ALGEBRAIC THEORY OF TIME-VARYING LINEAR SYSTEMS: A SURVEY

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Abstract: The development of the algebraic theory of time-varying linear systems is described. The class of systems considered consists of differential-algebraic equation in kernel presentation. This class encompasses time-varying state space, descriptor systems as well as Rosenbrock systems, and time-invariant systems in the behavioural approach.
One difference between time-varying and time-invariant systems is that, since the coefficients of the differential equations are time-varying function, the differential operator does not commute with the coefficients. However, the main difficulty is that solutions may exhibit a finite escape time. Hence there is a conflict between the class of time-varying coefficients and the class of admissible solution spaces. All contributions to time-varying systems have to cope with this.
As an efficient tool in linear, time-invariant system theory, Kalman introduced in the 1960s elementary module theory over principal ideal rings. This tool proved efficient also for time-varying systems. Although from then on, the field of time-varying linear systems has never been a “hot topic” in systems theory, there has been an ongoing evolution which led to a rather substantial theory. Not surprisingly, the theory is mainly restricted to linear systems and most results are on such properties as controllability, and not on stability. Recent results use successfully tools from module theory and homological algebra.

Keywords: Time-varying linear systems, skew polynomial rings, module theory

1. INTRODUCTION

1.1 An algebraic approach and solution spaces

Consider linear time-varying systems described by differential-algebraic equations of the form

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = \sum_{i=0}^{n} R_i(t)w^{(i)}(t) = 0, \qquad (1)$$

where

$$R(D) = \sum_{i=0}^{n} R_i D^i \in \mathcal{R}[D]^{g\times q} \cong \mathcal{R}^{g\times q}[D]$$

is a polynomial matrix in the indeterminate D with coefficient matrices R_i over a certain ring or field $\mathcal{R}$ of time-varying functions, defined on an interval $\mathbb{I} \subset \mathbb{R}$. The solution w belongs to a “suitable” solution space.

The polynomial ring $\mathcal{R}[D]$ is endowed with the multiplication rule

$$Df = fD + \dot{f}. \qquad (2)$$

This is a consequence of assuming the associative rule $(Df)g = D(fg)$ for all differentiable functions f, g which yields $(Df)(g) = \frac{\mathrm{d}}{\mathrm{d}t}f \cdot g + f \cdot \frac{\mathrm{d}}{\mathrm{d}t}g =$

$\left(\frac{\mathrm{d}}{\mathrm{d}t}f + fD\right)(g)$. The non-commutativity of the elements of $\mathcal{R}[D]$, in contrast to the commutative ring $\mathbb{R}[D]$ in the time-invariant case, is a considerable but not crucial difference. In the following we carefully distinguish between the algebraic indeterminate D and the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$.

For $R(D) \in \mathcal{R}[D]^{g\times q}$ and a solution space of time-varying functions $\mathcal{W}$ we study the behaviour given by the kernel representation

$$\ker R \;=\; \left\{ w \in \mathcal{W} \middle|\; R(\tfrac{\mathrm{d}}{\mathrm{d}\tau})w(\cdot) = 0 \right\}.$$

In analysing ker R, we have to cope with two basic difficulties: First, how can the system theoretic properties of the algebraic-differential system ker R, i.e. its behaviour, be described? Secondly, how is the algebraic object, i.e. the ring $\mathcal{R}[D]$, related to the analytic object, namely the solution space $\mathcal{W}$? For the answer of both questions the interplay between the coefficient ring $\mathcal{R}$ and the solution space $\mathcal{W}$ is fundamental. Loosely speaking, the more general the solution space is (e.g. distributions or even Sato's hyperfunctions), the more general the ring $\mathcal{R}$ is allowed. This is the essential difficulty for time-varying systems.

In Subsection 1.2 we present several subclasses of systems encompassing (1). In Subsection 1.3 we show that even if $\mathcal{R} = \mathbb{R}[D]$ the solution space exhibits some surprises.

The following sets will be used for the ring $\mathcal{R}$ or for candidates of solution spaces in the following.

