THE GELFAND FORMULA FOR LINEAR PARAMETER-VARYING AND LINEAR SWITCHING SYSTEMS

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Abstract: It is shown that the Gelfand formula holds for a large class of families of linear time-varying systems, encompassing in particular standard formulations of linear parameter-varying and linear switching systems. By this result, the uniform exponential growth rate may be approximated to arbitrary precision by the growth rate of periodic systems within the family. This result extends classical results in the area of linear inclusions. The basic tool in the proof is a recent construction of parameter dependent Lyapunov functions for the family of linear time-varying systems that exactly characterize the exponential growth rate.

Keywords: Lyapunov exponent, linear switching system, linear parameter-varying system, periodic system.

1. INTRODUCTION

In this paper we consider time-varying linear systems of the form

$$\dot{x} = A(t)x\,, \tag{1}$$

where $A : \mathbb{R} \to \mathcal{M}$ is a measurable map and $\mathcal{M}$ is a compact subset of real or complex matrices of a given dimension. We are interested not in one individual system but in the exponential growth rate of a set of systems that is described by a subset $\mathcal{A} \subset L^\infty(\mathbb{R}, \mathcal{M})$. The stability and spectral properties of such kinds of systems have been actively investigated over the past two decades, in particular for linear parameter varying and linear switching systems. The setup described in Section 2 encompasses a large subset of both these system classes. The case that $\mathcal{A} = L^\infty(\mathbb{R}, \mathcal{M})$, which is equivalent to the linear inclusion

$$\dot{x} \in \{Ax \mid A \in \mathcal{M}\}\,, \tag{2}$$

is also treated in the literature under the name of families of linear time-varying systems, linear parameter-varying systems, and linear switching systems (Colonius and Kliemann, 2000) and a complete Lyapunov function theory has been developed (Barabanov, 1988; Molchanov and Pyatnitskij, 1989; Wirth, 2002). One of the interesting results for the inclusion (2) is that the uniform exponential growth rate can be approximated arbitrarily well by growth rates associated to periodic systems. This result is sometimes called the Gelfand formula in reminiscence of the characterization of the spectral radius of a bounded linear operator as the infimum of norms of its powers. It has been obtained for (2) using different approaches, see e.g. (Berger and Wang, 1992; Colonius and Kliemann, 1993; Elsner, 1995). This result is of interest as it lays the foundation for various methods for the calculation of the growth rate. Also it provides insight into the complexity of the system class under consideration.

Here we generalize this result to a large class of systems, which includes in particular linear

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parameter-varying systems with bounds on the derivative of the parameter variation and linear switching systems with bounds on the length of intervals between switchings. The method of proof relies on a general construction of Lyapunov functions for this system class. These Lyapunov functions characterize the exponential growth rate in the irreducible case. This result complements constructions of Lyapunov functions for linear inclusions in (Barabanov, 1988; Molchanov and Pyatnitskij, 1989; Wirth, 2002), sometimes also called extremal norms in this area.
We proceed as follows. In the ensuing Section 2 we introduce the class of systems under consideration and define the corresponding uniform exponential growth rate. This is the quantity of interest in this paper. In Section 3 we describe the concatenation structure within the set of admissible parameter variations. These considerations yield the right notion of "parameter" for the parameterized Lyapunov functions. Finally, in Section 4 the Gelfand formula is proved. The paper concludes with some final comments in Section 5.

2. FAMILIES OF LINEAR TIME-VARYING SYSTEMS

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ denote the real or the complex field. We consider families of time-varying systems of the form

$$\dot{x}(t) = A(\theta(t))x(t)\,, \quad t \in \mathbb{R}\,, \tag{3}$$

where $\theta(\cdot) \in L^\infty(\mathbb{R}, \Theta)$ is an admissible parameter variation, $\emptyset \neq \Theta \subset \mathbb{K}^m$ and $A : \Theta \to \mathbb{K}^{n\times n}$ is continuous. For fixed $\theta(\cdot)$ the evolution operator generated by (3) is denoted by $\Phi_\theta(t, s)$, $t, s \in \mathbb{R}$.

