INFORMATION THEORETIC INTERPRETATIONS FOR H_∞ ENTROPY

Hui ZHANG[*], Youxian SUN[**]

National Laboratory of Industrial Control Technology, Institute of Modern Control Engineering, Department of Control Science & Engineering, Zhejiang University, Hang Zhou, 310027, P.R.China
Emails: [*] zhanghui_iipc@zju.edu.cn , [**] yxsun@iipc.zju.edu.cn

Abstract: Based on the studies on information transmission in discrete multivariable linear time invariant (LTI) system disturbed by stationary noise, relations within entropy rate, mutual information rate and H_∞ entropy are discussed in both general control problem and classic tracking problem. For the general control systems, equivalent relations between entropy rate and H_∞ entropy are formulated by using spectral factorization. For the classic tracking systems, performance of disturbance rejection is measured by a difference of entropy rates (or mutual information rate in Gaussian case). This performance function is proved to be bounded by the H_∞ entropy of the closed-loop transfer function from disturbance to output. These relations give information theoretic interpretations for the minimum entropy H_∞ control theory. Potential application of these relations is discussed.

Keywords: Entropy, information theory, linear system, stochastic systems, H-infinity control, spectral factorization, disturbance rejection.

1. INTRODUCTION

As a suboptimal robust control design method, the minimum entropy H_∞ control method has been developed in the past (Mustafa and Glover, 1990; Iglesias, 1990). It adopts an unintuitive function of closed-loop transfer matrix as its performance index. Let any transfer function matrix $\boldsymbol{F}(z) \in \boldsymbol{RH}_\infty$, where $\boldsymbol{RH}_\infty$ denotes the set of all stable and proper transfer functions, satisfy $\| \boldsymbol{F}(z) \|_\infty = \max_\omega \sigma_1[\boldsymbol{F}(e^{i\omega})] < \lambda$, where σ_1 denotes the largest singular value, then the entropy of $\boldsymbol{F}(z)$ in H_∞ control is defined as:

$$H(\boldsymbol{F}, \lambda) = \frac{-\lambda^2}{4\pi} \int_{-\pi}^{\pi} \ln \det[\boldsymbol{I} - \lambda^{-2} \boldsymbol{F}^*(e^{i\omega}) \boldsymbol{F}(e^{i\omega})] \mathrm{d}\omega \quad (1)$$

where $\boldsymbol{F}^*(e^{i\omega}) = \boldsymbol{F}^{\mathrm{T}}(e^{-i\omega})$. Function (1) is different from the concepts of Shannon entropy. We refer to it as the H_∞ entropy. The minimum entropy H_∞ control method is to find an admissible controller minimizing the H_∞ entropy of the closed-loop transfer function of system when its H_∞ norm is constrained. It is known that the H_∞ entropy is an index measuring the tradeoff between the H_∞ optimality and the H_2 optimality, and the deduced minimum entropy H_∞ controller is in fact the 'central controller' (Zhou, 1998). However, little has been done on the physical meaning of this measure in control systems.

On the other hand, information theoretic aspects of stochastic control systems have attracted a great deal of attention in the past (Weidemann, 1969; Engell, 1984; Stoorvogel and Schuppen, 1996; Zhang, 2003). As the key concepts of Shannon information theory and unified probabilistic measures, entropy (rate) and mutual information (rate) describe the information or uncertainty of random variables, and play important roles in stochastic control and estimation. Let sequences of discrete-time stochastic processes X and Y be $X^n = \{x_1, x_2, ..., x_n\}$ and $Y^n = \{y_1, y_2, ..., y_n\}$, respectively. The entropy rate of X (Ihara, 1993)

$$\overline{H}(X) = \lim_{n\to\infty} \frac{1}{n} H(X^n) \quad (2)$$

describes the per unit time information or uncertainty of X, where $H(X^n)$ is the entropy of X^n; the mutual information rate between X and Y (Ihara, 1993)

$$\bar{I}(X;Y) = \lim_{n\to\infty} \frac{1}{n} I(X^n;Y^n) \tag{3}$$

describes the time average information transmitted between processes X and Y, where $I(X^n;Y^n)$ is the mutual information of X^n and Y^n. Information rates (entropy rate and mutual information rate) were adopted as steady or time average criterion functions of stochastic control systems (Engell, 1984; Stoorvogel and Schuppen, 1996; Zhang, 2003).

