CONTROL OF PENDUBOT USING INPUT-OUTPUT FEEDBACK LINEARIZATION AND PREDICTIVE CONTROL

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Abstract: In applying nonlinear model-predictive control to fast unstable systems, the main difficulty is that the optimization cannot be finished within one sampling interval. To solve this problem, a cascade-control scheme has been proposed, where input-output feedback linearization forms the inner loop and nonlinear predictive control the outer loop. Thus, the nonlinear predictive control has to handle only the internal dynamics which might be slow. For example, in the case of pendubot, an underactuated mechanical system with two rotary joints, the input-output dynamics are fast while the internal dynamics are slow. The proposed cascade approach was applied successfully to the experimental pendubot setup. *Copyright*$^{©}$ *2005 IFAC*

Keywords: Input-output feedback linearization, Nonlinear model predictive control, Singular perturbation

1. INTRODUCTION

Predictive control is an effective approach for tackling problems with nonlinear dynamics, especially when the analytical computation of the control law is difficult (Morari and Lee, 1999; Rawlings *et al.*, 1994). This methodology is widely used in the process industry, where system dynamics are sufficiently slow to permit its implementation (Qin and Badgwell, 1997). In contrast, applications to fast systems are rather limited since it is often not possible to complete the optimization within one sampling interval, the duration of which is limited by Nyquist's sampling theorem.

An approach to use predictive control for the stabilization of fast unstable systems has been developed by the authors (Guemghar *et al.*, 2002;

Guemghar *et al.*, 2004). The methodology is based on input-output feedback linearization, model predictive control and singular-perturbation theory. The system is first input-output feedback linearized, separating the input-output system behavior that is fast, from the internal dynamics that is hopefully slow. Predictive control is then used to stabilize the internal dynamics, using the reference of outputs as the manipulated variables. This results in a cascade-control scheme, where the outer loop consists of a model predictive control of the internal dynamics, and the inner loop is the input-output linearization. Stability analysis of the cascade system is provided using results of singular-perturbation theory (Khalil, 1996).

The goal of this paper is to apply the above methodology on the pendubot (Spong *et al.*,

2001). The pendubot consists of two rotary joints, with the second one being unactuated. This system represents an interesting example for control, and has been widely used in the literature due to its nonlinear, unstable, and nonminimum-phase natures (Fantoni *et al.*, 2000; Yamada *et al.*, 2003; Zhang and Tarn, 2002; Ma and Su, 2002). Moreover, this system has all the properties that suit the cascade scheme mentioned above. It is nonlinear, unstable and has fast input-output dynamics. Also, when input-output feedback linearization is used, the internal dynamics is rather slow, so that the model predictive control can be used with a low re-optimization frequency. In addition, no analytical stabilizing feedback law can be formulated for the internal dynamics, justifying the use of predictive control for the stabilization of internal dynamics.

The paper is organized as follows: The next section describes the model of the pendubot. In Section 3, the cascade-control scheme is presented. Experimental results are given in Section 4, and Section 5 concludes the paper.

2. THE MODEL
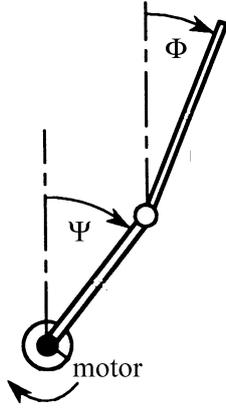
Fig. 1. Pendubot Struture

The pendubot (Figure 1) is a two-degree of freedom underactuated mechanical system consisting of an actuated rotating arm and an unactuated one. Let ψ denote the actuated coordinate, ϕ the unactuated one and τ the torque. The pendubot is described by the following set of differential equations (Favez, 1997):

$$J_1 \ddot{\psi} + J \cos(\psi - \phi)\ddot{\phi} + J \sin(\psi - \phi)\dot{\phi}^2$$
$$-g_1 \sin(\psi) + b\dot{\psi} + c_1 sign(\dot{\psi}) = \tau \quad (1)$$

$$J_2 \ddot{\phi} + J \cos(\psi - \phi)\ddot{\psi} - J \sin(\psi - \phi)\dot{\psi}^2$$
$$-g_2 \sin(\phi) = 0 \quad (2)$$

where g_1, g_2 are the gravity components, J, J_1, J_2 the inertia components, c_1 the Coulomb friction

coefficient, and b_1 the viscous friction coefficient. The physical values of the parameters obtained from measurements on the experimental setup are $g_1 = 2.2$ N, $g_2 = 0.36$ N, $J = 0.0185$ kg m^2, $J_1 = 0.22$ kg m^2, $J_2 = 0.019$ kg m^2, $c_1 = 1.12$ N m, $b = 0.066$ N s rad^{-1}.

