ADAPTIVE TUNING TO A BIFURCATION FOR NONLINEAR SYSTEMS WITH HIGH RELATIVE DEGREE

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Abstract: The problem of tuning of adjustable parameters of a nonlinear system to unknown values ensuring the desired bifurcation properties is introduced. An adaptive output feedback control algorithm providing solution of the problem for nonlinear systems with high relative degree from adjusted parameters to measured output is proposed. Several application examples are presented.

Keywords: adaptive control, observers, bifurcation control, oscillation.

1. INTRODUCTION

The paper deals with the problem of bifurcation control design, that is a relatively new area of the control theory. The typical task solving in this area consists in synthesis a controller providing the desired bifurcation properties for a given nonlinear system (see (Chen, *et al.*, 2003; Chen, *et al.*, 2000) and references therein). However, parameters of the control law would in complicated manner depend on parameters of the plant model. In practice these parameters of the plant can differ from the nominal values used during synthesis phase. Since bifurcation or resonance properties of system are sensitive to small changes in parameter values, even small parametric error of control may result in significant changes of the system behavior (Leung, *et al.*, 2004). Moreover, since the system near bifurcation point lies on the border of stability, small parametric error may even lead to instability of the system.

To overcome the above difficulties in paper (Efimov, and Fradkov, 2004) it was proposed to use adaptive control approach for tuning adjustable parameters of the system in order to guarantee its desired bifurcation or resonance properties. This solution was obtained for a subclass of Lurie systems excited by exogenous input and basing on output measurements. In the papers (Moreau, and Sontag, 2003; Moreau, *et al.*, 2003) the same problem, motivated by biological applications, was posed and solved for autonomous linear systems using state dynamical feedback. In the standard adaptive control theory for nonlinear systems with linear parameterization (Fomin, *et al.*, 1981; Fradkov, *et al.*, 1999; Krstić, *et al.*, 1995) it is usually assumed that it is possible to ensure its asymptotic stability without asymptotic convergence of parameters estimates to their desired values. Such assumption is not suitable for bifurcation control where for exactly tuned system only stability or forward completeness is guaranteed and asymptotic stability may be absent. The solution of work (Efimov, and Fradkov, 2004) was built on passification based adaptive observer design (Fradkov, 1995; Fradkov, *et al.*, 1999) for systems with relative degree $\{1,..,1\}$.

In this paper a extension of solution proposed in (Efimov, and Fradkov, 2004) is presented for the class of systems, which have relative degree higher than $\{1,..,1\}$ with persistently exciting available for measurements input. The result utilizes non-passification based adaptive observer design theory (Efimov, and Fradkov, 2003; Fradkov, *et al.*, 2002). In Section 2 the necessary statements and definitions are introduced. Main result is formulated in Section 3. Two application examples are included in Section 4. Some conclusions are given in Section 5.

2. STATEMENTS AND DEFINITIONS

Consider the following model of nonlinear system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{y})\mathbf{x} + \varphi(\mathbf{y}) + \mathbf{B}(\mathbf{y})(\mu - \mu_0) + \mathbf{d}, \ \mathbf{y} = \mathbf{C}\mathbf{x}, \quad (1)$$

which possesses non-passification based adaptive observer (Efimov, and Fradkov, 2003; Fradkov, *et al.*, 2002). Here $\mathbf{x} \in R^n$, $\mathbf{y} \in R^p$, $\mathbf{d} \in R^n$ are state space vector, measurable output and exciting input vectors; $\mu \in R^q$ is vector of adjustable parameters serving as estimate of unknown constant vector $\mu_0 \in R^q$. Functions $\mathbf{A}$, φ and $\mathbf{B}$ are continuous and locally Lipschitz. The signal $\mathbf{d}: R_+ \to R^n$, $R_+ = \{\tau \in R, \tau \geq 0\}$ is assumed Lebesgue measurable and essentially bounded function of time $t \geq 0$, i.e.

$$\|\mathbf{d}\| < +\infty,\ \|\mathbf{d}\| = \|\mathbf{d}\|_{[0,+\infty)},$$

$$\|\mathbf{d}\|_{[0,T)} = ess\sup\{|\mathbf{d}(t)|, t \in [0,T)\},$$

$|\cdot|$ denotes Euclidean vector norm. For continuous input $\mu(t)$ the solution of such system $\mathbf{x}(t)$ is well defined at the least locally for $t \in [0,T)$, $T < +\infty$. If $T = +\infty$, then such system is called forward complete (Fradkov, *et al.*, 1999).