This paper focuses on the information theoretic meaning of the H_∞ entropy in multivariable LTI control systems. Lemmas concerning information transmission will be introduced in section 2. In section 3, equivalent relations between entropy rate and H_∞ entropy in general LTI systems will be given by using spectral factorization. In section 4, another information theoretic interpretation for H_∞ entropy will be formulated in terms of relations between information rates and H_∞ entropy in classic tracking systems. It was stated in (Yue and Wang, 2003) that the H_∞ entropy is a measure of the mutual information between system input and output. Our analysis will point out that the H_∞ entropy of a closed transfer function is not exactly the mutual information between system input and output, but a lower bound of performance function measured by a difference of entropy rates (or mutual information rate in Gaussian case). Conclusion and discussion of potential applications of the obtained results will be given in section 5.

2. INFORMATION TRANSMISSION

From the viewpoint of information theory, any control system can be considered as an information or uncertainty transmission channel. In this section, two lemmas concerning information transmission will be introduced.

Lemma 1: Let $\boldsymbol{F}(z) \in \boldsymbol{RH}_\infty^{m\times m}$ be a square transfer function matrix of a discrete time MIMO LTI system, the stationary stochastic input $\boldsymbol{x}(k) \in \boldsymbol{R}^m$ (k=0,1, …) has spectral density $\boldsymbol{\Phi}_x(\omega)$, then the entropy rate of system output $\boldsymbol{y}(k) \in \boldsymbol{R}^m$ is

$$\overline{H}(\boldsymbol{y}) = \overline{H}(\boldsymbol{x}) + \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln|\det\boldsymbol{F}(e^{i\omega})|^2\, \mathrm{d}\omega\,, \tag{4}$$

where $\overline{H}(\boldsymbol{x})$ is the entropy rate of input:

$$\overline{H}(\boldsymbol{x}) = \frac{m}{2}\ln(2\pi e) + \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\boldsymbol{\Phi}_x(\omega)\, \mathrm{d}\omega\,. \tag{5}$$

Proof: See (Zhang, 2003).

The integral in equation (4) describes the variation of information (or uncertainty) after a signal transmitted through a LTI system. Denote it as

$$V(\boldsymbol{F}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln|\det\boldsymbol{F}(e^{i\omega})|^2\, \mathrm{d}\omega\,. \tag{6}$$

Lemma 2 (Pinsker, 1964): Suppose joint Gaussian stationary processes $\boldsymbol{x}(k) \in \boldsymbol{R}^n$, $\boldsymbol{y}(k) \in \boldsymbol{R}^m$, k=0,1,2,…, have spectral densities $\boldsymbol{\Phi}_x(\omega)$, $\boldsymbol{\Phi}_y(\omega)$, and $\boldsymbol{\psi}(k) = [\boldsymbol{x}^{\mathrm{T}}(k)\,\boldsymbol{y}^{\mathrm{T}}(k)]^{\mathrm{T}} \in \boldsymbol{R}^{n+m}$ has spectral density $\boldsymbol{\Phi}_\psi(\omega)$. Then, the mutual information rate of $\boldsymbol{x}(k)$, $\boldsymbol{y}(k)$ is

$$\bar{I}(\boldsymbol{x};\boldsymbol{y}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\frac{\det\boldsymbol{\Phi}_x(\omega)\det\boldsymbol{\Phi}_y(\omega)}{\det\boldsymbol{\Phi}_\psi(\omega)}\, \mathrm{d}\omega\,. \tag{7}$$

The relations between information rates and the H_∞ entropy will be discussed based on the above lemmas.

3. GENERAL CONTROL SYSTEMS

In this section, we will discuss the relations between entropy rate and H_∞ entropy in general control systems.

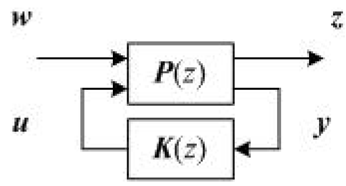

Figure 1: Standard close-loop system $\boldsymbol{F}_l(\boldsymbol{P},\boldsymbol{K})$

The general framework is illustrated by Figure 1, where $\boldsymbol{P}(z)$ and $\boldsymbol{K}(z)$ are respectively the generalized plant and controller with corresponding dimensions, $\boldsymbol{w}$, $\boldsymbol{u}$, $\boldsymbol{z}$ and $\boldsymbol{y}$ are the disturbance input, the control signal, the output and the measured signal, respectively. The closed-loop transfer function from $\boldsymbol{w}$ to $\boldsymbol{z}$ is formulated as the linear fractional transform (Zhou, 1998) $\boldsymbol{F}_l(\boldsymbol{P},\boldsymbol{K}) \in \boldsymbol{RH}_\infty^{n\times n}$. Let $\boldsymbol{F}_l(\boldsymbol{P},\boldsymbol{K}) = \boldsymbol{G}(z)$ for simplicity. In this standard system, the output $\boldsymbol{z}$ is to be minimized in the sense of a suitable performance criterion. In the present paper, $\boldsymbol{w}$ is assumed to be stationary sequence; $\boldsymbol{G}^{-1}(z) \in \boldsymbol{RH}_\infty^{n\times n}$.