2.1 State space formulation

Denoting $x = \begin{bmatrix} \phi & \dot{\phi} & \psi & \dot{\psi} \end{bmatrix}^T$, $u = \tau - b\dot{\psi} - c_1 sign(\dot{\psi})$, and considering ϕ as the output, the pendubot equations (1)-(2) can be rewritten as:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x) \quad (3)$$

where

$$f(x) = \begin{bmatrix} \dot{\phi} \\ -J_1\alpha - J\cos(\psi - \phi)\beta \\ \dot{\psi} \\ J\cos(\psi - \phi)\alpha + J_2\beta \end{bmatrix} \quad (4)$$

$$g(x) = \begin{bmatrix} 0 \\ -aJ\cos(\psi - \phi) \\ 0 \\ aJ_2 \end{bmatrix} \quad (5)$$

$$h(x) = \phi \quad (6)$$

and

$$a = \frac{1}{J_1 J_2 - J^2 \cos^2(\psi - \phi)}$$
$$\alpha = a(-g_2 \sin\phi - J\sin(\psi - \phi)\dot{\psi}^2) \quad (7)$$
$$\beta = a(g_1 \sin\psi - J\sin(\psi - \phi)\dot{\phi}^2)$$

2.2 Relative Degree of the Pendubot

Differentiating the output $h(x)$ successively with respect to time t, it follows that $L_g h(x) = 0$, where $L_g h(x) = \frac{\partial h}{\partial x} g(x)$ is the Lie derivative of $h(x)$ along g. However, the second derivative $L_{fg} h(x) = aJ \cos(\psi - \phi) \neq 0$, indicating that System (3) is of relative degree $r = 2$ (Isidori, 1989).

2.3 Nonminumun-phase Behavior of the Pendubot

To test the nonminimum-phase property of the pendubot, the output ϕ and its derivative are replaced in Equation (2) by their values at the equilibrium, $\phi = 0$, $\dot{\phi} = 0$. This yields:

$$\ddot{\psi} = \dot{\psi}^2 \tan\psi \quad (8)$$

whose solution is

$$\psi(t) = \arcsin\left(\dot{\psi}(0)\cos(\psi(0))\,t + \sin(\psi(0))\right) \quad (9)$$

where t is the time, $\psi(0)$ and $\dot{\psi}(0)$ the values of ψ and $\dot{\psi}$, respectively, at time $t = 0$. It can be seen that $\psi(t) \not\to 0$ as $t \to \infty$. So, the zero dynamics (9) is not asymptotically stable, and System (3) is nonminimum phase (Isidori, 1989). Also, the linearization of the zero dynamics around the origin (9) gives $\ddot{\psi} = 0$, i.e. a double integrator, which corresponds to slow internal dynamics.

3. CASCADE CONTROL OF THE PENDUBOT

3.1 Description of the Scheme

The control objective is to control the unactuated angle ϕ to the upright position and, at the same time, stabilize the pendulum angle ψ to the upright position. These two tasks are considered separately in the following control structure (Figure 2):
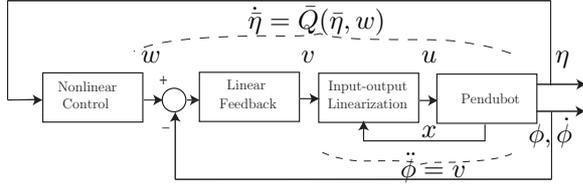
Fig. 2. Cascade-control scheme

(1) Inner loop: Input-output feedback linearization and linear feedback. First, System (3) is input-output feedback linearized into Byrnes-Isidori normal form using the following steps (Isidori, 1989):

- Apply a state feedback law that compensates the nonlinearities in the input-output behavior:

$$u = \frac{v - J_1\alpha - J\cos(\psi - \phi)\beta}{aJ\cos(\psi - \phi)} \qquad (10)$$

  where a, α and β are given in (7).
- Use the nonlinear transformation $z = T(x)$, $z = [\phi \; \dot{\phi} \; \eta^T]^T$, with $\eta = [\eta_1 \; \eta_2]^T$, $\eta_1 = [J\sin(\psi - \phi)$, $\eta_2 = J\dot{\psi}\cos(\psi - \phi) + J_2\dot{\phi}$, to express System (3) as:

$$\ddot{\phi} = v, \qquad \phi(0) = \phi_0, \; \dot{\phi}(0) = 0 \quad (11)$$
$$\dot{\eta} = Q(\eta, \phi, \dot{\phi}), \qquad \eta(0) = \eta_0 \qquad (12)$$