The problem is to find an algorithm of adjusting $\mu(t)$, ensuring forward completeness of the plant model (1), boundedness of regulator solutions and the limit relation

$$\lim_{t \to +\infty} \mu(t) = \mu_0 .$$

The posed problem differs from standard adaptive observer design problem due to presence of the feedback via μ in equation (1), i.e. it can be classified as adaptive observer based controller design. Opposite to paper (Efimov, and Fradkov, 2004) here we assume existence of high relative degree from input $\mu_0 - \mu$ to output $\mathbf{y}$ (i.e. only output time derivatives $\mathbf{y}^{(k)}$ of order $k > 1$ depends on input $\mu_0 - \mu$ in explicit fashion). An additional difficulty is in that the solutions of the system (1) may not be assumed bounded for some values of μ since it is not the case near bifurcation point $\mu = \mu_0$. It is supposed that signal $\mathbf{d}(t)$ can be directly measured or estimated, that is a realistic assumption for some examples (e.g. for some biological systems).

A s s u m p t i o n 1. *For any Lebesgue measurable and essentially bounded inputs $\mu(t)$, $\mathbf{d}(t)$ and any initial conditions $\mathbf{x}(0) \in R^n$ system (1) has well defined solution $\mathbf{x}(t)$ for all $t \geq 0$ (forward completeness property).* □

A s s u m p t i o n 2. *There exists locally Lipschitz continuous matrix function $\mathbf{K}(\mathbf{y})$, $\mathbf{y} \in R^p$, such, that solution of the system*

$$\dot{\mathbf{s}} = \mathbf{G}(\mathbf{y})\mathbf{s} + \mathbf{u},\ \mathbf{G}(\mathbf{y}) = \mathbf{A}(\mathbf{y}) - \mathbf{K}(\mathbf{y})\mathbf{C} \qquad (2)$$

for any initial conditions $\mathbf{s}_0 \in R^n$, any Lebesgue measurable $\mathbf{y}$ and any Lebesgue measurable and locally essentially bounded input $\mathbf{u}$ admits estimate:

$$|\mathbf{s}(t)| \leq d_1 |\mathbf{s}_0| + d_2 \|\mathbf{u}\|_{[0,t)},\ t \geq 0$$

for some d_1 and d_2 from R_+. □

A s s u m p t i o n 3. *There exist some continuously differentiable function $V: R^n \to R_+$ and matrix $\mathbf{L}$, such, that*

$$c_1 |\mathbf{x}|^2 \leq V(\mathbf{x}) \leq c_2 |\mathbf{x}|^2,\ |\mathbf{C}\mathbf{x}| \leq |\mathbf{L}\mathbf{x}|,$$

$$\partial V / \partial \mathbf{x}\, \mathbf{G}(\mathbf{y})\mathbf{x} \leq -c_3 |\mathbf{L}\mathbf{x}|^2$$

for any $\mathbf{x} \in R^n$ and $\mathbf{y} \in R^p$, $0 < c_1 \leq c_2$, $c_3 > 0.5$. □

Let us discuss the above assumptions. The Assumption 1 ensures existence of original system solutions for all $t \geq 0$, see also paper (Angeli, and Sontag, 1999) for necessary and sufficient conditions of forward completeness. Assumption 2 claims that it is possible to provide so-called bounded-input-bounded-state property for linear part of the system (1) by appropriate choice of output feedback gain matrix $\mathbf{K}$. Assumption 3 establishes conditions, under which system (2) is globally asymptotically stable with respect to part of variables $\mathbf{L}\mathbf{s}$ with known Lyapunov function (Rumyantsev, and Oziraner, 1987). Conditions of previous assumptions are enough to design adaptive observer for (1) (see also (Efimov, and Fradkov, 2003; Fradkov, *et al.*, 2002)):

$$\dot{\mathbf{z}} = \mathbf{A}(\mathbf{y})\mathbf{z} + \varphi(\mathbf{y}) + \mathbf{K}(\mathbf{y})(\mathbf{y} - \mathbf{C}\mathbf{z}) + \mathbf{D}; \qquad (3)$$

$$\dot{\mathbf{\Omega}} = \mathbf{G}(\mathbf{y})\mathbf{\Omega} - \mathbf{B}(\mathbf{y}); \qquad (4)$$

$$\dot{\boldsymbol{\eta}} = \mathbf{G}(\mathbf{y})\boldsymbol{\eta} - \mathbf{\Omega}\dot{\mu}, \qquad (5)$$

where $\mathbf{z} \in R^n$ is estimate of $\mathbf{x}$; $\boldsymbol{\eta} \in R^n$ and $\mathbf{\Omega} \in R^{n \times q}$ are auxiliary vector and matrix variables, which help us to overcome high relative degree obstruction; $\mathbf{D}: R_+ \to R^n$ is Lebesgue measurable and locally essentially bounded estimate of exciting input $\mathbf{d}$. To solve the posed problem it is suggested to adjust the estimates μ of unknown parameters μ_0 by the speed gradient algorithm:

$$\dot{\mu} = \gamma \mathbf{\Omega}^T \mathbf{C}^T (\mathbf{y} - \mathbf{C}\mathbf{z} + \mathbf{C}\boldsymbol{\eta}),\ \gamma > 0. \qquad (6)$$

Therefore, the proposed adaptive observer based controller is described by equations (3)–(6). Before we proceed let us introduce the following useful property proposed in (Efimov, and Fradkov, 2003).