In all that follows the set of admissible parameter variations is described by a quadruple $\Sigma = (h, \Theta, \Theta_1, A)$ denoting a bound on the dwell time, the parameter set, a set of restrictions for the derivative, and a map A. Our assumptions are:

(A1) $h \in (0, \infty]$,
(A2) $\Theta \subset \mathbb{K}^m$ is a finite disjoint union of compact, convex sets Ω_j, $j \in \{1, \dots, k\}$, if $h = \infty$ then $k = 1$, i.e. Θ is compact and convex,
(A3) Θ_1 is compact and convex,
(A4) $0 \in \Theta_1$,
(A5) $A : \Theta \to \mathbb{K}^{n\times n}$ is a continuous map from the parameter space to the space of matrices.

Sometimes it will be necessary to assume the following sharpened version of (A4). [2]

(A6) $0 \in \operatorname{ri} \Theta_1$ and $\operatorname{span} \Theta_1 \supset \operatorname{span}(\Omega_j - \eta_j)$, $j = 1, \dots, k$ where $\eta_j \in \Omega_j$ is arbitrary.

[2] Recall that the relative interior of a convex set $\mathcal{M}$, denoted by $\operatorname{ri} \mathcal{M}$, is the interior of $\mathcal{M}$ in the relative topology of the affine space space generated by $\mathcal{M}$, which is the smallest affine space containing $\mathcal{M}$.

With these assumptions we are able to define admissible parameter variations.

Definition 1. Let $\Sigma = (h, \Theta, \Theta_1, A)$ satisfy (A1)–(A5). If $h \in (0, \infty)$ a parameter variation $\theta : \mathbb{R} \to \Theta$ is called *admissible* (with respect to Σ), if there is an index set $\mathcal{I}_\theta \subset \mathbb{Z}$ and times $t_k, k \in \mathcal{I}_\theta$ such that

(i) $h \leq t_{k+1} - t_k$, for $k \in \mathcal{I}_\theta, k < \sup \mathcal{I}_\theta$,
(ii) for $k \in I_\theta, k < \sup I_\theta$ the function θ is absolutely continuous on the interval $[t_k, t_{k+1})$, and satisfies

$$\dot{\theta}(t) \in \Theta_1\,, \quad \text{a.e.} \tag{4}$$

(This condition also applies to $(-\infty, \min \mathcal{I}_\theta)$, $(\max \mathcal{I}_\theta, \infty)$ if $\min \mathcal{I}_\theta$, resp. $\max \mathcal{I}_\theta$, is finite.)

The set of admissible parameter variations is denoted by $\mathcal{U}$ or $\mathcal{U}(h, \Theta, \Theta_1, A)$, if dependence on the data needs to be emphasized.
If $h = \infty$ the set of admissible parameter variations is the set of absolutely continuous functions $\theta : \mathbb{R} \to \Theta$ satisfying (4) almost everywhere on $\mathbb{R}$.

Thus in brief, the constant h denotes the minimal distance between the discontinuities $t_k, k \in \mathcal{I}_\theta$ of θ, and in between discontinuities the variations of θ are bounded by Θ_1. In particular, the case $h = \infty$ corresponds to the absence of discontinuities. This is reflected in (A2): If there are no discontinuities, the parameter variations have no chance to move from one component Ω_j to another $\Omega_{j'}$. It is therefore no restriction to consider each individual component separately. Our system class encompasses linear switching systems with dwell time (see e.g. (Liberzon, 2003)) by letting $\Omega_j := \{A_j\}$ be singleton sets, and parameter-varying systems (see e.g. (Shamma and Athans, 1991)) by choosing $h = \infty$.

We now define the object of interest in this paper: the (uniform) exponential growth rate associated to system (3). Given the system Σ, define for $t \geq 0$ the sets of finite time evolution operators

$$\mathcal{S}_t(\Sigma) := \{\, \Phi_u(t, 0) \mid u \in \mathcal{U} \}\,, \quad \mathcal{S}(\Sigma) := \bigcup_{t\geq 0} \mathcal{S}_t(\Sigma).$$

We now introduce for $t > 0$ finite time growth constants given by

$$\widehat{\rho}_t(\Sigma) := \sup \left\{ \frac{1}{t} \log \|S\| \mid S \in \mathcal{S}_t(\Sigma) \right\}.$$

It is easy to see that (restricted to positive t) the function $t \longmapsto t\widehat{\rho}_t(\Sigma)$ is subadditive, so that the following limit exists

$$\widehat{\rho}(\Sigma) := \lim_{t\to\infty} \widehat{\rho}_t(\Sigma) = \inf_{t\geq 0} \widehat{\rho}_t(\Sigma). \tag{5}$$