$\mathcal{C}^N(M, \mathbb{R}^q)$	the set of N-times differentiable functions $f : M \to \mathbb{R}^q$, $M \subset \mathbb{R}$ an open set, $N \in \mathbb{N} \cup \{\infty\}$
$\mathcal{C}^\infty_{\mathrm{pw}}(\mathbb{R}^q)$	the set of piecewise $\mathcal{C}^\infty$-functions $f : \mathbb{R} \setminus \mathbb{T} \to \mathbb{R}^q$, $\mathbb{T} \subset \mathbb{R}$ discrete
$\mathcal{C}^\infty_t(\mathbb{R}^q)$	the set of locally $\mathcal{C}^\infty$-functions around $t \in \mathbb{R}$, i.e. functions $w \in \mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^q)$ for $\mathbb{I} \subset \mathbb{R}$ an open interval with $t \in \mathbb{I}$
$\mathcal{A}$	the ring of real analytic functions
$\mathcal{M}$	the quotient field of $\mathcal{A}$, i.e. the field of real meromorphic functions
$\mathcal{D}'(\mathbb{I}, \mathbb{R})$	the set of real valued distributions on $\mathbb{I} \subset \mathbb{R}$ an open interval

1.2 Examples of system classes

Consider the following subclasses of systems of (1).

(a) Time-varying *descriptor systems* of the form

$$\begin{aligned} E(t)\,\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) \;&=\; A(t)\,x(t) \;+\; B(t)\,u(t), \\ y(t) \;&=\; C(t)\,x(t) \;+\; F(t)\,u(t), \end{aligned} \tag{3}$$

with matrices E, A, B, C, F of appropriate dimension and defined over a ring of time-varying functions. If $E(\cdot) \equiv I_n$, then (3) describes a *state space system*; this is fairly standard, see for example the standard monograph (Rugh, 1996). However, if E is singular, then even for time-invariant matrices E, A, B, C, F the system (3) does not allow to speak of inputs, outputs, and states. To see this consider the variables $x_1, \ldots, x_4, u_1, u_2$ of the descriptor system (3) with

$$E = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix},\; A = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix},\; B = \begin{bmatrix} 0&0\\1&0\\0&1\\0&0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0&0&0&1 \end{bmatrix},\; F = 0_{1\times 2}\,.$$

Then an equivalent description is

$$u_2 = 0,\; \dot{x}_2 = x_1,\; y = x_4,\; \dot{x}_3 = x_2 + u_1\,,$$

and therefore u_2 is constrained to be 0 and cannot be freely chosen, as it could in the case of state space systems. The variables x_1 and x_4 can be viewed as input or state variables, the system description does not determine this. Note also that if we chose the input u_1 as a step function, then we would have to enlarge our solution space in order to allow that x_1 is a delta distribution. But even if we do so, then we have the problem that x_1 is not observable from the output y. This observation stresses to analyse (3) and in particular (1) from the behavioural viewpoint, where state-, output-, and input-variables are not distinguished.

(b) In (Ilchmann *et al.*, 1984) time-varying polynomial systems of the form

$$\begin{aligned} P(\tfrac{\mathrm{d}}{\mathrm{d}t})\,z(t) \;&=\; Q(\tfrac{\mathrm{d}}{\mathrm{d}t})\,u(t), \\ y(t) \;&=\; V(\tfrac{\mathrm{d}}{\mathrm{d}t})\,z(t) \;+\; W(\tfrac{\mathrm{d}}{\mathrm{d}t})\,u(t), \end{aligned} \tag{4}$$

where $P(D)$, $Q(D)$, $V(D)$ and $W(D)$ are matrices of size $r\times r$, $r\times m$, $p\times r$, $p\times m$, respectively, over $\mathcal{M}[D]$ are studied under the following assumptions:

(*i*) $P(D)$ represents a so called *full* operator, i.e. if z is a real analytic solution of $P(\frac{\mathrm{d}}{\mathrm{d}t})z = 0$ on some interval $\mathbb{I} \subset \mathbb{R}$, then this solution can be analytically extended to the whole of $\mathbb{R}$.

(*ii*) For every $u \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ with bounded support to the left, there exist some $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^r)$ and $y \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^p)$ so that (4) is satisfied.

Time-invariant polynomial systems, also called *Rosenbrock systems*, of the form (4), i.e. $P(D)$, $Q(D)$, $V(D)$ and $W(D)$ are matrices over $\mathbb{R}[D]$ and $\det P(\cdot) \neq 0$, were introduced in (Rosenbrock, 1970), and are well studied, see for example (Hinrichsen and Prätzel-Wolters, 1980; Wolovich, 1974).