D e f i n i t i o n 1. *Function $a: R_+ \to R$ is called (r,Δ)–positive in average (PA) if for any $t \geq 0$ and any $\delta \geq \Delta > 0$, $r > 0$,*

$$\int_t^{t+\delta} a(\tau)\, d\tau \geq r\delta . \qquad \square$$

In other words, time function $a(t)$ is (r,Δ)–PA, if its average value a_{av} on any large enough time interval $[t, t+\delta]$, $\delta \geq \Delta$,

$$a_{av} = \frac{1}{\delta} \int_t^{t+\delta} a(\tau)\, d\tau$$

is not smaller than some positive constant r. Importance of PA property is explained in the following lemma, which slightly modified version was proved in (Efimov, and Fradkov, 2003).

L e m m a 1. *Let us consider time-varying linear dynamical system*

$$\dot{p} = -a(t)p + b(t),\ t_0 \geq 0,$$

where $p \in R$, $p(t_0) \in R$ and functions $a: R_+ \to R$, $b: R_+ \to R$ are Lebesgue measurable, b is locally *essentially bounded, function a is (r,Δ)–PA for some $r>0$, $\Delta>0$ and essentially bounded from below, i.e. there exists $A\in R_+$, such, that:*

$$ess\inf\{a(t),\ t\ge t_0\}\ge -A\ .$$

Then solution of the system is defined for all $t\ge t_0$ and it admits estimate

$$|p(t)|\le\begin{cases}|p(t_0)|\,e^{A\Delta}e^{-r(t-t_0)+(A+r)\Delta}+\\ +\|b\|\max\{A^{-1}e^{At_0},r^{-1}e^{-rt_0}\},\ A\ne 0;\\ |p(t_0)|\,e^{-r(t-t_0)+r\Delta}+\\ +\|b\|\max\{\Delta,r^{-1}e^{-rt_0}\}, \qquad A=0.\end{cases}\ \square$$

It is possible to show that PA property is *equivalent* to persistent excitation (PE) property (Fomin, *et al.*, 1981; Fradkov, *et al.*, 1999; Loria, *et al.*, 2002) under some mild conditions (for example, in scalar case). Recall that the essentially bounded $n\times q$ matrix-function $\mathbf{B}(t)$ is called PE if there exist positive constants L and ϑ, such, that for any $t\ge 0$

$$\int_t^{t+L}\mathbf{B}(s)\mathbf{B}(s)^T\,ds\ge\vartheta\mathbf{I}_n\,,$$

where $\mathbf{I}_n$ is identity matrix with dimension $n\times n$. Note, that if $\mathbf{B}(\mathbf{y}(t))^T$ in (1) possesses PE property then, according to (4) signal $\mathbf{\Omega}$ should possess the same property. The above idea is equivalent to the following supposition.

A s s u m p t i o n 4. *The smallest singular value $a(t)$ of matrix function $\mathbf{C}^T\,\mathbf{\Omega}^T(t)$ is (r,Δ)-PA for some $r>0$, $\Delta>0$.* □

Like in classical adaptive control theory (Fomin, *et al.*, 1981; Fradkov, *et al.*, 1999; Krstić, *et al.*, 1995) this assumption will be used to justify convergence of the parametric error to zero.

3. MAIN RESULTS

Before we formulate and substantiate our main results a note should be done about measurements of output signal $\mathbf{y}(t)$. Typically, in practice these measurements are available with some noise:

$$\mathbf{y}_p(t)=\mathbf{y}(t)+\mathbf{p}(t)\,,$$

where $\mathbf{p}$ is a Lebesgue measurable and locally essentially bounded function of time. It is obvious, that such noise $\mathbf{p}$ presence in measurement channel seriously complicates functioning of the system as well as corresponding proofs. So at first part we will consider the case without measurement noise $\mathbf{p}$, and after that present result for a subclass of system (1) for noisy case.

3.1. Tuning without measurement noise

To formulate results of this paragraph we should impose additional requirement on boundedness properties of matrix function $\mathbf{B}(\mathbf{y}(t))$.

T h e o r e m 1. *Let for the system (1) Assumptions 1, 2 and 4 hold, $\|\mathbf{d}\|<+\infty$, $\|\mathbf{D}\|<+\infty$ and $\mathbf{B}(\mathbf{y}(t))\le\mathbf{B}$ for all $t\ge 0$. Then solution of system (1) is forward complete and solution of (3)–(6) is bounded. Furthermore, if Assumption 3 is satisfied and $\mathbf{D}(t)=\mathbf{d}(t)$ for almost all $t\ge 0$, then the following limit relation holds:*

$$\lim_{t\to+\infty}\mu(t)=\mu_0 \qquad (7)$$

and $\mathbf{L}(\mathbf{x}(t)-\mathbf{z}(t))\to 0$ for $t\to+\infty$.