The minimum entropy H_∞ control method is to find a admissible controller minimizing the H_∞ entropy of system closed-loop transfer function when the H_∞ norm of system is constrained. Suppose

$$||\boldsymbol{G}(z)||_\infty < \eta\,, \tag{8}$$

then the H_∞ entropy of $\boldsymbol{G}$ under constraint (8) is

$$\begin{aligned} &H(\boldsymbol{G},\eta) \\ &= \frac{-\eta^2}{4\pi}\int_{-\pi}^{\pi} \ln\det[\boldsymbol{I} - \eta^{-2}\boldsymbol{G}^*(e^{i\omega})\boldsymbol{G}(e^{i\omega})]\mathrm{d}\omega. \end{aligned} \tag{9}$$

In the framework of information theoretic method, the entropy rate of $\boldsymbol{z}$, which measures the time average uncertainty of system output, is a suitable criterion function of system performance. From Lemma 1 we have

$$\begin{aligned}\overline{H}(\boldsymbol{z}) &= \overline{H}(\boldsymbol{w}) + \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln|\det\boldsymbol{G}(e^{i\omega})|^2 \,\mathrm{d}\omega \\ &= \overline{H}(\boldsymbol{w}) + V(\boldsymbol{G}),\end{aligned} \tag{10}$$

where $\overline{H}(\boldsymbol{z}), \overline{H}(\boldsymbol{w})$ are entropy rates of $\boldsymbol{z}$ and $\boldsymbol{w}$, respectively. The connection between these two methodologies will be analyzed in the following by using spectral factorization.

Suppose $\boldsymbol{G}(z)$ has a state-space realization as

$$\boldsymbol{G}(z) = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}, \tag{11}$$

which means $\boldsymbol{G}(z) = \boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$, where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are constant matrices with corresponding dimensions, and $\boldsymbol{D}$ is invertible. The condition (8) allows a spectral factorization (Tsai, 1993) as

$$\boldsymbol{I} - \eta^{-2}\boldsymbol{G}^*(z)\boldsymbol{G}(z) = \boldsymbol{U}^*(z)\boldsymbol{U}(z), \tag{12}$$

where $\boldsymbol{U}(z), \boldsymbol{U}^{-1}(z) \in \boldsymbol{RH}_\infty^{n\times n}$, $\|\boldsymbol{U}\|_\infty < 1$, and $\boldsymbol{U}$ has the state space realization

$$\boldsymbol{U}(z) = \{\boldsymbol{A}, \boldsymbol{B}, -\boldsymbol{Z}^{\frac{1}{2}}\boldsymbol{F}, \boldsymbol{Z}^{\frac{1}{2}}\}, \tag{13}$$

where

$$\begin{aligned}&\boldsymbol{F} = -\boldsymbol{Z}^{-1}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A} - \eta^{-2}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{C}), \\ &\boldsymbol{Z} = \boldsymbol{R} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{B},\ \boldsymbol{R} = \boldsymbol{I} - \eta^{-2}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} > 0,\end{aligned}$$

and $\boldsymbol{X} = \boldsymbol{X}^{\mathrm{T}} \geq 0$ is the stabilizing solution of the discrete algebraic Ricciti equation

$$\begin{aligned}&(\boldsymbol{A} + \eta^{-2}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{C})^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{I} + \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{X})^{-1} \\ &(\boldsymbol{A} + \eta^{-2}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{C}) - \eta^{-2}\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{I} + \eta^{-2}\boldsymbol{D}\boldsymbol{R}^{-1}\boldsymbol{D}^{\mathrm{T}})\boldsymbol{C} \\ &-\boldsymbol{X} = 0.\end{aligned} \tag{14}$$

Let $\boldsymbol{\varLambda}(z) = \boldsymbol{U}^{-1}(z)$. For $\boldsymbol{Z}$ in (13) is invertible, $\boldsymbol{\varLambda}$ has the realization

$$\boldsymbol{\varLambda}(z) = \{\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}, -\boldsymbol{B}\boldsymbol{Z}^{-\frac{1}{2}}, -\boldsymbol{F}, \boldsymbol{Z}^{-\frac{1}{2}}\}. \tag{15}$$

Then,

$$[\boldsymbol{I} - \eta^{-2}\boldsymbol{G}^*(z)\boldsymbol{G}(z)]^{-1} = \boldsymbol{\varLambda}^*(z)\boldsymbol{\varLambda}(z). \tag{16}$$

Suppose $\boldsymbol{\varLambda}$ is the closed-loop transfer function of a system driven by a stationary process $\boldsymbol{\alpha}(k)$, and the output is $\boldsymbol{\beta}(k)$. Then the entropy rate of $\boldsymbol{\beta}(k)$ is

$$\begin{aligned}\overline{H}(\boldsymbol{\beta}) &= \overline{H}(\boldsymbol{\alpha}) + \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln|\det\boldsymbol{\varLambda}(e^{i\omega})|^2 \,\mathrm{d}\omega \\ &= \overline{H}(\boldsymbol{\alpha}) + V(\boldsymbol{\varLambda}),\end{aligned} \tag{17}$$

where $\overline{H}(\boldsymbol{\alpha})$ is the entropy rate of $\boldsymbol{\alpha}(k)$.