(c) Time-invariant polynomial systems in the so called kernel representation $\ker R$ have been introduced by Willems in (Willems, 1981); see also (Willems, 1986*a*; Willems, 1986*b*; Willems, 1987) and the textbook (Polderman and Willems, 1998).

1.3 Examples of time-varying scalar differential equations

To understand a fundamental difference between time-varying and time-invariant linear differential equations consider the following examples for scalar $r(D) \in \mathcal{R}[D]$ and the ring of polynomials $\mathcal{R} = \mathbb{R}[t]$.

(i) Let $r(D) = tD + 1$. Then the function $t \mapsto w(t) = t^{-1}$ is a meromorphic solution of $r(\frac{\mathrm{d}}{\mathrm{d}t})w = t\frac{\mathrm{d}}{\mathrm{d}t}w + w = 0$. The point 0 is the only zero of the leading coefficient $t \mapsto t$ of $r(D)$, and 0 is also a pole of $t \mapsto w(t)$. Therefore, for every open interval $\mathbb{I} \subset \mathbb{R}$ with $0 \notin \mathbb{I}$,

$$\begin{aligned} \dim\ \ker_{\mathcal{M}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) &= \dim\ \ker_{\mathcal{A}_{|_{\mathbb{I}}}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) \\ &= \dim\ \ker_{\mathcal{D}'(\mathbb{I},\mathbb{R})} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) \\ &= 1 = \deg r(D). \end{aligned}$$

For the meromorphic solution space, its dimension equals the degree of $r(D)$. This is not true in general as illustrated by the following Example (ii). However, it can be shown that there exists a distribution $W \in \mathcal{D}'(\mathbb{R},\mathbb{R})$ such that W coincides with the regular distribution generated by w on $\mathbb{R} \setminus \{0\}$ for all test functions with support excluding $\{0\}$ or, more formally,

$$\begin{aligned} \ker_{\mathcal{A}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) &= \ker_{\mathcal{C}^\infty(\mathbb{R},\mathbb{R})} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) \\ &= \{0\} \subsetneq \ker_{\mathcal{D}'(\mathbb{R},\mathbb{R})} r(\tfrac{\mathrm{d}}{\mathrm{d}t}). \end{aligned}$$

(ii) Let $r(D) = t^2D + 1$. The function $t \mapsto w(t) = e^{1/t}$ solves $r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. The point 0 is again the only zero of the leading coefficient $t \mapsto t^2$ of $r(D)$, and 0 is also a pole of $t \mapsto w(t)$. But w is not meromorphic and the singularity at $t = 0$ differs from (i) as follows: no matter whether the solution w in (i) approaches 0 from the left or right, the limit at $t = 0$ does not exist; whereas, for the solution w in the present example, we have $\lim_{t\to 0-} w(t) = 0$ and $\lim_{t\to 0+} w(t) = \infty$. Hence,

$$\ker_{\mathcal{M}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \{0\}.$$

For every open interval $\mathbb{I} \subset \mathbb{R}$ with $0 \notin \mathbb{I}$ we have

$$\dim\ \ker_{\mathcal{M}_{|_{\mathbb{I}}}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = 1 = \deg r(D).$$

(iii) Let $r(D) = tD - 1$. The function $t \mapsto w(t) = t$ solves $r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and

$$\dim\ \ker_{\mathcal{A}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = 1 = \deg r(D).$$

Note that again the point $t = 0$ is the only zero of the leading coefficient $t \mapsto t$ of $r(D)$, but this time the zero does not produce a pole of the solution, the solution w is even a real analytic function on $\mathbb{R}$. However, the solution is not as arbitrary as for time-invariant systems, since $w(0) = 0$ is the only value at $t = 0$.

(iv) Let $r(D) = 2tD - 1$. The functions $t \mapsto w_+(t) = \sqrt{t}$ and $t \mapsto w_-(t) = \sqrt{-t}$ solve $r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ on $(0,\infty)$, $(-\infty,0)$, respectively. For every open interval $\mathbb{I} \subset \mathbb{R}$ with $0 \notin \mathbb{I}$, we have

$$\dim\ \ker_{\mathcal{A}_{|_{\mathbb{I}}}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = 1 = \deg r(D).$$

However,

$$\ker_{\mathcal{M}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \{0\}.$$

The real analytic solution w_+ on $(0,\infty)$ cannot be continued to $(-\varepsilon,\infty)$ for any $\varepsilon > 0$.