P r o o f. Let us consider differential equations described dynamics of observer estimation error

$$\mathbf{e}=\mathbf{x}-\mathbf{z}\,,$$

which can be written as follows:

$$\dot{\mathbf{e}}=\mathbf{G}(\mathbf{y}(t))\mathbf{e}+\mathbf{B}(\mathbf{y}(t))(\mu(t)-\mu_0)+\mathbf{e}_d(t)\,, \quad (8)$$

where $\mathbf{e}_d(t)=\mathbf{d}(t)-\mathbf{D}(t)$ is exciting input $\mathbf{d}$ estimation error (by conditions $\|\mathbf{e}_d\|<+\infty$). Let us also analyze the following auxiliary error signal:

$$\delta=\mathbf{e}+\mathbf{\eta}+\mathbf{\Omega}(\mu-\mu_0)\,, \qquad (9)$$

which behavior obeys differential equation:

$$\dot\delta=\mathbf{G}(\mathbf{y}(t))\delta+\mathbf{e}_d(t)\,. \qquad (10)$$

So, applying Assumption 2 it is possible to obtain boundedness of variables δ and $\mathbf{\Omega}$. Rewriting equation (6) for $\tilde\mu=\mu-\mu_0$ and tacking in mind (9), it is possible receive

$$\dot{\tilde\mu}=\gamma\mathbf{\Omega}^T\mathbf{C}^T\mathbf{C}\,(\delta-\mathbf{\Omega}\tilde\mu). \qquad (11)$$

From Assumption 4 the smallest eigenvalue $a(t)$ of matrix $\mathbf{\Omega}(t)^T\mathbf{C}^T\mathbf{C}\mathbf{\Omega}(t)$ is PA and signal $\mathbf{\Omega}^T\mathbf{C}^T\mathbf{C}\delta$ is bounded, that according to result of Lemma 1 means boundedness of $\tilde\mu$. From Assumption 1 system (1) is forward complete. Application of Assumption 2 to the system (8) justifies boundedness of variable $\mathbf{e}$. Variable $\mathbf{\eta}$ is also bounded since it is a part of (9), where all other variables are bounded. Thus we obtain boundedness of the regulator solution. If Assumption 4 holds and $\mathbf{e}_d(t)=0$ for almost all $t\ge 0$, then system (10) is globally asymptotically stable with respect to variable $\mathbf{L}\delta$. Therefore, signal $\mathbf{\Omega}(t)^T\mathbf{C}^T\mathbf{C}\delta(t)$ converges to zero while $t\to+\infty$. Applying the result of Lemma 1 to (11) it is possible to obtain (7). Since the variable $\mathbf{e}$ is bounded, it has non empty closed and compact set of ω-limit values. In this set the system (8) can be reduced to

$$\dot{\mathbf{e}}=\mathbf{G}(\mathbf{y}(t))\mathbf{e}$$

and desired conclusion follows by recollecting Assumption 4 for the system. ■

If the boundedness of signal $\mathbf{B}(\mathbf{y}(t))$ or $\mathbf{y}(t)$ ($\mathbf{x}(t)$) is not assumed, then it is not possible to justify boundedness of regulator solution, but it is still possible to prove (7). Probably in some applications, like adaptive tuning of resonance regimes, requirement of boundedness of overall system solution $\mathbf{X}=(\mathbf{x}^T\mathbf{z}^T\mathbf{\Omega}^T\mathbf{\eta}^T\mu^T)^T$ is not natural and it reduces possible applicability of proposed approach. So let us consider the next result, where the boundedness of $\mathbf{B}(\mathbf{y}(t))$ is not assumed. Recall that a continuous function $\chi:R_+\to R_+$ is from class $\mathcal{K}$ if it is monotonously increasing and $\chi(0)=0$.

T h e o r e m 2. *Let for the system (1) Assumptions 1–4 hold,* $\|\mathbf{d}\|<+\infty$, $\mathbf{D}(t)=\mathbf{d}(t)$ *for almost all* $t\geq 0$ *and*

$$|\mathbf{B}(\mathbf{y}(t))|\leq\chi_1(|\mathbf{X}(0)|)+\chi_2\left(\|\mathbf{d}\|_{[0,t)}\right)+ \\ +\chi_2\left(\|\tilde{\mu}\|_{[0,t)}\right)+\chi_3(|\mathbf{X}(0)|)\,e^{\chi_4 t} \tag{12}$$

for all $t\geq 0$, *for some* χ_1, χ_2, $\chi_3\in\mathcal{K}$ *and positive constant* χ_4. *Assume that by appropriate choice of matrix function* $\mathbf{K}$ *from Assumption 2 it is possible to assign arbitrary constant* c_3 *from Assumption 3, where additionally*

$$c_1|\mathbf{L}\mathbf{x}|^2\leq V(\mathbf{x})\leq c_2|\mathbf{L}\mathbf{x}|^2.$$