From equations (6), (9), (16) and (17), the following theorem can be gotten directly.

Theorem 1: For the LTI system $\boldsymbol{G}$ shown by Figure 1 with state space realization (11), $\|\boldsymbol{G}(z)\|_\infty < \eta$, then there exists a system $\boldsymbol{\varLambda}$ with state space realization (15) satisfying the spectral factorization (16), so that

$$\eta^{-2}H(\boldsymbol{G}, \eta) = V(\boldsymbol{\varLambda}). \tag{18}$$

That is, minimizing the H_∞ entropy of system $\boldsymbol{G}$ under constraint (8) is equivalent to minimizing the entropy rate of the output of system $\boldsymbol{\varLambda}$ when it is driven by a stationary process.

Theorem 1 gives an information theoretic interpretation for the H_∞ entropy of system $\boldsymbol{G}$. On the other hand, if we focus on the information theoretic character of $\boldsymbol{G}$, we can get a dual conclusion.

Let the inverse of $\boldsymbol{G}$ be

$$\boldsymbol{M}(z) = \boldsymbol{G}^{-1}(z) = \{\boldsymbol{A}_M, \boldsymbol{B}_M, \boldsymbol{C}_M, \boldsymbol{D}_M\}, \tag{19}$$

where $\boldsymbol{A}_M = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$, $\boldsymbol{B}_M = -\boldsymbol{B}\boldsymbol{D}^{-1}$, $\boldsymbol{C}_M = \boldsymbol{D}^{-1}\boldsymbol{C}$, $\boldsymbol{D}_M = \boldsymbol{D}^{-1}$. Let σ_1, σ_n denote the largest and smallest singular values, respectively. For $\sigma_i(\boldsymbol{G}) > 0$, $i = 1, ..., n$, there always exists a constant μ so that $0 < \mu < \min_\omega \sigma_n(\boldsymbol{G}(e^{i\omega}))$, then $\max_\omega \sigma_1^{-1}(\boldsymbol{M}(e^{i\omega})) = \min_\omega \sigma_n(\boldsymbol{G}(e^{i\omega})) > \mu$. Hence,

$$\|\boldsymbol{M}\|_\infty < \mu^{-1}. \tag{20}$$

From the spectral factorization theorem (Tsai, 1993), condition (20) allows one has

$$\boldsymbol{I} - \mu^2\boldsymbol{M}^*(z)\boldsymbol{M}(z) = \boldsymbol{W}^*(z)\boldsymbol{W}(z), \tag{21}$$

where $\boldsymbol{W}(z), \boldsymbol{W}^{-1}(z) \in \boldsymbol{RH}_\infty^{n\times n}$,

$$\|\boldsymbol{W}\|_\infty < 1, \tag{22}$$

and $\boldsymbol{W}$ has the state space realization

$$\boldsymbol{W}(z) = \{\boldsymbol{A}_M, \boldsymbol{B}_M, -\boldsymbol{Z}_M^{\frac{1}{2}}\boldsymbol{F}_M, \boldsymbol{Z}_M^{\frac{1}{2}}\}, \tag{23}$$

where

$$\begin{aligned}&\boldsymbol{F}_M = -\boldsymbol{Z}_M^{-1}(\boldsymbol{B}_M^{\mathrm{T}}\boldsymbol{X}_M\boldsymbol{A}_M - \mu^2\boldsymbol{D}_M^{\mathrm{T}}\boldsymbol{C}_M), \\ &\boldsymbol{Z}_M = \boldsymbol{R}_M + \boldsymbol{B}_M^{\mathrm{T}}\boldsymbol{X}_M\boldsymbol{B}_M,\ \boldsymbol{R}_M = \boldsymbol{I} - \mu^2\boldsymbol{D}_M^{\mathrm{T}}\boldsymbol{D}_M > 0,\end{aligned}$$

and $\boldsymbol{X}_M = \boldsymbol{X}_M^{\mathrm{T}} \geq 0$ is the stabilizing solution of the discrete algebraic Ricciti equation

$$(A_M + \mu^2 B_M R_M^{-1} D_M^{\mathrm{T}} C_M)^{\mathrm{T}} X_M (I + B_M R_M^{-1} B_M^{\mathrm{T}} X_M)^{-1}$$
$$(A_M + \mu^2 B_M R_M^{-1} D_M^{\mathrm{T}} C_M)$$
$$-\mu^2 C_M^{\mathrm{T}} (I + \mu^2 D_M R_M^{-1} D_M^{\mathrm{T}}) C_M - X_M = 0. \tag{24}$$