It is furthermore well known that an alternative way to describe $\hat{\rho}$ is given by

$$\widehat{\rho}(\Sigma) = \inf\{\beta \in \mathbb{R} \mid \exists M \geq 1 : \|\Phi_u(t,0)\| \leq Me^{\beta t} \text{ for all } u \in \mathcal{U}, t \geq 0\}\,.$$

For this reason the quantity $\hat{\rho}(\Sigma)$ is called *uniform exponential growth rate* of the family of linear time-varying systems of the form (3) given by Σ. An alternative way to define exponential growth is to employ a trajectory-wise definition. In this case we define the Lyapunov exponent corresponding to an initial condition $x_0 \in \mathbb{K}^n \setminus \{0\}$ and $u \in \mathcal{U}$ by

$$\lambda(x_0, u) := \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_u(t,0)x_0\|\,, \tag{6}$$

and define as exponential growth rate $\kappa(\Sigma) := \sup\{\lambda(x,u) \mid 0 \neq x \in \mathbb{K}^n\,, u \in \mathcal{U}\}$. Using (Colonius and Kliemann, 2000, Prop. 5.4.15) it can be shown that $\kappa(\Sigma) = \hat{\rho}(\Sigma)$, see (Wirth, 2004).

Similar statements hold for the linear inclusion (2). As the uniform exponential growth rate can be defined trajectory wise, it is interesting to look for easier subclasses of trajectories which allow for its approximation. One such class is the set of solutions of periodic systems and it is our main result in Section 4 that this set is indeed sufficient.

3. CONCATENATION, IRREDUCIBILITY AND LYAPUNOV NORMS

In this section we briefly describe the main results from (Wirth, 2004) which are essential in order to obtain the desired result. We assume the system $\Sigma = (h, \Theta, \Theta_1, A)$ to be given. For ease of notation we will therefore suppress the dependence on these data of $\hat{\rho}(\Sigma)$, $\mathcal{S}_t(\Sigma)$, etc.

Our problem in studying the family of linear time-varying systems if compared to the case of linear inclusions may be viewed as follows: Simple concatenation of admissible parameter variations does in general not result in an admissible parameter variation. In contrast for every admissible parameter variation $u_1 \in \mathcal{U}$ and $t \geq 0$ there is a certain subset of $u_2 \in \mathcal{U}$ for which the following concatenation is also admissible

$$(u_1 \diamond_t u_2)(s) := \begin{cases} u_1(s)\,, & s < t \\ u_2(s-t)\,, & t \leq s \end{cases}\,. \tag{7}$$

It is easy to see, that this subset depends on the continuous extension of u_1 at t from the left and, in the case $h \in (0,\infty)$, on the difference between the time instance t and the largest discontinuity of u_1 smaller than t. To denote these quantities we define for $u \in \mathcal{U}$

$$u(t^-) := \lim_{s \nearrow t} u(s) \tag{8}$$

and

$$\tau^-(u,t) := \min\{h, t - \max\{t_k \mid t_k < t \text{ where } t_k \text{ is a discontinuity of } u\}\}\,.$$

We first treat the case $h \in (0,\infty)$. Let $t_0(u)$ denotes the smallest positive discontinuity of a parameter variation u. We want to define the set of admissible parameter variations that are concatenable to u_1 at t. To this end we define for $(\theta,\tau) =: \omega \in \Theta \times [0,h)$ the set of concatenable parameter variations by

$$\mathcal{U}(\omega) := \mathcal{U}(\theta,\tau) := \{u \in \mathcal{U} \mid u(0) = \theta \text{ and } h \leq t_0(u) + \tau\}\,,$$

here τ represents the time elapsed since the last discontinuity. For $\tau = h$ and $\omega = (\theta, h)$ let

$$\mathcal{U}(\omega) := \{u \in \mathcal{U} \mid u(0) = \theta \text{ or } h \leq t_0(u)\}\,.$$

Note that with this definition we clearly have $\mathcal{U} = \cup_{\omega\in\Theta\times[0,h]} \mathcal{U}(\omega)$ as every admissible parameter variation is continuous on some interval $[0,\tau]$.