Then there exists matrix function $\mathbf{K}$, *such, that system (1), (3)–(6) is forward complete and (7) holds.*

P r o o f. By conditions $\mathbf{e}_d(t)=0$ for almost all $t\geq 0$. Let us consider Lyapunov function

$$W(\mu,\delta)=V(\delta)+0.5\gamma^{-1}(\mu-\mu_0)^T(\mu-\mu_0),$$

which time derivative by virtue of Assumptions 3 and 4 and equations (10) and (11) takes form

$$\dot{W}=\partial V/\partial\delta\,\mathbf{G}(\mathbf{y})\delta+\tilde{\mu}^T\,\boldsymbol{\Omega}^T\mathbf{C}^T\mathbf{C}\,(\delta-\boldsymbol{\Omega}\tilde{\mu})\leq \\ \leq -c_3|\mathbf{L}\delta|-a(t)|\tilde{\mu}|^2+\tilde{\mu}^T\,\boldsymbol{\Omega}^T\mathbf{C}^T\mathbf{C}\delta\leq \\ \leq -r_1|\mathbf{L}\delta|-r_2\,a(t)|\tilde{\mu}|^2,$$

for some positive r_1 and r_2. Thus, variables $\mathbf{L}\delta$ and $\tilde{\mu}$ are bounded, that with Assumption 1 means forward completeness property for system (1). Now it is possible to note, that according to (12) inputs of systems (4) and (8) are defined for all $t\geq 0$. Applying Assumption 2 to these systems one can obtain forward completeness property of overall system. To base goal limit relation (7) it is possible to use results of Lemma 1, if we apply it to system (11) with convergent to zero signal $\boldsymbol{\Omega}(t)^T\mathbf{C}^T\mathbf{C}\delta(t)$. Using results from Rumyantsev, and Oziraner (1987) it is possible to claim that the estimate

$$|\mathbf{C}\delta(t)|\leq\alpha_0|\delta(0)|e^{-\alpha t} \tag{13}$$

is satisfied due to for system (10) with $\mathbf{e}_d=0$ Assumption 3 is fulfilled, where $\alpha_0>0$, $\alpha=\kappa\alpha_1>0$, $\alpha_1>0$. Coefficient $\kappa>0$ is dependent on form of matrix function $\mathbf{K}(\mathbf{y})$ and by conditions of the Theorem it is possible to increase value of κ by appropriate choice of $\mathbf{K}(\mathbf{y})$. According to Assumption 2 and (12) variable $\boldsymbol{\Omega}(t)$ possesses estimate:

$$|\boldsymbol{\Omega}(t)|\leq d_1|\boldsymbol{\Omega}(0)|+d_2\left[\chi_1(|\mathbf{X}(0)|)+\right. \\ +\chi_2\left(\|\mathbf{d}\|_{[0,t)}\right)+\chi_2\left(\|\tilde{\mu}\|_{[0,t)}\right)+ \\ \left.+\chi_3(|\mathbf{X}(0)|)\,e^{\chi_4 t}\right]$$

for all $t\geq 0$. Therefore, multiplying the right hand side of above estimate on right side of (13) we obtain, that signal $\boldsymbol{\Omega}(t)^T\mathbf{C}^T\mathbf{C}\delta(t)$ would asymptotically vanish if $\kappa>\chi_4\,\alpha_1^{-1}$. ■

It is worth to note, that if system (1) is forward complete, then according to results of paper (Angeli, and Sontag, 1999) the following estimate holds:

$$|\mathbf{x}(t)|\leq\gamma_1(|\mathbf{x}(0)|)+\gamma_2\left(\|\mathbf{d}\|_{[0,t)}\right)+ \\ +\gamma_2\left(\|\tilde{\mu}\|_{[0,t)}\right)+\gamma_3(t)+\gamma_4$$

for some γ_1, γ_2, $\gamma_3\in\mathcal{K}$ and positive constant γ_4. Comparing this estimate with (12) it is possible to conclude, that (12) is a mild technical assumption, which can be viewed as a corollary of forward completeness property of system (1). The main difference between (12) and above estimate is the character of dependence on time argument. For example, signal $|\mathbf{B}(\mathbf{y}(t))|$ grows not faster than exponentially for essentially bounded μ and $\mathbf{d}$ if the following series of inequalities hold for system (1) for some positive constants α_i, $i=\overline{0,4}$:

$$|\mathbf{A}(\mathbf{y})|\leq\alpha_0,\ |\varphi(\mathbf{y})|\leq\alpha_1|\mathbf{x}|+\alpha_2,\ |\mathbf{B}(\mathbf{y})|\leq\alpha_3|\mathbf{x}|+\alpha_4,$$

here and further the norm of matrix $\mathbf{A}$ is defined by the maximum singular value.