Then,

$$[I - W^*(z)W(z)]^{-1} = \mu^{-2} G^*(z) G(z). \tag{25}$$

By definition (1), the H_∞ entropy of system W under condition (22) is

$$H(W,1) = \frac{-1}{4\pi} \int_{-\pi}^{\pi} \ln \det[I - W^*(e^{i\omega}) W(e^{i\omega})] \mathrm{d}\omega. \tag{26}$$

Then, the following theorem can be obtained from equations (6), (10), (25) and (26).

Theorem 2: Suppose for the system G shown by Figure 1 with state space realization (11), $\min_\omega \sigma_n(\mathbf{G}) > \mu > 0$, then there exists a system W with state space realization (23) satisfying the spectral factorization (25), so that

$$V(G) = H(W,1) + \frac{1}{4\pi} \ln \mu^{2n}. \tag{27}$$

That is, minimizing the entropy rate of the output of G is equivalent to minimizing the H_∞ entropy of W under constraint (22).

4. CLASSIC TRACKING SYSTEMS

In this section, the measure of system performance in classic LTI tracking problem will be discussed firstly, and then relations within entropy rate, mutual information rate and H_∞ entropy will be studied.

Consider the discrete multivariable LTI system shown by Figure 2:

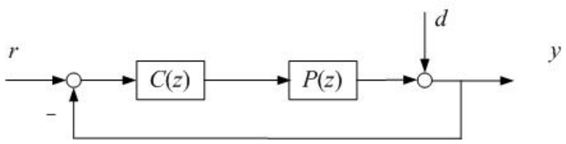

Figure 2. LTI tracking system with disturbance

where r, d, $y \in R^n$ are the reference input, disturbance and output, respectively; $C(z)$ and $P(z)$ are $n \times n$ proper transfer function matrices of controller and plant, respectively. The closed-loop system is well-posed and stable. The reference r and the disturbance d are mutual independent stationary random sequences with rational positive spectral densities $\Phi_r(\omega)$, $\Phi_d(\omega)$, respectively. Suppose $\Phi_d(\omega) = I$ for convenience.

The output signal y consists of two parts, the signal transmitted from reference r, and the signal transmitted from disturbance d. Denote them as y_r and y_d, respectively. The output closed-loop transfer functions are $S(z) = [I + L(z)]^{-1}$, $T(z) = I - S(z)$, where $L(z) = P(z)C(z)$. Then $y(z) = T(z)r(z) + S(z)d(z)$, where $y(z)$, $r(z)$ and $d(z)$ are the z-transformation of y, r and d, respectively. Then, the spectrum of output is $\Phi_y(\omega) = T(e^{i\omega})\Phi_r(\omega)T^*(e^{i\omega}) + S(e^{i\omega})\Phi_d(\omega)S^*(e^{i\omega})$, and the mutual spectral densities of pairs (r, y) and (d, y) are respectively $\Phi_{ry}(\omega) = \Phi_r(\omega)T^*(e^{i\omega})$, $\Phi_{dy}(\omega) = \Phi_d(\omega)S^*(e^{i\omega})$. Let $y_r(z) = T(z)r(z)$ be the z-transformation of y_r.

From the spectral factorization theorem (Ljung, 1999), there exists a rational matrix function of z, $F_y(z)$, with all zeros and poles of $\det|F_y(e^{i\omega})|$ inside the unit circle such that $\Phi_y(\omega) = F_y^*(e^{i\omega})F_y(e^{i\omega})$. In other words, y can be described as the output of a stable minimum phase system $F_y(z)$ driven by a white noise with spectral density I. From Lemma 1 we have

$$\begin{aligned} \bar{H}(y) &= \frac{n}{2}\ln 2\pi e + \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\ |\det F_y(e^{i\omega})|^2\ \mathrm{d}\omega \\ &= \frac{n}{2}\ln 2\pi e + \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \det \Phi_y(\omega)\ \mathrm{d}\omega, \end{aligned} \tag{28}$$

$$\begin{aligned} \bar{H}(y_r) &= \bar{H}(r) + \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\ |\det T(e^{i\omega})|^2\ \mathrm{d}\omega \\ &= \frac{n}{2}\ln 2\pi e \\ &\quad + \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \det[T(e^{i\omega})\Phi_r(\omega)T^*(e^{i\omega})]\mathrm{d}\omega. \end{aligned} \tag{29}$$

Transmission processes of signals from r and d to y are also information and uncertainty transmission processes. The aim of system design is to make y track reference r and reject disturbance d. As we know, the entropy rate $\bar{H}(y)$ describes the total uncertainty of system output, while $\bar{H}(y_r)$ is the information of r obtained by the output. In this consideration, the difference between these two entropy rates, $\bar{H}(y) - \bar{H}(y_r)$, is a measure of uncertainty in system output caused by disturbance.