(v) Consider $r(D) = (1-t^2)^2\,D + 2t$. The function

$$t \mapsto w(t) = \begin{cases} e^{-(1-t^2)^{-1}}, & t \in (-1,1) \\ 0, & t \in \mathbb{R} \setminus (-1,1) \end{cases}$$

satisfies $w \in \ker_{\mathcal{C}^\infty} r(\frac{\mathrm{d}}{\mathrm{d}t})$, is not real analytic and has compact support. This is impossible for time-invariant, scalar, inhomogeneous differential equations.

(vi) Let $r(D) = t^3D + 1$. Then the function $t \mapsto w(t) = \exp\{1/2t^2\}$ solves $r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ on every open interval $\mathbb{I} \subset \mathbb{R}$ with $0 \notin \mathbb{I}$. However, in contrast to Example (i), it may be shown that

$$\ker_{\mathcal{A}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \{0\} = \ker_{\mathcal{D}'(\mathbb{R},\mathbb{R})} r(\tfrac{\mathrm{d}}{\mathrm{d}t}).$$

In other words, there does not exist any distribution in $\mathcal{D}'(\mathbb{R},\mathbb{R})$ which coincides with the regular distribution generated by w on $\mathbb{R} \setminus \{0\}$ for all test functions with support excluding $\{0\}$. □

The above examples may give an impression of the different kind of problems already introduced by scalar differential equations with real polynomials as coefficients.

2. SYSTEM THEORETIC CONCEPTS

Since solutions of $R(\frac{\mathrm{d}}{\mathrm{d}t})w(\cdot) = 0$ may even in the scalar case exhibit a finite escape time, see the examples in Sub-section 1.3, system theoretic concepts are defined locally.

Let $R(D) \in \mathcal{R}[D]^{g \times q}$ and $\mathcal{W}_t$ be a set of time-varying functions defined in an open neighbourhood around $t \in \mathbb{R}$, of sufficient smoothness, and of appropriate dimension. Then the local behaviour at $t \in \mathbb{R}$ is

$$\ker_t R = \left\{ w \in \mathcal{W}_t \middle| R(\tfrac{\mathrm{d}}{\mathrm{d}\tau}) w(\cdot) = 0 \right\} .$$

Local controllability is now defined as a property of local solutions respectively trajectories.

Definition 1. For $R(D) \in \mathcal{R}[D]^{g \times q}$, the local behaviour $\ker_t R$ is called *locally controllable at* $t \in \mathbb{R}$ if, and only if, for every $w^1, w^2 \in \ker_t R$ and every $t_0 \in (-\infty, t) \cap \mathrm{dom}\ w^1$ there exist $t_1 \in \mathrm{dom}\ w^2 \cap (t, \infty)$ and $w \in \ker_t R$ such that

$$w(t) = \begin{cases} w^1(t), & t \in (-\infty, t_0] \cap \mathrm{dom}\ w^1 \\ w^2(t), & t \in [t_1, \infty) \cap \mathrm{dom}\ w^2. \end{cases}$$

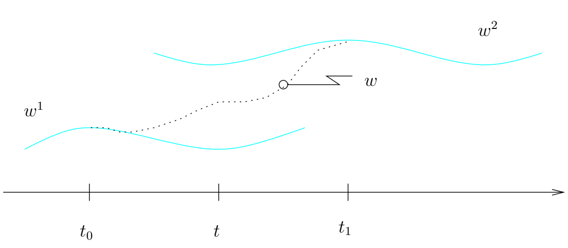

Fig. 1. Local controllability at t

□

Loosely speaking, controllability means that any two trajectories $w^1, w^2 \in \ker_t R$ can be connected by another trajectory $w \in \ker_t R$ so that in finite time w^1 moves via w into w^2. A similar notion of controllability via trajectories was introduced in (Hinrichsen and Prätzel-Wolters, 1980) for time-invariant Rosenbrock systems with of the form (4). For time-invariant systems of the form (1), the concept of controllability coincides with the one introduced by Willems (Willems, 1981), see also (Polderman and Willems, 1998, Sect. 5.2).