The interpretation of the set $\mathcal{U}(\theta,\tau)$ is the following. Consider a parameter variation u_1 defined on the interval $(-\infty, t)$ and the concatenation (7). If a discontinuity of u_1 occurs in the interval $(t-h, t)$, then admissible concatenations in t have to result in a continuous function in t. This requires $u_1(t) = u_2(0)$. Additionally, u_2 has to wait for a time span of length at least $h - \tau^-(u_1,t)$ until it is allowed to have a discontinuity, so $t_0(u_2) \geq h - \tau^-(u_1,t)$ is also necessary. If there is no discontinuity of u_1 in $(t-h,t)$, equivalently if $\tau^-(u_1,t) = h$, then we can either introduce a discontinuity at t, which requires $t_0(u_2) \geq h$, or if continuity is preserved by $u_1(t) = u_2(0)$, there is no restriction on $t_0(u_2)$. In all for $u_2 \in \mathcal{U}$ the concatenation $u_1 \diamond_t u_2$ defines an admissible parameter variation if and only if

$$u_2 \in \mathcal{U}(u_1(t^-), \tau^-(u_1,t))\,.$$

Note that for $0 \leq \tau_1 < \tau_2 \leq h$ we have

$$\mathcal{U}(\theta,\tau_1) \subset \mathcal{U}(\theta,\tau_2)\,.$$

Furthermore, it should be noted that the sets $\mathcal{U}(\theta, 0)$ are not really needed for concatenation purposes but are included for continuity reasons.

In the case $h = \infty$ there is no need to account for discontinuities. We thus define for $\theta \in \Theta$ the set

$$\mathcal{U}(\theta) := \{u \in \mathcal{U} \mid u(0) = \theta\}\,.$$

For the sake of a unified notation, we define

$$\Pi(\Theta, h) := \begin{cases} \Theta \times [0,h]\,, & \text{if } h \in (0,\infty)\,, \\ \Theta\,, & \text{if } h = \infty\,. \end{cases}$$

In all we have introduced notation just to be able to make the following statement, which is now obvious.

Lemma 2. Consider a system $\Sigma = (h, \Theta, \Theta_1, A)$ satisfying $(A1) - (A5)$ and let $u_1, u_2 \in \mathcal{U}$. The concatenation (7) yields an admissible parameter variation $u_1 \diamond_t u_2$, if and only if there exists $\omega \in \Pi(\Theta, h)$, such that

$$(u_1(t^-), \tau^-(u_1, t)) = \omega \text{ and } u_2 \in \mathcal{U}(\omega) \ .$$

For each $\omega \in \Pi(\Theta, h)$ and $t \geq 0$ we define the set of evolution operators "starting in ω" by

$$\mathcal{S}_t(\omega) := \{ \Phi_u(t,0) \mid u \in \mathcal{U}(\omega)\}. \quad (9)$$

Similarly, we define for $\omega, \zeta \in \Pi(\Theta, h)$ and for $t \geq 0$ the sets of evolution operators "starting in ω and ending at ζ" by

$$\mathcal{R}_t(\omega,\zeta) := \{\Phi_u(t,0) \,|\, u \in \mathcal{U}(\omega)\,, \text{ and } \forall u_2 \in \mathcal{U}(\zeta) \\ u \diamond_t u_2 \in \mathcal{U}(\omega)\,\}\,.$$

Thus by definition if $R \in \mathcal{R}_s(\omega, \zeta)$ and $S \in \mathcal{S}_t(\zeta)$, then $SR \in \mathcal{S}_{t+s}(\omega)$. We now define

$$\mathcal{S}(\omega) := \bigcup_{t \geq 0} \mathcal{S}_t(\omega)\,, \quad \text{respectively}$$
$$\mathcal{R}(\omega,\zeta) := \bigcup_{t \geq 0} \mathcal{R}_t(\omega,\zeta)\,.$$

Note that the definition entails that for every $\omega \in \Pi(\Theta, h)$ the set $\mathcal{R}(\omega, \omega)$ is a semigroup.

Remark 3. It is useful to keep in mind the following remark on parameter variations connecting two points $\omega, \zeta \in \Theta \times [0, h]$. If $h \in (0, \infty)$, then for all $\omega, \zeta \in \Pi(\Theta, h)$ the set $\mathcal{R}_{2h}(\omega, \zeta)$ is not empty. For if $\omega = (\theta, \tau), \zeta = (\eta, \sigma)$, then it suffices to define $u(s) = \theta, 0 \leq s < h$ and $u(s) = \eta, h \leq s \leq 2h$, which defines an admissible parameter variation in $\mathcal{U}(\omega)$. Similarly, if $h = \infty$ and (A6) holds then for a suitable constant $\bar{c}$ we have that $\mathcal{R}_{\bar{c}}(\omega, \zeta) \neq \emptyset$ for all $\omega, \zeta \in \Pi(\Theta, h)$.