From equations (28) and (29), we have

$$\begin{aligned} &\bar{H}(y) - \bar{H}(y_r) \\ &\quad = \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \det \Phi_y(\omega)\ \mathrm{d}\omega \\ &\quad\quad - \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \det[T(e^{i\omega})\Phi_r(\omega)T^*(e^{i\omega})]\mathrm{d}\omega \\ &\quad = \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \frac{\det \Phi_y(\omega)}{\det[T(e^{i\omega})\Phi_r(\omega)T^*(e^{i\omega})]}\mathrm{d}\omega. \end{aligned}$$

For

$$T(e^{i\omega})\Phi_r(\omega)T^*(e^{i\omega}) = \Phi_y(\omega) - S(e^{i\omega})S^*(e^{i\omega}),$$

then

$$\bar{H}(\boldsymbol{y})-\bar{H}(\boldsymbol{y_r})$$
$$=-\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det[\boldsymbol{I}-\boldsymbol{\Phi}_{\boldsymbol{y}}^{-1}(\omega)\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})]\,\mathrm{d}\omega.$$

Now let us consider the special case when the reference $\boldsymbol{r}$ and disturbance $\boldsymbol{d}$ are joint Gaussian. Let $\xi(k)=[\boldsymbol{d}^{\mathrm{T}}(k)\ \ \boldsymbol{y}^{\mathrm{T}}(k)]^{\mathrm{T}}$. $\boldsymbol{\xi}$ has the spectrum $\boldsymbol{\Phi}_{\xi}(e^{i\omega})=\begin{bmatrix}\boldsymbol{\Phi}_d(\omega) & \boldsymbol{\Phi}_{dy}(\omega)\\ \boldsymbol{\Phi}_{dy}^*(\omega) & \boldsymbol{\Phi}_y(\omega)\end{bmatrix}$, then

$$\det\boldsymbol{\Phi}_{\xi}(\omega)$$
$$=\det\boldsymbol{\Phi}_{\boldsymbol{d}}(\omega)\det[\boldsymbol{\Phi}_{\boldsymbol{y}}(\omega)-\boldsymbol{\Phi}_{\boldsymbol{dy}}^*(\omega)\boldsymbol{\Phi}_{\boldsymbol{d}}^{-1}(\omega)\boldsymbol{\Phi}_{\boldsymbol{dy}}(\omega)]$$
$$=\det\boldsymbol{\Phi}_{\boldsymbol{y}}(\omega)\det[\boldsymbol{I}-\boldsymbol{\Phi}_{\boldsymbol{y}}^{-1}(\omega)\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})].$$

From Lemma 2 we have that the mutual information rate between $\boldsymbol{d}$ and $\boldsymbol{y}$ is

$$\bar{I}(\boldsymbol{d};\boldsymbol{y})=\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\frac{\det\boldsymbol{\Phi}_{\boldsymbol{d}}(\omega)\det\boldsymbol{\Phi}_{\boldsymbol{y}}(\omega)}{\det\boldsymbol{\Phi}_{\xi}(\omega)}\,\mathrm{d}\omega$$
$$=\frac{-1}{4\pi}\int_{-\pi}^{\pi}\ln\det[\boldsymbol{I}-\boldsymbol{\Phi}_{\boldsymbol{y}}^{-1}(\omega)\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})]\,\mathrm{d}\omega.$$

We conclude the above analysis in the following lemma.

