INFORMATIONAL SETS IN MODEL PROBLEMS OF AIRCRAFT TRACKING [1]

V.S.Patsko * A.A.Fedotov * S.I.Kumkov * S.G.Pyatko **

* *Institute of Mathematics and Mechanics, S.Kovalevskaya str., 16, Ekaterinburg, 620219, Russia, e-mail: patsko@imm.uran.ru*
** *Academy of Civil Aviation, Pilotov str., 38, St. Petersburg, 196210, Russia, e-mail: boss@nita.ru*

Abstract: Model problems of aircraft tracking are considered for the case when an aircraft moves in the horizontal plane under conditions of uncertainty in measurements of its geometric coordinates. Estimation of the phase state is implemented by means of the informational sets. The results of numerical constructing the informational sets in the three- and four-dimensional phase spaces are discussed.

Keywords: Guaranteed estimation, Informational sets, Aircraft tracking, Numerical algorithms

1. INTRODUCTION

The paper deals with formulations of model problems of aircraft tracking when an aircraft moves in the horizontal plane x, y.

It is assumed that the current information about the aircraft motion comes in the form of measurements of its position in the plane x, y, and some geometric constraints on the error of measurements are known. The set $H(t_*)$ of uncertainty is put in correspondence to each measurement coming at the instant t_*. This set is the totality of all phase states consistent with the obtained measurement and the given constraint on its error. We assume also that the uncertainty set is cylindrical on the phase variables that differ from the coordinates x, y and has a convex projection in the plane x, y. It means that coordinates x, y are only measurable. The totality of all phase states consistent with all uncertainty sets constructed up to the instant t inclusively constitutes the informational set $I(t)$ at the instant t.

In a discrete scheme of observation with a grid of instants t_j, each informational set $I(t_j)$ is constructed by intersection of the forecast set $G(t_j)$ with the uncertainty set $H(t_j)$. The set $G(t_j)$ is the attainability set of the system under consideration at the instant t_j constructed on the basis of the informational set $I(t_{j-1})$ calculated for the instant t_{j-1} of the previous observation. If no measurements come at the instant t_j, then $I(t_j) = G(t_j)$.

Three model problems on constructing the informational sets are considered in the work. All these problems concern the theory of determinate guaranteed estimation (Krasovskii and Subbotin, 1974; Kurzhanskii, 1977; Chernousko, 1994; Milanese *et al.*, 1996; Kurzhanski and Valyi, 1997).

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Note that, as a rule, solution of particular applied problems causes essential difficulties and demands some special treatment. In problems under consideration, the forecast sets are not convex. As a consequence, the informational sets are not convex, too. The absence of convexity leads to a serious difficulties for elaboration of numerical algorithms. In the work, a variant of approximation of the forecast sets from above is suggested. Namely, instead of the sets $G(t_j)$ some sets $\mathbf{G}(t_j)$ are constructed such that $\mathbf{G}(t_j) \supset G(t_j)$. The sets $\mathbf{G}(t_j)$ in the whole are not convex, but in the space of the phase variables they have convex sections by two-dimensional planes parallel to the plane of the coordinates x, y. Such a feature allows to realize easily the intersection $\mathbf{G}(t_j) \cap H(t_j) =: \mathbf{I}(t_j)$. The set $\mathbf{I}(t_j)$ obtained approximates the true informational set $I(t_j)$ from above. In the sequel, speaking about a forecast set and an informational set, we shall mean the sets $\mathbf{G}(t)$ and $\mathbf{I}(t)$.

Some parts of this work were published in (Patsko *et al.*, 2003; Kumkov *et al.*, 2003; Ganebny and Fedotov, 2004).

2. THREE-DIMENSIONAL INFORMATIONAL SETS IN A PROBLEM OF AIRCRAFT TRACKING WITH UNKNOWN CONTROL LAW

2.1 Dynamics description

Let the dynamics of aircraft motion be described by the following system of ordinary differential equations:

$$\begin{aligned} \dot{x} &= V\cos\varphi, \\ \dot{y} &= V\sin\varphi, \\ \dot{\varphi} &= ku/V; \\ V &= \text{const} > 0, \quad k = \text{const} > 0, \quad |u| \le 1. \end{aligned} \tag{1}$$

Here, x and y are the geometric coordinates, φ is the heading of the velocity vector, u is an unknown aircraft control. The angle φ is counted counterclockwise from the horizontal axis (Fig. 1). System (1) is the simplest for description of an aircraft motion in the horizontal plane (Isaacs, 1965; Hamza *et al.*, 1971; Pecsvaradi, 1972).