In the following the most important assumption is that of irreducibility of $A(\Theta)$. Recall that a set of matrices $\mathcal{M} \subset \mathbb{K}^{n \times n}$ is called irreducible, if only the trivial subspaces 0 and $\mathbb{K}^n$ are invariant under all $A \in \mathcal{M}$ and reducible otherwise.

Remark 4. (i) Note that the set of systems Σ for which $A(\Theta)$ is irreducible is open and dense in the set of all systems in the appropriate topology, see (Wirth, 2004).
(ii) If $A(\Theta)$ is reducible we can find a similarity transformation T such that for all $\theta \in \Theta$ the tranformed matrix $TA(\theta)T^{-1}$ is of the form

$$\begin{bmatrix} A_{11}(\theta) & A_{12}(\theta) & \dots & A_{1d}(\theta) \\ 0 & A_{22}(\theta) & \dots & A_{2d}(\theta) \\ & \ddots & \ddots & \vdots \\ 0 & & 0 & A_{dd}(\theta) \end{bmatrix}, \quad (10)$$

where the sets $A_{ii}(\Theta)$ are irreducible or $\{0\}$. It is an easy exercise to show that in this case $\hat{\rho}(A, \mathcal{U}) = \max_{i=1,\dots,d} \hat{\rho}(A_i, \mathcal{U})$, where $A_i : \Theta \to \mathbb{K}^{n_i \times n_i}$ is the map $\theta \mapsto A_{ii}(\theta)$. Having said this, it is clear, that for the analysis of $\hat{\rho}$ with respect to one system we can assume irreducibility without loss of generality.

It can now be shown that if $A(\Theta)$ is irreducible and our standing assumptions hold there exists a family of norms parameterized by $\omega \in \Pi(\Theta, h)$. This family of norms characterizes the exponential growth rate of the system Σ. Also the following result shows that for irreducible systems the growth rate may be realized instantaneously from any initial condition, if the growth is measured with respect to the family of norms.

Theorem 5. (Wirth, 2004) Consider system (3) with (A1)-(A5). Assume that $A(\Theta)$ is irreducible. Then there exists an equicontinuous family of norms $v_\omega, \omega \in \Pi(\Theta, h)$, such that for any $\omega \in \Pi(\Theta, h)$ it holds that

(i) For all $u \in \mathcal{U}(\omega), t \geq 0$ and all $x \in \mathbb{K}^n$ it holds that

$$v_\zeta(\Phi_u(t,0)x) \leq e^{\hat{\rho}t}\, v_\omega(x)\,, \quad (11)$$

whenever $\Phi_u(t, 0) \in \mathcal{R}_t(\omega, \zeta)$ for $\zeta \in \Pi(\Theta, h)$. In particular, for all $t \geq s \geq 0$ it holds that

$$v_{u(t^-),\tau^-(u,t)}(\Phi_u(t,0)x) \\ \leq e^{\hat{\rho}(t-s)}\, v_{u(s^-),\tau^-(u,s)}(\Phi(s,0)x)\,.$$

(ii) For every $x \in \mathbb{K}^n$, $\omega \in \Pi(\Theta, h)$, and every $t \geq 0$, there exist $u \in \mathcal{U}(\omega)$ and a piecewise continuous map $\zeta : [0, t] \to \Pi(\Theta, h)$, with $\zeta(0) = \omega$, and such that for all $s \in [0, t]$ we have

$$v_{\zeta(s)}(\Phi_u(s,0)x) = e^{\hat{\rho}s}\, v_\omega(x)\,.$$

If $h = \infty$, then ζ may be chosen to be continuous. If $h < \infty$ and $\omega = (\theta, \tau) \in \Theta \times [0, h)$, the function ζ may be chosen, so that its discontinuities on $[0, t)$ coincide with those of u. Otherwise, ζ may have one further discontinuity at 0.

4. THE GELFAND FORMULA

In this section we give an application of the existence of the parameterized Lyapunov functions we have described so far. One of the classical results in the analysis of families of linear time-varying systems is that under certain conditions the exponential growth rate can be approximated by just considering the subset of periodic systems within the family.

