necessary and sufficient condition for the almost stabilization. This is due to the fact that the proposition (4) does not work for almost stabilization.

4. A CONSTRUCTIVE METHOD

Since, the proof of proposition 6 is not constructive, one may extend the construction of Sontag

[3] A function $v : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is *radially unbounded* if $\lim_{\|y\| \to +\infty} v(t,y) = +\infty$ uniformly in t.

to obtain an explicit feedback. One needs for a DCLF of (3) that there exists a positive definite function $\tilde{W} : \mathcal{V} \to \mathbb{R}$ with the property that for all $t \in \mathbb{R}$ and all $x \in \mathcal{V}$,

$$a(t,x)^2 + b\,(t,x)^2 \geq \tilde{W}(x). \qquad (5)$$

Under the existence of a DCLF satisfying the property (5), one obtains an explicit feedback control using the universal formula of Sontag in Sontag (1989).

Proposition 8. Suppose that for $1 \leq i \leq m$ the functions f_i are C^k (with $k \geq 1$). If there exists a DCLF for the system (3) which is C^k and satisfies the *property (5),* then the system (3) is almost $C^{k-1}-$stabilizable. If in addition the DCLF satisfies the small control property, then the system (3) is $C^{k-1}-$stabilizable. Moreover, if the assumptions hold globally and if the DCLF is radially unbounded, then system (3) is almost globally $C^{k-1}-$stabilizable.

Proof. Suppose there exists a C^k-DCLF $V : \mathbb{R} \times \mathcal{V} \to \mathbb{R}_+$. Let

$$E = \left\{(p,q) \in \mathbb{R}^2 : p < 0 \text{ or } q > 0\right\}$$

and φ defined by

$$\varphi\,(p,q) = \begin{cases} \dfrac{p + \sqrt{p^2+q^2}}{q} & \text{if } \ q \neq 0 \\ 0 & \text{if } \ q = 0 \end{cases}.$$

Equation $H\,(p,q,z) = qz^2 - 2pz - q^3 = 0$ holds with $z = \varphi\,(p,q)$ for $(p,q) \in E$. Moreover, $\frac{\partial H}{\partial z}\,(p,q,z) = 2\,(qz - p)$ and hence,

$$\frac{\partial H}{\partial z}\,(p,q,\varphi\,(p,q)) = \begin{cases} \sqrt{p^2+q^2} & \text{if } q \neq 0 \\ -2p & \text{if } q = 0 \end{cases}$$

is non zero for each $(p,q) \in E$. Thus, the implicit function theorem ascertains that φ is smooth on E. As V is a control Lyapunov function, then we know that $(a(t,x), b\,(t,x)) \in E$ for all $(t,x) \in \mathbb{R} \times \mathcal{V}\backslash\,\{0\}$. Thus, we define the feedback control by:

$$u_i\,(t,x) = \begin{cases} w_i\,(t,x) & \text{if } (t,x) \in \mathbb{R} \times \mathcal{V}\backslash\,\{0\} \\ 0 & \text{if } (t,x) \in \mathbb{R} \times \{0\} \end{cases}. \quad (6)$$

where $w_i\,(t,x) = -b_i\,(t,x)\,\varphi\,(a(t,x), b\,(t,x))$. u_i is C^{k-1} on $\mathbb{R} \times \mathcal{V}\backslash\,\{0\}$. With this feedback, one obtains for all $(t,x) \in \mathbb{R} \times \mathcal{V}\backslash\,\{0\}$

$$\frac{\partial V}{\partial t} + \left\langle \frac{\partial V}{\partial x}, f_{u(t,x)}\,(t,x)\right\rangle = -\sqrt{a(t,x)^2 + b\,(t,x)^2} < 0.$$

One knows that $\frac{\partial V}{\partial t}(t,0) = 0$ for all $t \in \mathbb{R}$, so, V is a Lyapunov function for the closed-loop system $(4)-(6)$, and using the Lyapunov theorem one knows that the origin of the closed loop system $(4)-(6)$ is stable. Moreover, since V satisfies the property (5), V is a Lyapunov function for the closed-loop system $(4)-(6)$ with a negative definite derivative. So, using the Lyapunov theorem, the origin of the closed loop system $(4)-(6)$ is asymptotically stable. Moreover, V is decrescent so the origin is uniformly asymptotically stable for the closed loop system $(4)-(6)$. Now, suppose that V satisfies the small control property. One wants to show that $\lim\limits_{\|x\| \to 0} u(t,x) = 0$ uniformly in t. For $t \in \mathbb{R}$, one knows that $\frac{\partial V}{\partial x}(t,0) = 0$ and $x \mapsto \|B\,(t,x)\|$ tends to zero uniformly in t. So adding the small control property to the previous remark, for each $\epsilon > 0$ there exists $\delta > 0$, for all $t \in \mathbb{R}$ and all $x \in \mathcal{V}\backslash\,\{0\} \cap \mathcal{B}^n_\delta$, there exists $u \in \mathbb{R}^m$ such that:

i) $\|u\|_m < \epsilon$
ii) $a(t,x) + \langle B\,(t,x)\,, u\rangle < 0$
iii) $\|B\,(t,x)\|_m < \epsilon$.