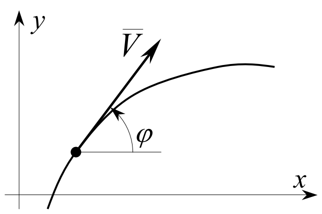

Fig. 1. Coordinate system

2.2 The main ideas for constructing the three-dimensional informational sets

Let us describe now the basic elements of the numerical construction of the informational sets $\mathbf{I}(t)$ for system (1). The set $\mathbf{I}(t)$ is represented by means of some fixed grid of nodes on the coordinate φ in the range $[0, 2\pi)$ and a collection of convex polygons $\mathbf{I}_\varphi(t)$ where each polygon corresponds to its node. Each forecast set $\mathbf{G}(t)$ is represented in the similar way.

Let t_*, t^* be two neighbor instants of measuring, $t_* < t^*$, and the set $\mathbf{I}(t_*)$ is already built. To find the set $\mathbf{I}(t^*)$, let us first construct the forecast set $\mathbf{G}(t^*)$. To make it, the interval $[t_*, t^*]$ is divided by an equidistant grid with a step Δ. A control $u(\cdot)$ is assumed to be a piece-wise on this partition. Under this, in each interval $[t_i, t_{i+1})$ of the partition the control u can take values $u = -1,\ 0,\ 1$.

The choice of these values is stipulated by the fact that for construction of the extremal motions giving the frontier of the attainability set (i.e., the forecast set) in correspondence with the Pontryagin maximum principle, the marginal control actions $u = \pm 1$ are used. The value $u = 0$ is realized on degenerated extremal motions.

Put $t_1 := t_*$ and $\mathbf{G}(t_1) := \mathbf{I}(t_*)$. The forecast set $\mathbf{G}(t_{i+1})$ is built on the basis of the set $\mathbf{G}(t_i)$ in the following way. At the instant t_i corresponding to the set $\mathbf{G}(t_i)$, we have a collection of non-empty nodes on the coordinate φ. A convex set $\mathbf{G}_\varphi(t_i)$ corresponds to each node. Applying the controls $u = -1,\ 0,\ 1$, we obtain three nodes $\bar{\varphi} = \varphi + \Delta k u/V$ of the intermediate grid at the instant t_{i+1}. The sets transferred to these nodes have the form

$$\mathbf{G}_{\bar{\varphi}}(t_{i+1}, u) = \begin{cases} \mathbf{G}_\varphi(t_i) + \dfrac{V^2}{ku}\begin{pmatrix} \sin\bar{\varphi} - \sin\varphi \\ \cos\varphi - \cos\bar{\varphi} \end{pmatrix}, \\ \qquad\qquad \text{if} \quad u = \pm 1, \\ \mathbf{G}_\varphi(t_i) + \Delta \cdot V \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \\ \qquad\qquad \text{if} \quad u = 0. \end{cases}$$

It is seen that a number of nodes of the intermediate grid can increase thrice on the step Δ. These intermediate nodes are grouped with respect to the nodes of the standard grid. Instead of each group of transferred sets with close values $\bar{\varphi}$, one set (section) is introduced that corresponds to a node of the standard grid. This section is constructed as a convex hull of the union of sets from the group under consideration. The collection of the resultant sets constitutes the set $\mathbf{G}(t_{i+1})$. Acting in the way described, we come to the instant t^* and obtain the set $\mathbf{G}(t^*)$.

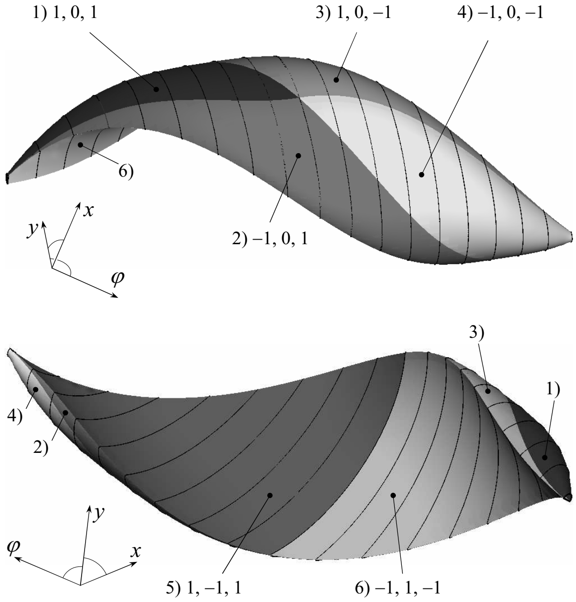

Fig. 2. Structure of the frontier of the attainability set at $t = \pi \cdot (V/k)$