In the case of linear inclusions one way to define the exponential growth rate is via the long term behavior of the maximal spectral radius of evolution operators. In our case periodicity of the underlying parameter variation is the natural assumption, which is analyzed in the sequel.

For $t \in \mathbb{R}_+$ we define the set of evolution operators corresponding to periodic $u \in \mathcal{U}$ by

$$\mathcal{P}_t := \bigcup_{\omega \in \Pi(\Theta, h)} \mathcal{R}_t(\omega, \omega) \ .$$

Then we may define the normalized supremum over the spectral radii by

$$\bar{\rho}_t := \sup \left\{ \frac{1}{t} \log r(S) \mid S \in \mathcal{P}_t \right\}$$

and the supremum of the exponential growth rates obtainable by periodic parameter variations is defined by

$$\bar{\rho} := \limsup_{t \to \infty} \bar{\rho}_t \, .$$

As it is clear that $\bar{\rho}_t \leq \hat{\rho}_t$ for all $t \geq 0$, we obtain immediately that $\bar{\rho} \leq \hat{\rho}$. We intend to show that these quantities are equal. To this end we need the following lemma.

Lemma 6. Consider system (3) with (A1)–(A5). Assume that $A(\Theta)$ is irreducible and let one of the following assumptions be satisfied

(a) $h \in (0, \infty)$,
(b) $h = \infty$ and (A6) is satisfied.

Then there exist $\omega \in \Theta \times [0, h]$, $x \in \mathbb{K}^n, v_\omega(x) = 1$ and a sequence $S_k \in \mathcal{R}_{t_k}(\omega, \omega)$, $t_k \geq 1$ with

$$e^{-\hat{\rho} t_k} S_k x \to x \, .$$

PROOF. We may assume that $\hat{\rho} = 0$ by considering the shifted function $A - \hat{\rho} I$, if necessary. Pick an arbitrary $\omega_0 \in \Theta \times [0, h]$ and $z \in \mathbb{K}^n$ such that $v_{\omega_0}(z) = 1$. By Proposition 5 (ii) there exist a ω_1 and $S_1 \in \mathcal{R}_1(\omega_0, \omega_1)$ such that $v_{\omega_1}(S_1 z) = v_{\omega_0}(z) = 1$. Applying this argument again there exist ω_2 and $S_2 \in \mathcal{R}_1(\omega_1, \omega_2)$ such that $v_{\omega_2}(S_2 S_1 z) = 1$. Repeating this argument inductively we obtain sequences ω_k and $\{S_k\}_{k \in \mathbb{N}}$ with

$$v_{\omega_k}(S_k S_{k-1} \cdot \ldots \cdot S_1 z) = 1 \, , \quad k \in \mathbb{N} \, .$$

As $\Theta \times [0, h]$ is compact there exists a convergent subsequence $\omega_{k_l} \to \omega$ for some $\omega \in \Theta \times [0, h]$. Applying (Wirth, 2004, Corollary 6.7) we may without loss of generality assume that $z_{k_l} := S_{k_l} S_{k_l - 1} \cdot \ldots \cdot S_1 z \to x \neq 0$. We denote $T_{k_l} := S_{k_l} S_{k_l - 1} \cdot \ldots \cdot \cdot S_{k_{l-1}} \in \mathcal{R}(\omega_{k_{l-1}}, \omega_{k_l})$. After relabeling we return to the index k.
Now by (Wirth, 2004, Lemma 4.6 (vi)) and using the assumptions (a) and (b) the map $(\omega, \zeta) \to \mathcal{R}_t(\omega, \zeta)$ is upper semicontinuous uniformly in t (which is crucial, as we have no control over the length of the intervals needed to define the sequence $\{T_k\}$). Thus by convergence of $\omega_k \to \omega$ and for every $\epsilon > 0$ there exists a k_0 such that for every $k \geq k_0$ there exists an $R_k \in \mathcal{R}(\omega, \omega)$ with $v_\omega(T_k - R_k) < \epsilon$ and so that $v_\omega(z_k - x) \leq \epsilon$. Then we obtain that

$$\begin{aligned} v_\omega(R_k x - x) &\leq v_\omega(R_k - T_k) v_\omega(x) + \\ &\quad v_\omega(T_k x - T_k z_k) + v_\omega(z_{k+1} - x) \\ &\leq \epsilon \left(v_\omega(x) + v_\omega(T_k) + 1 \right) \, . \end{aligned}$$