Definition 2. Let $[R_1(D), R_2(D)] \in \mathcal{R}[D]^{g \times (q_1 + q_2)}$ and $t \in \mathbb{R}$. Then $w_2 \in \mathcal{C}_t^\infty(\mathbb{R}^{q_2})$ is called *locally observable at* $t \in \mathbb{R}$ *from* $w_1 \in \mathcal{C}_t^\infty(\mathbb{R}^{q_1})$ for $t \in \mathbb{R}$ if, and only if,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ \tilde{w}_2 \end{bmatrix} \in \ker_t [R_1, R_2]$$

implies that

$$\forall \tau \in \mathrm{dom}\ w_2 \cap \mathrm{dom}\ \tilde{w}_2 : \ w_2(\tau) = \tilde{w}_2(\tau) .$$

□

It can be shown that, under suitable assumptions, the concepts of local controllability and observability are adjoint as for time-invariant systems.

The generalization of autonomous sub-behaviour, see for example (Polderman and Willems, 1998, p. 67) for time-invariant systems, is given as follows.

Definition 3. Let $R(D) \in \mathcal{R}[D]^{g \times q}$ and $t \in \mathbb{R}$. A local sub-behaviour $\mathfrak{B}_t \subset \ker_t R$ is called *autonomous* if, and only if, for any $w^1, w^2 \in \mathfrak{B}_t$ with $w_1 \equiv w_2$ on some open interval $\mathbb{I} \subset \mathrm{dom}\ w^1 \cap \mathrm{dom}\ w^2$ with $t \in \mathbb{I}$ it follows that $w_1 \equiv w_2$ on $\mathrm{dom}\ w^1 \cap \mathrm{dom}\ w^2$. □

3. EARLY ALGEBRAIC CONTRIBUTIONS

As an efficient tool in linear, time-invariant system theory, (Kalman *et al.*, 1969) used elementary module theory over principal ideal rings. These tools have also been applied to time-varying systems. An early algebraic contribution on time-varying systems of the form (4) with $V \equiv 0$ and $W \equiv 0$ is given by (Ylinen, 1975). The ring $\mathcal{R}$ is a certain ring of endomorphisms. Results on minimal transfer matrices, minimal realization, interconnection and observability are achieved. However, the system class is rather restrictive. In later contribution, (Ylinen, 1980) assumes that the ring $\mathcal{R}$ is a subring of $\mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^q)$, it must not contain zero divisors of $\mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^q)$, and $[P, Q]$ must be row equivalent to a matrix in upper triangular form with coefficients in $\mathcal{R}$ and monic diagonal elements. In this set-up, it can be shown that a polynomial matrix over the ring $\mathcal{R}$ can only be transformed in this normal form if any local behaviour is a global behaviour. Controllability is treated and characterized in terms of coprimeness of P and Q in (4).

In (Kamen, 1976) the ring $\mathcal{R}$ is assumed to be Noetherian. Under this hypothesis, a state space realization of (4) with monic P can be constructed. The Noether conditiion seems to be rather restrictive , see examples given in (Kamen, 1976). The ring of real analytic function is not Noetherian.

4. AN ALGEBRAIC APPROACH

In (Fliess, 1990) matrices over the ring of linear differential operators $\mathcal{R}[D]$ is considered, where $\mathcal{R}$ denotes a differential field. Linear dynamics are finitely generated left $\mathcal{R}[D]$-modules. The dynamics are proved to be controllable if, and only if, they are a free left $\mathcal{R}[D]$-module. Observability and its duality to controllability is also shown.

This contribution is merely on the algebraic side, the solution space is not specified.
In the same set-up with $\mathcal{R}$ specified to be the quotient field $\mathcal{M}$ of real meromorphic functions, (Fliess *et al.*, 1993) investigate descriptor systems of the form (3). Under a similar assumption as in Sub-section 1.2 (b)(ii), the index of a transfer function is investigated.
In (Rudolph, 1996) contributions to duality of systems in the set-up of (Fliess, 1990) for systems in generalized state space representation are given, however the solution space is not specified either.