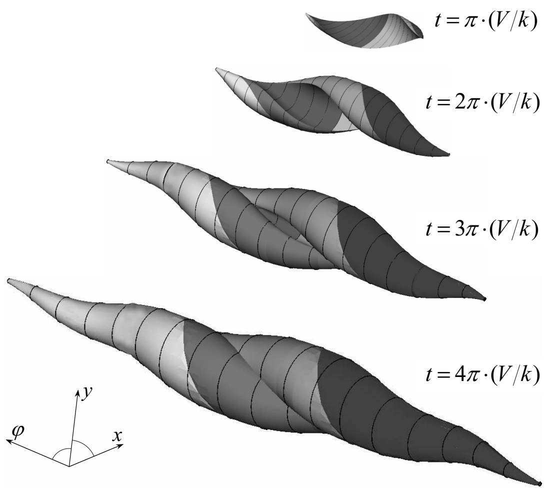

Fig. 3. Evolution of the attainability set

If a measurement comes at the instant t^*, the set $H(t^*)$ of its uncertainty is built. The set is cylindrical on φ and in the whole is determined by its projection $H^{\#}(t^*)$ in the plane x, y.

The informational set $\mathbf{I}(t^*)$ is constructed by means of intersection of each section $\mathbf{G}_\varphi(t^*)$ of the forecast set $\mathbf{G}(t^*)$ with the set $H^{\#}(t^*)$. The result of non-empty intersections constitutes the informational set $\mathbf{I}(t^*)$. Thus, in the plane x, y, we work with convex sets $\mathbf{G}_\varphi(t)$, $\mathbf{I}_\varphi(t)$, and $H^{\#}(t)$. To represent them, convex polygons approximating the sets from above are used.

2.3 Comparison with the exact constructions

To find out the character of errors stipulated by the operation of convexification, let us compare the attainability set $G(t)$ of system (1) at a fixed instant t with the set $\mathbf{G}(t)$ calculated by means of

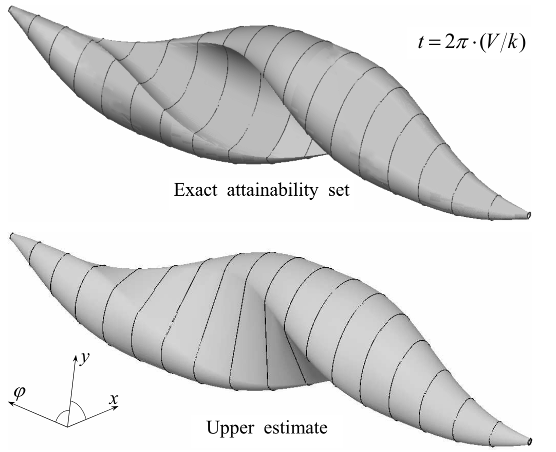

Fig. 4. Comparison with the exact attainability set at $t = 2\pi \cdot (V/k)$

the algorithm suggested for building the forecast set, in which the mentioned operation of convexification is used on each time step Δ. Assume that at the instant $t_0 = 0$ the initial set of the aircraft positions is a point in the three-dimensional phase space x, y, φ, i.e., at the initial instant, the geometric position and the heading of the velocity vector are given. Assume also that the value of the coordinate φ is considered on the infinite axis $(-\infty, \infty)$ (i.e., φ is not transformed by the modulo 2π). In this connection, for constructing the sets $\mathbf{G}(t)$ we use a grid on φ, which expands in time.

In paper (Patsko *et al.*, 2003), the following statement was proved.

Theorem. *Each point of the frontier of the attainability set $G(t)$ of system* (1) *can be achieved by means of a piece-wise control $u(\cdot)$ with not more than two switches. Under this, in the case of two switches, it is sufficient to deal only with the following six variants of sequential controls*:

1) 1, 0, 1; 2) −1, 0, 1; 3) 1, 0, −1;

4) −1, 0, −1; 5) 1, −1, 1; 6) −1, 1, −1.