This implies that there exists a sequence $\{R_k\} \subset \mathcal{R}(\omega, \omega)$ with $R_k x - x \to 0$, as desired. $\square$

We also need the following semi-continuity property of $\hat{\rho}$, for which we omit the proof. We denote the space of systems

$$\mathcal{L} := \{ \Sigma := (h, \Theta, \Theta_1, A) \mid \Sigma \text{ satisfies (A1) – (A5)} \}$$

and endow it with the product topology inherited from $(0, \infty]$, the space of compact (resp. convex, compact) sets endowed with the Hausdorff topology, and the topology on $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^{n \times n})$, which we take to be the topology of locally uniform convergence.

Proposition 7. (Wirth, 2004) The map $\hat{\rho} : \mathcal{L} \to \mathbb{R}$,

$$(h, \Theta, \Theta_1, A) \mapsto \hat{\rho}(h, \Theta, \Theta_1, A)$$

is upper semicontinuous.

Before we can state the main result of this section we need a further observation for the case $h = \infty$.

Proposition 8. Consider $\Sigma = (h, \Theta, \Theta_1, A)$ satisfying (A1)-(A5) and assume furthermore that $h = \infty$. Let Θ_2 be the largest convex set contained in Θ_1 such that $0 \in \operatorname{ri} \Theta_2$. Then

$$\hat{\rho}(\infty, \Theta, \Theta_1, A) = \hat{\rho}(\infty, \Theta, \Theta_2, A) \, .$$

PROOF. Clearly, $\hat{\rho}(\infty, \Theta, \Theta_1, A)$ is greater or equal to $\hat{\rho}(\infty, \Theta, \Theta_2, A)$, we show the opposite.
If $0 \in \operatorname{ri} \Theta_1$, there is nothing to show. Otherwise denote by X_2 the linear subspace generated by Θ_2 and denote by $X_2^\perp$ its orthogonal complement. Choose $\theta(\cdot) \in \mathcal{U}$ such that for some $x_0 \neq 0$ we have

$$\hat{\rho}(\infty, \Theta, \Theta_1, A) = \lambda(x_0, \theta(\cdot)) \, .$$

This choice is possible using Fenichel's uniformity lemma, see (Colonius and Kliemann, 2000, Prop. 5.4.15) and (Wirth, 2004).
Now θ may be decomposed as $\theta = \theta_1 + \theta_2$ such that $\dot{\theta}_1 : \mathbb{R}_+ \to X_2^\perp$ and $\dot{\theta}_2 : \mathbb{R}_+ \to \Theta_2$. Furthermore, as 0 is contained in the boundary of Θ_1, there exists a supporting hyperplane X in 0, which has to contain X_2. Hence there is a vector $d \neq 0$ such that $\left\langle d, \dot{\theta}_1(t) \right\rangle \geq 0$ and $\left\langle d, \dot{\theta}_2(t) \right\rangle \equiv 0$ for all $t \geq 0$. Now Θ is compact and so $\langle d, \theta \rangle$ is bounded over $\theta \in \Theta$. This implies that the expression

$$c := \langle d, \theta(0) \rangle + \int_0^\infty \langle d, \dot{\theta}_1(t) \rangle dt = \lim_{t \to \infty} \langle d, \theta(t) \rangle$$

is well defined. If we introduce the set $\Theta_c := \{ \eta \in \Theta \mid \langle d, \eta \rangle = c \}$ we see that

$$\operatorname{dist}(\theta(t), \Theta_c) \to 0 \, , \text{ for } t \to \infty \, .$$

Thus for the set $\Theta_{c,\epsilon} := \{\eta \in \Theta \mid \mathrm{dist}\,(\eta, \Theta_c) \le \epsilon\}$ we obtain $\theta(t) \in \Theta_\epsilon$ for all t large enough. This implies that for all $\epsilon > 0$ we have for t large enough that

$$\hat\rho(\infty, \Theta, \Theta_1, A) \ge \hat\rho(\infty, \Theta_{c,\epsilon}, \Theta_1, A) \ge \\ \lambda(\Phi_\theta(t,0)x_0, \theta(t+\cdot)) = \lambda(x_0, \theta(\cdot))$$

so that equality holds throughout. By Lemma 7 it follows that

$$\hat\rho(\infty, \Theta_c, \Theta_1, A) \ge \lim_{\epsilon\to 0} \hat\rho(\infty, \Theta_{c,\epsilon}, \Theta_1, A) = \\ \hat\rho(\infty, \Theta, \Theta_1, A)\,.$$