An important contribution by (Fröhler and Oberst, 1998) has the following background:
In Example (i) and (vi) in Sub-section 1.3 we have seen that even if the coefficients of $\mathcal{R}[D]$ are simple polynomials in t, not every solution exists on the whole of $\mathbb{R}$ and, more importantly, even if distributions on $\mathbb{R}$ are allowed as solutions, then not every local solution can be extended to such a distribution. Hence enlarging the solution space to allow for distributions on $\mathbb{R}$ does not necessarily resolve the problem, even in the simple case when the coefficients of the time-varying systems are polynomials. However, if the solution space is enlarged even further to allow for Sato's hyperfunctions, i.e. generalized distributions introduced in (Sato, 1960), then (Fröhler and Oberst, 1998) do present a nice theory. They consider systems of the form (1) respectively behaviour in the kernel representation $\ker R$, where the coefficient matrices of the polynomial $R(D)$ are defined over rational analytic functions

$$\frac{f(\cdot)}{g(\cdot)} \quad \text{for } f, g \in \mathbb{C}[t] \text{ with } g(t) \neq 0 \text{ for all } t \in \mathbb{I}.$$

Note that by multiplication with a least common multiple of all denominators of the coefficients, the coefficients of $R(D)$ are polynomials. Based on the seminal paper of extensive length by (Oberst, 1990), where an algebraic analytic approach is developed to show a categorical duality between the solution spaces of linear partial differential equations with constant coefficients and certain polynomial modules associated to them, a generalization to time-varying but ordinary differential equations is achieved by (Fröhler and Oberst, 1998). However, if the set of coefficients of $\mathcal{R}[D]$ is enlarged to real analytic coefficients and not only polynomials in t, then their result does not hold true in general.

5. THE RING $\mathcal{M}[D]$

The skew polynomial ring $\mathcal{M}[D]$ has been introduced by (Ilchmann *et al.*, 1984) to describe time-varying linear systems of the form (4). This ring does not contain any zero divisors, is simple (in the sense that the only two sided ideals are the trivial once), and it admits right- and left-Euclidian division. Therefore the following Teichmüller-Nakayama normal form can be achieved for matrices over $\mathcal{M}[D]$.

Theorem 4. (*Teichmüller-Nakayama normal form*) Any $R(D) \in \mathcal{M}[D]^{g\times q}$ with $\mathrm{rk}_{\mathcal{M}[D]} R(D) = l$ can be factorized into

$$R(D) = U(D)^{-1} \begin{bmatrix} I_{l-1} & 0 & 0 \\ 0 & r(D) & 0 \\ 0 & 0 & 0_{(g-l)\times(q-l)} \end{bmatrix} V(D)^{-1}$$

where $U(D)$ and $V(D)$ are $\mathcal{M}[D]$-unimodular matrices of sizes g and q, respectively, and $r(D) \in \mathcal{M}[D]$ is non-zero, unique up to similarity, and of unique degree.

A proof and an interesting historical description of the development of the above normal form can be found in (Cohn, 1971, Ch. 8). Two elements $q_1, q_2 \in \mathcal{M}[D]$ are *similar* if, and only if, $q_1 a = b q_2$ for some $a, b \in \mathcal{M}[D]$ for which q_1 and b (q_2 and a) are left (right) coprime. For example, $a(D) = D$ and $b(D) = D - 1/t$ are similar: $[D + (t^2-1)/t]a(D) = b(D)\,[D+t]$ and $D + (t^2-1)/t$, $b(D)$ are right coprime, $a(D)$, $D+t$ are left coprime. Moreover, this example shows that a unique factorisation of the ring elements cannot be expected. However, (Ore, 1933) shows that the degree of similar polynomials coincide. The latter property is crucial for determining dimensions of solution spaces.

The Teichmüller-Nakayama normal form is the essential tool in (Ilchmann *et al.*, 1984) to study time-varying Rosenbrock systems of the form (4). The solution space is the set of $\mathcal{C}^\infty$-functions on the whole time axis, but this is ensured by the assumption that $\mathrm{im}\, Q(\frac{\mathrm{d}}{\mathrm{d}t}) \subset \mathrm{im}\, P(\frac{\mathrm{d}}{\mathrm{d}t})$ and, most importantly, that $P(D)$ is a "full" operator, i.e. every local analytic solution of $P(\frac{\mathrm{d}}{\mathrm{d}t})z = 0$ is extendable to a global analytic solution on the whole of $\mathbb{R}$. Controllability and observability are characterized in terms of coprimeness of matrices. In the same set-up, (Ilchmann, 1985) and (Ilchmann, 1989) derive results on indices (controllability, minimal, geometric, dynamical) and give a complete set of invariants to characterize system equivalence. The system class do encompass state space systems, however the hypothesis of full generators is a rather restrictive assumption.