In paper (Dubins, 1957, pp. 515), necessary conditions of optimality were obtained in the problem of time minimization for system (1). These conditions have the same form as ones formulated in the theorem. But generally speaking, not all frontier points of the attainability set $G(t)$ correspond to the solution of the time minimization problem. Thus, the theorem formulated above does not follow from the results of paper (Dubins, 1957).

This theorem allows to construct numerically the frontier of the attainability set $G(t)$ in the three-dimensional phase space with sufficient accuracy.

Figure 2 shows (from two view-points) the frontier of the attainability set $G(t)$ at the instant $t = \pi \cdot (V/k)$. Parts of the frontier realized on controls of different character are shown in gray of various depth.

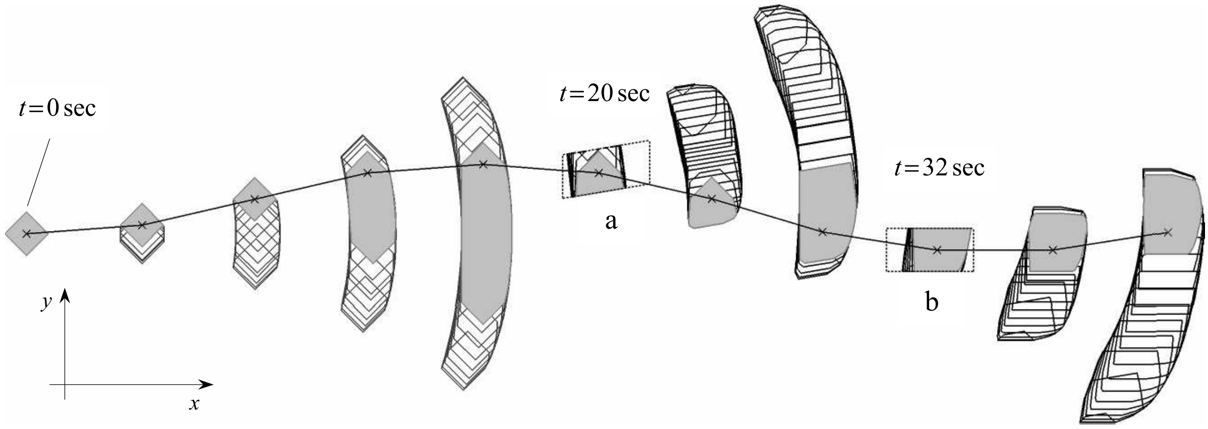

Fig. 5. Motion of the informational set, projection in the plane x, y

In Fig. 3, the sets $G(t)$ are given (from a single view-point) for four instants $t = h\pi \cdot (V/k)$, $h = 1, 2, 3, 4$. Evolution of the structure of the attainability set frontier can be seen exactly. When the time grows, the external frontier "enwraps" the internal one, and a cameo-shell-wise body is created.

Figure 4 illustrates (at the instant $t = 2\pi \cdot (V/k)$) the attainability set $G(t)$ and its approximation $\mathbf{G}(t)$ from above calculated by the suggested algorithm for constructing the forecast set.

Underline that the true attainability set $G(t)$ is regarded to be unknown when the set $\mathbf{G}(t)$ is calculated. It is seen that the error stipulated by the operation of convexification appears only from one side of the surface constraining this set. In the matter, each φ-section of the approximating set coincides with the convex hull of the corresponding φ-section of the exact attainability set.

In paper (Patsko *et al.*, 2003), the pair illustrations of the sets $G(t)$ and $\mathbf{G}(t)$ are given for the instants $t = 3\pi \cdot (V/k)$ and $t = 4\pi \cdot (V/k)$. These pictures show that the approximation from above for the set $G(t)$ by means of the set $\mathbf{G}(t)$ is sufficiently good.

2.4 Simulation of the informational set motion

Consider the results that illustrate dynamics of the informational sets $\mathbf{I}(t)$ on time. The following parameters were taken for this simulation: $V = 400$ m/sec, $k = 15$ m/sec^2, $\Delta = 1$ sec. The constraint on the number of sides of the convex polygons used was taken as 60.

The initial informational set $\mathbf{I}(0)$ has only one φ-section of the square-wise form. Thus, at the initial instant $t_0 = 0$, the heading of the velocity vector is regarded to be known.