The second point implies that:

$$|a(t,x)| < \|B\,(t,x)\|_m\,\|u\|_m < \|B\,(t,x)\|_m\,\epsilon.$$

• <u>First case</u>: $a(t,x) > 0$ for $0 < \|x\|_n < \delta$. For $(p,q) \in E$ such that $q > 0$, we have:

$$\varphi\,(p,q) = \frac{p + \sqrt{p^2+q^2}}{q} \leq \frac{2\,|p| + |q|}{q} = 2\frac{|p|}{q} + 1.$$

Thus, in this case:

$$\begin{aligned} \|u(t,x)\|_m &= \|B\,(t,x)\|_m\,\varphi\,(a(t,x), b(t,x)) \\ &\leq \|B\,(t,x)\|_m \left(2\frac{\epsilon}{\|B\,(t,x)\|_m} + 1\right) \leq 3\epsilon. \end{aligned}$$

• <u>Second case</u>: $a(t,x) \leq 0$ for $0 < \|x\|_{\mathbb{R}^n} < \delta$. We have

$$\begin{aligned} a(t,x) + \sqrt{a(t,x)^2 + b(t,x)^2} &\leq a(t,x) + |a(t,x)| + |b(t,x)| \\ &= b(t,x). \end{aligned}$$

The previous inequality implies that:

$$\varphi\,(a(t,x), b(t,x)) = \frac{a(t,x) + \sqrt{a(t,x)^2 + b(t,x)^2}}{b(t,x)} \leq 1$$

This leads to:

$$\begin{aligned} \|u(t,x)\|_m &= \|B\,(t,x)\|_m\,\varphi\,(a(t,x), b(t,x)) \\ &\leq \|B\,(t,x)\|_m \leq \epsilon \end{aligned}$$

Finally, noting that for all $(t', x') \in \mathbb{R} \times \mathcal{V}\backslash\,\{0\}$ and $t \in \mathbb{R}$:

$$\|u(t',x') - u(t,0)\|_m = \|u(t',x')\|_m\,,$$

we conclude that u is continuous on $\mathbb{R} \times \mathcal{V}$. Concerning the proof of the global stabilization one can just repeat the proof in proposition 6. ■

Remark 9. For autonomous systems $\dot{x} = f_0(x) + \sum\limits_{i=1}^{m} f_i(x)u_i$, the formula (6) is universal with $a(x) = \left\langle \frac{\partial V}{\partial x}(x), f_0(x)\right\rangle$ and $b_i\,(x) = \left\langle \frac{\partial V}{\partial x}(x), f_i(x)\right\rangle$ for $1 \leq i \leq m$, this is not the case for non autonomous systems in the proposition (8). We have to add the property of uniformity (5).

To illustrate the proposition (8), let us consider a two dimensional system.

Example 10. Let the system

$$\begin{cases} \dot{x}_1 = -x_1 - \dfrac{x_2}{1+t^2} \\ \dot{x}_2 = x_1 - \dfrac{1+t+2t^2+2t^3+t^4}{1+t^2} x_2 + u \end{cases} \tag{7}$$

with $x \in \mathbb{R}^2$, $t \geq 0$ and the smooth positive definite decrescent function $V(t,x) = x_1^2 + \frac{x_2^2}{1+t^2} \leq x_1^2 + x_2^2$. One sees that $a(t,x) = -2x_1^2 - \frac{2(t+1)^2}{1+t^2} x_2^2$ and $B(t,x) = \frac{2x_2}{1+t^2}$ for all $t \geq 0$. For $x \neq 0$, $\inf_{u \in \mathbb{R}} [a(t,x) + B(t,x)u] \leq -2x_1^2 - 2x_2^2 < 0$, so V is a smooth DCLF for the system (7). Moreover, $a(t,x)^2 + b(t,x)^2 \geq 4x_1^4 + 4x_2^4 = \tilde{W}(x)$, so using the proposition (8), one knows that the system is smoothly stabilizable by the smooth feedback control

$$u(t,x) = \frac{\theta(x_1,x_2) - \sqrt{\theta(x_1,x_2)^2 + \frac{4x_2^4}{(1+t^2)^2}}}{x_2} \tag{8}$$

where $\theta(x_1,x_2) = x_1^2(1+t^2) + x_2^2(t+1)^2$. This leads to the simulation on figure 1.

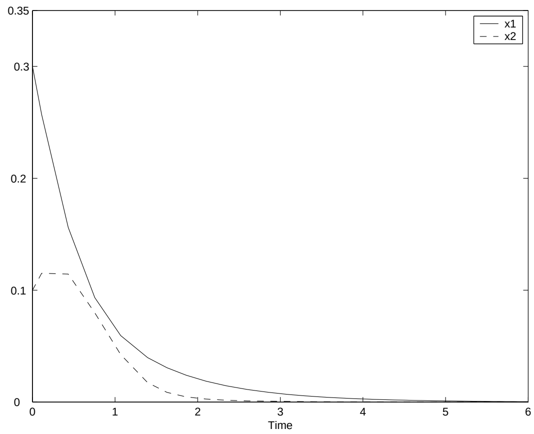

Fig. 1. Simulation of (7) with control (8)

5. CONCLUSION

A necessary and sufficient condition for the stabilization problem is given for time-varying affine systems together with an extension of Sontag's formula. These results shows the difference between autonomous and non autonomous affine systems in the stabilization problem.

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