3.2. Case with presence of measurement noise

Let us suppose that regulator measures plant (1) output with some noise:

$$\mathbf{y}_p(t)=\mathbf{y}(t)+\mathbf{p}(t), \tag{14}$$

where $\mathbf{p}$ is a Lebesgue measurable and locally essentially bounded signal. In such situation equations of regulator (3)–(6) should be rewritten with substitution (14) as follows:

$$\dot{\mathbf{z}}=\mathbf{A}(\mathbf{y}_p)\mathbf{z}+\varphi(\mathbf{y}_p)+\mathbf{K}(\mathbf{y}_p)(\mathbf{y}_p-\mathbf{C}\mathbf{z})+\mathbf{D}; \tag{15}$$

$$\dot{\boldsymbol{\Omega}}=\mathbf{G}(\mathbf{y}_p)\boldsymbol{\Omega}-\mathbf{B}(\mathbf{y}_p); \tag{16}$$

$$\dot{\boldsymbol{\eta}}=\mathbf{G}(\mathbf{y}_p)\boldsymbol{\eta}-\boldsymbol{\Omega}\dot{\mu}; \tag{17}$$

$$\dot{\mu}=\gamma\boldsymbol{\Omega}^T\mathbf{C}^T\left(\mathbf{y}_p-\mathbf{C}\mathbf{z}+\mathbf{C}\boldsymbol{\eta}\right),\ \gamma>0. \tag{18}$$

Let us rewrite equation for estimation error (8):

$$\dot{\mathbf{e}}=\mathbf{G}(\mathbf{y}_p)\mathbf{e}+\mathbf{B}(\mathbf{y})(\mu-\mu_0)+\mathbf{e}_d-\mathbf{K}(\mathbf{y}_p)\mathbf{p}+ \\ +(\varphi(\mathbf{y})-\varphi(\mathbf{y}_p))+(\mathbf{A}(\mathbf{y})-\mathbf{A}(\mathbf{y}_p))\,\mathbf{x} \tag{19}$$

and equation (10) for δ dynamics:

$$\dot{\delta}=\mathbf{G}(\mathbf{y}_p)\delta+(\mathbf{B}(\mathbf{y})-\mathbf{B}(\mathbf{y}_p))(\mu-\mu_0)+ \\ +\mathbf{e}_d-\mathbf{K}(\mathbf{y}_p)\mathbf{p}+(\varphi(\mathbf{y})-\varphi(\mathbf{y}_p))+ \\ +(\mathbf{A}(\mathbf{y})-\mathbf{A}(\mathbf{y}_p))\,\mathbf{x}. \tag{20}$$

If one would apply technique used above to prove Theorems 1 and 2 in this case, then presence of variable $\mathbf{x}$ in equations (19) and (20) would prevent it. A way to overcome this obstacle consists in supposition, that matrix functions $\mathbf{A}(\mathbf{y})$ and $\mathbf{B}(\mathbf{y})$ are constant matrices.

T h e o r e m 3. *Let for the system (1) Assumptions 1, 2 and 4 hold;* $\|\mathbf{d}\|<+\infty$, $\|\mathbf{D}\|<+\infty$, $\|\mathbf{p}\|<+\infty$; $\mathbf{A}(\mathbf{y})=\mathbf{A}$, $\mathbf{B}(\mathbf{y})=\mathbf{B}$ *and function* φ *is globally Lipschitz continuous. Then solution of system (1) is forward complete and solution of (15)–(18) is bounded. Furthermore, if Assumption 3 is satisfied and* $\mathbf{D}(t)=\mathbf{d}(t)$, $\mathbf{p}(t)=0$ *for almost all* $t\geq 0$, *then (7) holds and* $\mathbf{L}(\mathbf{x}(t)-\mathbf{z}(t))\to 0$ *for* $t\to+\infty$.

P r o o f. In this case differential equations (19) and (20) take forms:

$$\dot{\mathbf{e}} = \mathbf{G}\,\mathbf{e} + \mathbf{B}\left(\mu - \mu_0\right) + \mathbf{e}_d - \mathbf{K}\,\mathbf{p} + \left(\varphi(\mathbf{y}) - \varphi(\mathbf{y}_p)\right); \quad (21)$$

$$\dot{\delta} = \mathbf{G}\,\delta + \mathbf{e}_d - \mathbf{K}\,\mathbf{p} + \left(\varphi(\mathbf{y}) - \varphi(\mathbf{y}_p)\right), \quad (22)$$

where it was assumed, that in this case matrix function $\mathbf{K}(\mathbf{y}) = \mathbf{K}$ and the same property holds for matrix $\mathbf{G}(\mathbf{y}) = \mathbf{G}$. Let us introduce auxiliary signal