The converse equality holds because $\Theta_c \subset \Theta$. Furthermore, we have

$$\hat\rho(\infty, \Theta_c, \Theta_1, A) = \hat\rho(\infty, \Theta_c, \Theta_2, A)\,,$$

as admissible parameter variation with derivative in Θ_1 that remains in Θ_c has to satisfy $\langle d, \theta(t)\rangle \equiv 0$, so that $\langle d, \dot\theta(t)\rangle = 0$ almost everywhere, whence $\dot\theta(t) \in \Theta_2$, a.e. This completes the proof. □

Theorem 9. Consider a system Σ of the form (3) satisfying (A1)–(A5). Then

$$\bar\rho(\Sigma) = \hat\rho(\Sigma)\,. \tag{12}$$

PROOF. Again we may assume that $\hat\rho = 0$.
If $h = \infty$ and $(A6)$ does not hold, then we may first assume that $0 \in \mathrm{ri}\,\Theta_1$ using Proposition 8. Let $X = \mathrm{span}\,\Theta_1$. Then with the notation $\Theta_z := \Theta \cap (z + X)$ we may write

$$\Theta = \bigcup_{z\in X^\perp} \Theta_z\,,$$

and because each (nonempty) Θ_z is invariant under parameter variations with derivative in Θ_1 we see that

$$\hat\rho(\infty, \Theta, \Theta_1, A) = \sup_{z, \Theta_z \ne \emptyset} \hat\rho(\infty, \Theta_z, \Theta_1, A)\,.$$

Thus if we can show the assertion for each of the terms on the right, it follows also for $(\infty, \Theta, \Theta_1, A)$. Note that (A6) is satisfied for $(\infty, \Theta_z, \Theta_1, A)$, so that we may from now on assume that $h \in (0,\infty)$ or (A6) is satisfied.
Furthermore, if $A(\Theta)$ is not irreducible, then there exists a regular $T \in \mathbb{K}^{n\times n}$ such that all matrices $A_0 \in A(\Theta)$ can be transformed to upper block triangular form as in (10). Then is is easy to see that

$$\begin{aligned}\hat\rho(\Sigma) &= \max_{i=1,\dots,d} \hat\rho(\infty, \Theta, \Theta_1, A_i) \quad \text{and} \\ \bar\rho(\Sigma) &= \max_{i=1,\dots,d} \bar\rho(\infty, \Theta, \Theta_1, A_i)\,.\end{aligned} \tag{13}$$

Hence, we only need to show (12) for each of the blocks.
So assume now that $A(\Theta)$ is irreducible and that $h \in (0,\infty)$ or (A6) holds. By Lemma 6 there exist $\omega \in \Theta, x \in \mathbb{K}^n, v_\omega(x) = 1$ and a sequence $S_k \in \mathcal{R}(\omega,\omega)$ such that $S_k x - x \to 0$. Then we have by (Elsner, 1995, Lemma 2) for the eigenvalues $\lambda_i(k)$ of S_k that

$$0 \le \min_{1\le i\le n} 1 - |\lambda_i(k)| \le \min_{1\le i\le n} |1 - \lambda_i(k)| \\ \le C\|S_k x - x\|^{1/n}\,,$$

where C is a constant only depending on the upper bound of $\|S_k\|$. Denoting by $\tilde\lambda_k$ an eigenvalue of S_k for which the minimum on the left is attained we see that $|\tilde\lambda_k| \to 1$ as $k \to \infty$. As we have $|\tilde\lambda_k| \le 1$ and $t_k \ge 1$ we obtain $\bar\rho \ge 1/t_k \log|\tilde\lambda_k| \ge \log|\tilde\lambda_k|$, and it follows that $\bar\rho \ge 0$. □

5. CONCLUSIONS

In this paper we have studied certain classes of families of linear parameter varying systems that are basically described by constraints on the distance between discontinuities and on the derivative in the time between discontinuities. For these classes it has been shown that periodic systems exhibit the same growth behavior as the overall system.

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