For instants $t \geq 0$, the true motion of the phase point in the three-dimensional space x, y, φ is simulated. The measurements are simulated around this motion. For each measurement, the corresponding uncertainty set with the prescribed form of its projection $H^{\#}(t)$ in the plane x, y is built.

The general view of evolution of the informational sets is shown in Fig. 5 for the time interval $[0, 40]$ sec in projection in the plane x, y. The true trajectory is marked in the solid line. The measurements come at the instants 20 and 32 sec. We assume that the projections $H^{\#}(20)$ and $H^{\#}(32)$ of the uncertainty sets are a parallelogram (a) and a rectangle (b). For simplicity, not all but only every second section is drawn in the informational sets. For the same reason, the informational sets are given for each fourth time step only: $\mathbf{I}(0)$, $\mathbf{I}(4)$, $\mathbf{I}(8)$, …, $\mathbf{I}(40)$. The crosses mark the positions of the true point at these instants. In the informational sets, the sections that are most close (on φ) to the corresponding true values are marked in gray.

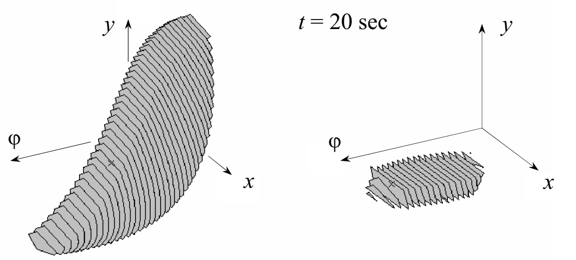

Fig. 6. Informational set before and after a measurement

Figure 6 shows obtaining the informational set at the instant 20 sec in the three-dimensional space. Two sets are drawn: the forecast set before taking into account the measurement and the informational set after this operation.

3. FOUR-DIMENSIONAL INFORMATIONAL SETS

In the second formulation of the problem, the velocity V can vary. The dynamics of motion is described as follows:

$$\begin{aligned} &\dot{x} = V\cos\varphi, \\ &\dot{y} = V\sin\varphi, \\ &\dot{\varphi} = ku/V, \\ &\dot{V} = w; \\ &V \geq \text{const} > 0, \quad k = \text{const} > 0, \\ &|u| \leq 1, \quad \mu_1 \leq w \leq \mu_2, \end{aligned} \tag{2}$$

where u and w are unknown controls restricted by geometrical constraints. The control w, in particular, can be interpreted as a variable that takes into account the uncertainty of relations describing variation of the velocity V. Assume that $\mu_1 < 0$, $\mu_2 > 0$.

When the dynamics is described by (2), the main difficulty is in the fact that the informational sets have to be built in the four-dimensional space but not in the three-dimensional one as for system (1). In this case, the grid on the variable V is introduced in addition to the grid in the coordinate φ. When the forecast sets $\mathbf{G}(t)$ and the informational sets $\mathbf{I}(t)$ are constructed, some convex polygon in the plane x, y is put in correspondence to each node of the two-dimensional grid. For constructing the set $\mathbf{G}(t)$, piece-wise controls with values $u = -1, 0, 1$ and $w = \mu_1, \mu_2$ are used.

In Fig. 7, a typical four-dimensional forecast set $\mathbf{G}(t)$ is shown. It is represented in the form of a totality of three-dimensional sets, which are sections (layers) on the coordinate V. The sections are drawn for three values of V in gray of various depth. Their "banana-wise" form is typical for the dynamics under investigation.

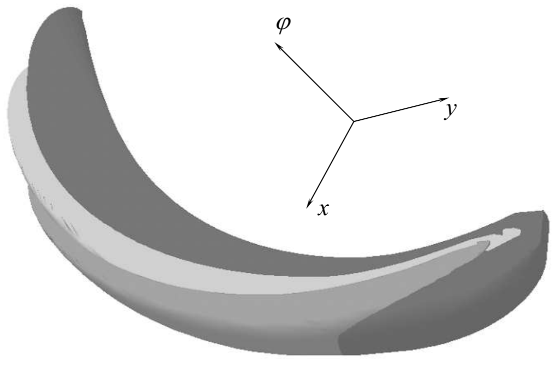

Fig. 7. Representation of the four-dimensional forecast set by means of the three-dimensional sections