$$\mathbf{N}(t) = \left[\varphi\left(\mathbf{y}(t)\right) - \varphi\left(\mathbf{y}_p(t)\right)\right] - \mathbf{K}\,\mathbf{p}(t) + \mathbf{e}_d(t),$$

that is essentially bounded. Indeed,

$$\left|\mathbf{N}(t)\right| \le \left(\left|L_\varphi\right| + \left|\mathbf{K}\right|\right)\|\mathbf{p}\| + \|\mathbf{e}_d\|,$$

where L_φ is Lipschitz constant of function φ. Further, applying Assumption 2 to systems (16) and (22) it is possible to recover boundedness of variables δ and $\mathbf{\Omega}$. Rewriting equation (18) in form (11), one can receive

$$\dot{\tilde{\mu}} = \gamma\,\mathbf{\Omega}^T\mathbf{C}^T\left[\mathbf{p} + \mathbf{C}\left(\delta - \mathbf{\Omega}\,\tilde{\mu}\right)\right]. \quad (23)$$

From Assumption 4 the smallest eigenvalue $a(t)$ of matrix $\mathbf{\Omega}(t)^T\mathbf{C}^T\mathbf{C}\mathbf{\Omega}(t)$ is PA and signal $\mathbf{\Omega}^T\mathbf{C}^T\left[\mathbf{p} + \mathbf{C}\delta\right]$ is bounded, that according to result of Lemma 1 means boundedness of $\tilde{\mu}$. From Assumption 1 system (1) is forward complete. Now applying to system (21) Assumption 2 it is possible to base boundedness of variable $\mathbf{e}$. Variable $\boldsymbol{\eta}$ is also bounded due to it is a part of (9), where all other variables are bounded. Thus we obtain boundedness of regulator (15)–(18) solution.

If Assumption 4 holds and $\mathbf{e}_d(t) = 0$, $\mathbf{p}(t) = 0$ for almost all $t \ge 0$, then system (22) is globally asymptotically stable with respect to variable $\mathbf{L}\,\delta$. Therefore, signal $\mathbf{\Omega}(t)^T\mathbf{C}^T\mathbf{C}\,\delta(t)$ asymptotically converges to zero. Further applying to (23) result of Lemma 1 it is possible obtain (7). Relation

$$\lim_{t\to+\infty} \mathbf{L}\left(\mathbf{x}(t) - \mathbf{z}(t)\right) = 0$$

can be established by the same arguments as in proof of Theorem 1. ■

R e m a r k 1. Note, that in all theorems only forward completeness of system (1) was used. This property does not contradict to fulfillment of any other stability property for system (1). In fact one can additionally impose some stability properties for system (1) to use proposed approach to obtain classical problem solution. For example, let in case of Theorem 3 nominal part of system (1)

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \varphi(\mathbf{y})$$

be asymptotically stable. Then it is also globally exponentially asymptotically stable. Hence, for any bounded additive disturbance trajectories of nominal part of system (1) are bounded. If all conditions of Theorem 3 are satisfied, then due to (7) asymptotic behavior of system (1) is determined by properties of exciting signal $\mathbf{d}$ and we recover the classical adaptive stabilization problem for system (1) with disturbance input and measurement noise. In such a case the result of Theorem 3 presents a new solution of adaptive stabilization problem for system (1). The same remark is valid for all other theorems. □

R e m a r k 2. The results of the paper were obtained with utilizing of adaptive observer design method borrowed from (Efimov, and Fradkov, 2003; Fradkov, *et al.*, 2002). It is possible to simplify obtained solution excluding from consideration auxiliary variable $\boldsymbol{\eta}$ if one would modify equations (3)–(6) in the following way:

$$\dot{\mathbf{z}} = \mathbf{A}(\mathbf{y})\,\mathbf{z} + \varphi(\mathbf{y}) + \mathbf{B}(\mathbf{y})\,\mu + \mathbf{K}(\mathbf{y})\left(\mathbf{y} - \mathbf{C}\,\mathbf{z}\right) + \mathbf{D};$$

$$\dot{\mathbf{\Omega}} = \mathbf{G}(\mathbf{y})\,\mathbf{\Omega} - \mathbf{B}(\mathbf{y});$$

$$\dot{\mu} = \gamma\,\mathbf{\Omega}^T\mathbf{C}^T\left(\mathbf{y} - \mathbf{C}\,\mathbf{z} - \mathbf{C}\,\mathbf{\Omega}\,\mu\right), \ \gamma > 0. \quad (24)$$

Differential equation described observer estimation error $\mathbf{e} = \mathbf{x} - \mathbf{z}$ has form:

$$\dot{\mathbf{e}} = \mathbf{G}\left(\mathbf{y}(t)\right)\mathbf{e} - \mathbf{B}\left(\mathbf{y}(t)\right)\mu_0 + \mathbf{e}_d(t),$$

and dynamics of auxiliary error $\delta = \mathbf{e} - \mathbf{\Omega}\,\mu_0$ yields to (10), while (24) can be rewritten as (11). For this regulator with this new δ it is possible to repeat without any modifications all proofs of all theorems with saving of results. □

In order to illustrate verification of the conditions of theorems and assumptions below some examples are presented.