Lemma 3: For the stable tracking system shown in Figure 2,

$$\bar{H}(\boldsymbol{y})-\bar{H}(\boldsymbol{y_r})=\frac{-1}{4\pi}\int_{-\pi}^{\pi}\ln\det[\boldsymbol{I}-\boldsymbol{\Phi}_{\boldsymbol{y}}^{-1}(\omega)\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})]\,\mathrm{d}\omega. \tag{30}$$

Specially, when the reference $\boldsymbol{r}$ and disturbance $\boldsymbol{d}$ are joint Gaussian,

$$\bar{H}(\boldsymbol{y})-\bar{H}(\boldsymbol{y_r})=\bar{I}(\boldsymbol{d};\boldsymbol{y}). \tag{31}$$

The above lemma makes it clearer that $\bar{H}(\boldsymbol{y})-\bar{H}(\boldsymbol{y}_r)$ is a measure of uncertainty in system output caused by disturbance: When $\boldsymbol{r}$ and $\boldsymbol{d}$ are joint Gaussian, this function is exactly the mutual information rate between $\boldsymbol{d}$ and $\boldsymbol{y}$, which describes the uncertainty about $\boldsymbol{d}$ contained in $\boldsymbol{y}$. It is demonstrated by equation (31) that, when the spectrum of output is constrained, the smaller $\bar{I}(\boldsymbol{d};\boldsymbol{y})$ is, the larger $\bar{H}(\boldsymbol{y}_r)$ is. As we have known, $\bar{H}(\boldsymbol{y}_r)$ describes the information about reference $\boldsymbol{r}$ obtained by output. Hence, as a measure of disturbance rejection, $\bar{H}(\boldsymbol{y})-\bar{H}(\boldsymbol{y}_r)$ also reflects the tracking performance.

We now consider the relation between information rates and the H_∞ entropy, which gives a performance bound of disturbance rejection.

For $\boldsymbol{\Phi}_y(\omega)$ is positive, there exists a unitary matrix $\boldsymbol{V}\in\mathbf{C}^{n\times n}$, $\boldsymbol{V}^*\boldsymbol{V}=\boldsymbol{I}$ such that $\boldsymbol{\Phi}_y=\boldsymbol{V}^*\boldsymbol{\Sigma}\boldsymbol{V}$, where $\boldsymbol{\Sigma}=\mathrm{diag}[\sigma_1^2,\ldots,\sigma_n^2]$, $\sigma_i, i=1,2,...,n$ are the singular values of $\boldsymbol{F}_y(z)$. More, $\det\boldsymbol{\Phi}_y=\det(\boldsymbol{V}^*\boldsymbol{\Sigma}\boldsymbol{V})=\det\boldsymbol{\Sigma}$.

Let $\gamma=\|\boldsymbol{F}_y(z)\|_\infty$, then $\det\boldsymbol{\Phi}_y=\prod_{i=1}^{n}\sigma_i^2\le\gamma^{2n}$. So,

$$\det[\boldsymbol{\Phi}_y^{-1}(\omega)\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})]\ge\gamma^{-2n}\det[\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})],$$

i.e.,

$$\det[\boldsymbol{I}-\boldsymbol{\Phi}_{\boldsymbol{y}}^{-1}(\omega)\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})]\le\det[\boldsymbol{I}-\gamma^{-2}\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})].$$

Hence,

$$\begin{aligned}&\frac{-1}{4\pi}\int_{-\pi}^{\pi}\ln\det[\boldsymbol{I}-\boldsymbol{\Phi}_{\boldsymbol{y}}^{-1}(\omega)\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})]\,\mathrm{d}\omega\\&\quad\ge\frac{-1}{4\pi}\int_{-\pi}^{\pi}\ln\det[\boldsymbol{I}-\gamma^{-2}\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})]\,\mathrm{d}\omega.\end{aligned} \tag{32}$$

For

$$\boldsymbol{\Phi}_{\boldsymbol{y}}(\omega)=\boldsymbol{T}(e^{i\omega})\boldsymbol{\Phi}_{\boldsymbol{r}}(\omega)\boldsymbol{T}^*(e^{i\omega})+\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})$$
$$\ge\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega}),$$

then $\boldsymbol{\Phi}_y^{-1}(\omega)\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})\le\boldsymbol{I}$. It is known that entropy rates of stationary processes exist, so $\bar{H}(\boldsymbol{y})-\bar{H}(\boldsymbol{y}_r)<+\infty$. If $\boldsymbol{\Phi}_y^{-1}(\omega)\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})=\boldsymbol{I}$, it can be deduced from Lemma 3 that $\bar{I}(\boldsymbol{d};\boldsymbol{y})=+\infty$ when $\boldsymbol{r}$ and $\boldsymbol{d}$ are joint Gaussian. However, in this case $\bar{H}(\boldsymbol{y})-\bar{H}(\boldsymbol{y}_r)=\bar{I}(\boldsymbol{d};\boldsymbol{y})$. Hence,

$$\gamma^{-2}\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})\le\boldsymbol{\Phi}_y^{-1}(\omega)\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})<\boldsymbol{I}.$$

So,

$$\|\boldsymbol{S}(z)\|_\infty<\gamma. \tag{33}$$

By definition (1), the H_∞ entropy of transfer function $\boldsymbol{S}(z)$ under the condition (33) is

$$H(\boldsymbol{S},\gamma)=\frac{-\gamma^2}{4\pi}\int_{-\pi}^{\pi}\ln\det[\boldsymbol{I}-\gamma^{-2}\boldsymbol{S}(e^{i\omega})\boldsymbol{S}^*(e^{i\omega})]\,\mathrm{d}\omega. \tag{34}$$

From Lemma 3 and inequality (32), the following result is concluded.