To overcome this assumption, in (Ilchmann *et al.*, 2000) a first approach in the spirit of the present paper is presented for scalar systems. This approach is developed in detail in (**?**). Since the zeros and poles of real meromorphic function is a discrete subset of $\mathbb{R}$, this carries over the set of points in $\mathbb{R}$ where the elements of $\ker R$ may have a finite escape time. Therefore, an almost global theory is developed. Again, the main tool is the

Teichmüller-Nakayama normal form. It is shown that $\ker R$ is *controllable almost everywhere*, i.e. $\ker_t R$ is locally controllable for almost all $t \in \mathbb{R}$, if, and only if, $R(D)$ is right invertible; which is also equivalent to having an image representation, i.e. there exists $M(D) \in \mathcal{M}[D]^{q\times m}$ such that, for almost all $t \in \mathbb{R}$,

$$\operatorname{im}_t M := \{w \in \mathcal{C}_t^\infty(\mathbb{R}^q) \| \, \exists\, l \in \mathcal{C}_t^\infty(\mathbb{R}^m) \;:\; \forall\, \tau \in \text{dom } w \cap \text{ dom } l \;:\; w(\tau) = M(\tfrac{\mathrm{d}}{\mathrm{d}t})\, l(\tau)\}.$$

For $[R_1(D), R_2(D)] \in \mathcal{R}[D]^{g\times(q_1+q_2)}$ associated to $\begin{bmatrix} w_1 \\ \tilde{w}_2 \end{bmatrix} \in \ker[R_1, R_2]$, it is shown that w_2 is observable from w_1, i.e. $w_2 \in \mathcal{C}_t^\infty(\mathbb{R}^{q_2})$ is locally observable from $w_1 \in \mathcal{C}_t^\infty(\mathbb{R}^{q_1})$ for almost all $t \in \mathbb{R}$, if, and only if, R_2 is left invertible.
Furthermore, it is shown that the behaviour can be written as the direct sum of the controllable behaviour an an arbitrary maximal autonomous behaviour.

6. THE ALGEBRAIC APPROACH REVISITED

Based on the findings in (**?**), a much more elegant algebraic approach in the spirit of (Fröhler and Oberst, 1998) has been developed by (Zerz, 2005). The main tool is again the Teichmüller-Nakayama normal form and $\ker R$ is considered as a subset of $\mathcal{C}_{\text{pw}}^\infty(\mathbb{R}^q)$ for $R(D) \in \mathcal{M}[D]^{g\times q}$. The main result is that the left $\mathcal{M}[D]$–module $\mathcal{C}_{\text{pw}}^\infty(\mathbb{R}^q)$ is an injective cogenerator. Once this result has been established, system theoretic consequences follow: the characterization of equivalence of behaviours; a relationship between kernel and image representation; a characterization of autonomy of $\ker R$ in terms of the rank of $R(D)$ and in terms of a module to be torsion; the characterization of the possibility of an image representation in terms of freeness of a module, and more.

7. DESCRIPTOR SYSTEMS

A completely different approach results from the study of differential-algebraic equations introduced in (Brenan *et al.*, 1996; Griepentrog and März, 1986). A general solvability theory for non-square linear time-varying systems was first given in (Kunkel and Mehrmann, 1993) and analysed for control problems in a behavioural context in (Byers *et al.*, 1997; Kunkel *et al.*, 2001; Rath, 1997), see also (Kunkel and Mehrmann, 2001) for the general nonlinear case.

In (Campbell *et al.*, 1991) controllability and observability have been studied in terms of derivative arrays, see also (Dai, 1989). In (Byers *et al.*, 1997) a first behaviour like approach to systems (3) with analytic coefficients has been discussed. A more general approach that allows for larger classes of coefficients and that can be implemented also numerically has been introduced in (Kunkel *et al.*, 2001) and generalized partially to the nonlinear case in (Kunkel and Mehrmann, 2001).

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