4. APPLICATIONS

1. Resonance tuning of a pendulum. Let us consider the pendulum equations

$$\dot{x}_1 = x_2;$$

$$\dot{x}_2 = (\mu - \omega^2)\sin(x_1) - \varepsilon\,x_2 + d(t),$$

where $\omega > 0$ is unknown natural frequency of the pendulum; $\varepsilon > 0$ is small friction factor; $\mu \in R$ is the adjusted parameter, as before, that is introduced to tune pendulum frequency to the desired value; $d(t) = \sin(\omega_d\,t)$ is a sinusoidal signal with known exciting frequency $\omega_d > 0$. The problem is to tune the resonance regime of the pendulum to diminish the tuning error $\mu = \omega^2 - \omega_d^2$ when natural frequency of the pendulum and frequency of external input d coincide and pendulum solution exhibits oscillations with infinite amplitude (for sufficient small values of ε). Such a problem is important for practice, since resonance regime with growing amplitude arises only for coinciding frequencies of the pendulum and the input d. It can be generated by designer for the case of known frequency ω_d, while the natural frequency of the pendulum ω may depend on uncertain and unpredictable external factors and its exact value is unmeasured. Note, that if we extend state space of the plant (1) by dynamical system generating exogenous disturbance d, then this extended system possesses the bifurcation for $\tilde{\mu} = 0$. That explains application of task of resonance regime stabilization in this paper as examples.

Let $y = x_1$ (in paper (Efimov, and Fradkov, 2004) such example was previously considered for case of full state measurements and $\varepsilon = 0$) and pendulum equations can be rewritten as follows

$$\dot{x}_1 = x_2;$$

$$\dot{x}_2 = -\omega_d^2 \sin(x_1) - \varepsilon x_2 + (\mu - \mu_0)\sin(x_1) + d(t),$$

where $\mu_0 = \omega^2 - \omega_d^2$. It is possible to see that in this case matrix function $\mathbf{B}(\mathbf{y})$ is bounded and it is the case of Theorem 1. Appealing to Remark 2 adaptive regulator takes form

$$\dot{z}_1 = z_2;$$

$$\dot{z}_2 = -\varepsilon z_2 - \left(\omega_d^2 - \mu\right)\sin(y) + K(y - z_1) + D(t);$$

$$\dot{\Omega}_1 = \Omega_2; \quad \dot{\Omega}_2 = -\varepsilon\Omega_2 - K\Omega_1 - \sin(y); \quad \dot{\mu} = \gamma\Omega_1\left(y - z_1 - \Omega_1\mu\right).$$

It is possible to show, that conditions of the Theorem hold in this example, especially PA condition is satisfied for sinusoidal function of runaway argument.

2. Filtering of steady component of input exciting signal for a pendulum. Let us again consider a pendulum equations:

$$\dot{x}_1 = x_2;$$

$$\dot{x}_2 = -\varepsilon x_2 - \omega^2 \sin(x_1) + (\mu - \mu_0) + d(t),$$

where $\omega > 0$ is known natural frequency of the pendulum; $\varepsilon > 0$ is small friction factor; $d(t) = \sin(\omega_d t)$ is a sinusoidal signal with known exciting frequency $\omega_d > 0$; $\mu \in R$ is the adjusted parameter, that is introduced to annihilate possible steady (parasitic) component of the exciting input. Let again $y = x_1$ and this task admits all conditions of Theorem 3. So, let output measurements are available with additive noise $y_p(t) = x_1(t) + p(t)$. Recollecting Remark 2, equations of adaptive regulator take form:

$$\dot{z}_1 = z_2;$$

$$\dot{z}_2 = -\varepsilon z_2 - \omega^2 \sin(y_p) + \mu + K(y_p - z_1) + D(t);$$

$$\dot{\Omega}_1 = \Omega_2; \quad \dot{\Omega}_2 = -\varepsilon\Omega_2 - K\Omega_1 - 1; \quad \dot{\mu} = \gamma\Omega_1\left(y_p - z_1 - \Omega_1\mu\right),$$

where $D(t) = d(t) + p(t)$. Results of computer simulations for both examples are not included due to space limitation.

5. CONCLUSION

An adaptive output feedback controller is proposed which tunes a nonlinear uncertain dynamical system to its bifurcation point under some mild conditions. This solution is based on theory of adaptive observers design proposed in (Efimov, and Fradkov, 2003; Fradkov, *et al.*, 2002). The result differs from the previously proposed in (Efimov, and Fradkov, 2004) in that it is applicable to the systems with relative degree greater than one. Besides, a more general form of nonlinear system is considered. Possibility of noisy measurements is taken into account.

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