Theorem 3: For the stable tracking system shown in Figure 2,

$$\gamma^2[\bar{H}(\boldsymbol{y})-\bar{H}(\boldsymbol{y}_r)]\ge H(\boldsymbol{S},\gamma). \tag{35}$$

Specially, when the reference $\boldsymbol{r}$ and disturbance $\boldsymbol{d}$ are joint Gaussian,

$$\gamma^2 I(\boldsymbol{d};\boldsymbol{y})\ge H(\boldsymbol{S},\gamma). \tag{36}$$

Because γ reflects the 'size' of system output, $\gamma^2[\overline{H}(\boldsymbol{y})-\overline{H}(\boldsymbol{y}_r)]$ (or $\gamma^2\bar{I}(\boldsymbol{d};\boldsymbol{y})$ in Gaussian case) is a more suitable measure of disturbance rejection than $\overline{H}(\boldsymbol{y})-\overline{H}(\boldsymbol{y}_r)$ (or, $\bar{I}(\boldsymbol{d};\boldsymbol{y})$). Theorem 3 shows that the H_∞ entropy of the sensitivity function is an lower bound of the performance function ($\gamma^2[\overline{H}(\boldsymbol{y})-\overline{H}(\boldsymbol{y}_r)]$) of disturbance rejection. Stoorvogel and Schuppen (1996) gave an equivalent relation between mutual information rate and the H_∞ entropy for identification problem in Gaussian case, while Theorem 3 formulates a general relation between entropy rate and the H_∞ entropy for tracking problem in the case of general stationary processes.

5. CONCLUSION AND DISCUSSION

By investigating the information transmission in discrete multivariable LTI system disturbed by stationary noise, relations within entropy rate, mutual information rate and H_∞ entropy were formulated, and information interpretations of minimum entropy H_∞ control were stated. For the general regulation problem, there are equivalent relations between minimizing entropy rate and minimizing the H_∞ entropy for systems satisfying spectral factorization. For the tracking problem, the H_∞ entropy of the closed-loop transfer function from disturbance to output is a lower bound of performance function of disturbance rejection measured by information rates. These relations formulate the information descriptive property of the H_∞ entropy.

Information theoretic methods are very useful in analysis and design of stochastic control systems. On the other hand, the minimum entropy H_∞ control theory is a suboptimal robust design method in the field of H_∞ control, and is not limited to the case of stochastic system. However, Theorems 1, 2 and 3 state somewhat interesting connections between these two methodologies. In our viewpoint, the obtained results are valuable for control system analysis and design.

There are several potential applications of these theorems:

○ If we consider the regulation problem in the classic framework shown in Figure 2 when reference $\boldsymbol{r}$ is assumed to be constant, then the limit of disturbance rejection performance (measured by $\overline{H}(\boldsymbol{y})$) can be formulated by using Bode integral (Zhou, 1998). That is

$$\overline{H}(\boldsymbol{y})=\overline{H}(\boldsymbol{d})+\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln|\det\boldsymbol{S}(e^{i\omega})|^2\,\mathrm{d}\omega\geq\overline{H}(\boldsymbol{d}).$$

However, for the general control system illustrated in Figure 1, Bode integral is not applicable. In this case, Theorem 2 may become a useful device to analyze the system performance measured by entropy rate. This analysis, in our view, will concern the largest and smallest singular values of systems $\boldsymbol{G}$ and $\boldsymbol{W}$.

○ Using the computation of H_∞ entropy (Mustafa, 1990; Iglesias, 1990) based on state space realization, we can get explicit formulations of entropy rate in equations (10) and bounds of information rates in (35) and (36), in terms of state space parameters. That is, Theorems 2 and 3 give time domain computation method for these performance functions.

○ It can be seen from Theorem 3 that, as a performance function, $\gamma^2[\overline{H}(\boldsymbol{y})-\overline{H}(\boldsymbol{y}_r)]$ is 'looser' than the H_∞ entropy. This illumines us that the existence and procedure of the solution to the following problem may become subjects of further consideration: Minimizing $\overline{H}(\boldsymbol{y})-\overline{H}(\boldsymbol{y}_r)$ (or, $\bar{I}(\boldsymbol{d};\boldsymbol{y})$ in the Gaussian case) under the constraint of $\det\boldsymbol{\Phi}_y\leq\gamma^{2n}$.

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