4. THE CASE OF AIRCRAFT MOTION UNDER AUTOPILOT CONTROL

In the framework of the third formulation of the problem, the phase vector is three-dimensional just as in system (1). We assume that the aircraft moves in the horizontal plane with a constant value of velocity under autopilot control, whose law is known with some uncertainty. The dynamics equations have the following form:

$$\begin{aligned} &\dot{x} = V\cos\varphi, \\ &\dot{y} = V\sin\varphi, \\ &\dot{\varphi} = R(k_1 y + k_2\varphi) + v; \\ &k_1 = \text{const} < 0, \quad k_2 = \text{const} < 0, \quad |v| \leq \nu. \end{aligned} \tag{3}$$

Here, k_1 and k_2 are the autopilot coefficients, R is the law of the autopilot action, v is a disturbance restricted by a geometric constraint. Introduction of this disturbance takes into account the uncertainty of data about the autopilot. The law R is described by the following formula:

$$R(z) = \begin{cases} z, & |z| \leq M, \\ M, & z > M, \\ -M, & z < -M \end{cases} \tag{4}$$

where $z = k_1 y + k_2\varphi$. For an appropriate choice of the coefficients, the law stabilizes the aircraft motion with respect to the direct trace oriented along the axis x.

In paper (Ganebny and Fedotov, 2004), for typical collection of parameters $V = 250$ m/sec, $k_1 = -0.0000112$ 1/(m sec), $k_2 = -0.0333333$ 1/sec, and $M = 0.008$ 1/sec, the dependence of size of the forecast set $\mathbf{G}(t)$ on the value ν restricting the disturbance was investigated. It was found that in the range $0 \leq \nu \leq 0.45M$ the size of the forecast set stabilizes on the coordinates y, φ when the time t increases, and the size grows linearly on ν. For $\nu > 0.45M$, the stabilization is broken, and the growth becomes non-linear. Figure 8 illustrates the curve of size of the set $\mathbf{G}(t)$ on the coordinate y at the instants $t = 300$ sec and $t = 600$ sec.

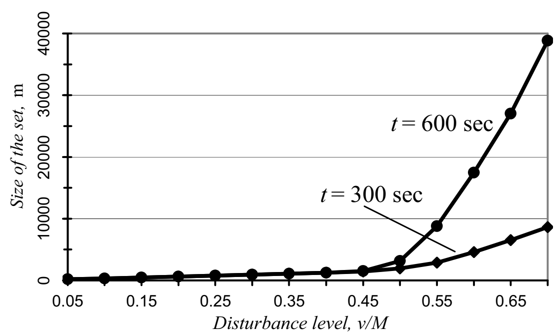

Fig. 8. Size of the set $\mathbf{G}(t)$ on the coordinate y

Evolution of the informational set is shown in Fig. 9. Here, the uncertainty sets at the instants $t = 12$ sec and $t = 42$ sec are circles, the informational set $\mathbf{I}(0)$ has several φ-sections of the circle form. In the informational sets, the sections that are most close (on φ) to the true values are marked in gray.

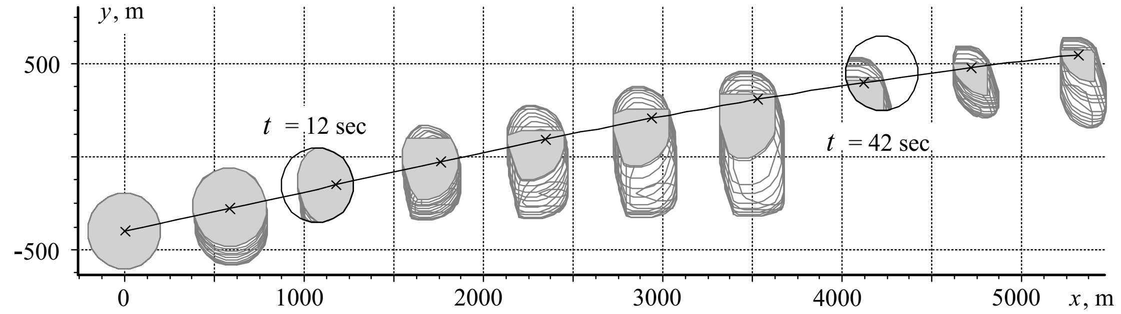

Fig. 9. Autopilot control; motion of the informational set, projection in the plane x, y

5. CONCLUSION

The work is devoted to numerical construction and investigation of the informational sets in model problems of aircraft tracking for the case when the aircraft moves in the horizontal plane. The algorithm elaborated allows to construct the informational sets in the real-time mode and can be used in contemporary air traffic control systems.

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