  with

$$Q(\eta, \phi, \dot{\phi}) = \begin{bmatrix} -\dot{\phi}\sqrt{J^2 - \eta_1^2} + \eta_2 - J_2\dot{\phi} \\ \dfrac{\eta_1\dot{\phi}}{\sqrt{J^2 - \eta_1^2}}\left(\eta_2 - \dot{\phi}J_2\right) \\ + g_2\sin(\phi) \end{bmatrix} \quad (13)$$

  where Q is the nonlinear function defining the dynamics of η. The particular choice of η_2

makes the dynamics independent of $\ddot{\phi}$. Also, η_1 is chosen to be the integral of η_2 when $\phi = 0$.

The system output ϕ is controlled by a linear feedback control that computes the new input v defined in Equation (10):

$$v = -\frac{1}{\epsilon^2}(\phi - \phi_{ref}) - \frac{2}{\epsilon}(\dot{\phi} - \dot{\phi}_{ref}) \qquad (14)$$

where ϕ_{ref} and $\dot{\phi}_{ref}$ are references for the output ϕ and its derivative $\dot{\phi}$, respectively, and are determined by the outer loop of the cascade control. $\epsilon \to 0$ is a small parameter. The closed-loop poles of the linearized resulting subsystem correspond to a double pole at $-\frac{1}{\epsilon}$. The gains are chosen this way since, for any choice of $\epsilon > 0$, the closed-loop subsystem is stable and ϵ can be used as a single tuning parameter.

(2) Outer loop: Stabilization of the internal dynamics using model predictive control.

As can be seen, the internal dynamics of the pendubot (12) depend on both ϕ and its derivative $\dot{\phi}$. However, the parameter ϵ being small, quasi-steady-state assumption can be made, which leads to $\phi \to \phi_{ref}$ and $\dot{\phi} \to \dot{\phi}_{ref}$ in (12)-(13). Then, the trajectories $(\phi_{ref}, \dot{\phi}_{ref})$ will used to stabilize the internal dynamics.

In general, input-output linearization decouples the input-output behavior from the internal dynamics, i.e. η has no effect on the output ϕ. On the other hand, the quasi-steady-state assumption decouples the internal dynamics from the input-output behavior, i.e. ϕ has no effect on the output η, though the profile of ϕ_{ref} (an independent variable) is used to control η. Thus, the two subsystems can be handled separately.

The internal dynamics under quasi-steady-state assumption can be written as:

$$\dot{\bar{\eta}} = \bar{Q}(\bar{\eta}, w), \qquad \bar{\eta}(0) = \bar{\eta}_0 \qquad (15)$$

with

$$\bar{Q}(\bar{\eta}, w) = \begin{bmatrix} \bar{\eta}_2 - w\left(\sqrt{J^2 - \bar{\eta}_1^2} + J_2\right) \\ g_2\sin(\bar{\eta}_3) + \dfrac{\bar{\eta}_1 w}{\sqrt{J^2 - \bar{\eta}_1^2}}\left(\bar{\eta}_2 - wJ_2\right) \\ w \end{bmatrix} (16)$$

where $\bar{\eta} = [\bar{\eta}_1 \; \bar{\eta}_2 \; \bar{\eta}_3]^T$, $\bar{\eta}_1 = J\sin(\psi - \phi_{ref})$, $\bar{\eta}_2 = J\dot{\psi}\cos(\psi - \phi_{ref}) + J_2\dot{\phi}_{ref}$, $\bar{\eta}_3 = \phi_{ref}$ and $w = \dot{\phi}_{ref}$. Note that it is important to add an additional state ϕ_{ref} since it is treated as an independent variable. Its derivative w is considered as the manipulated variable for stabilization.

An analytical solution for w that stabilizes the internal dynamics under quasi-steady-state assump-

tion cannot be computed. So, predictive control is used to compute numerically the value of a stabilizing w^*:

$$w^* = \arg\min_{w([t,t+T])} \left\{ \frac{1}{2}\bar{\eta}(t+T)^T P\,\bar{\eta}(t+T) \right. \quad (17)$$

$$\left. + \frac{1}{2}\int_t^{t+T} \left(\bar{\eta}(\tau)^T S\,\bar{\eta}(\tau) + R\,w^2(\tau)\right) d\tau \right\}$$

$$s.t. \quad \dot{\bar{\eta}} = \bar{Q}(\bar{\eta},w) \qquad \bar{\eta}(t) = \bar{\eta}_t$$

$$w(\cdot) \in \mathcal{Y},\; \bar{\eta}(\cdot) \in \mathcal{N},\; \bar{\eta}(t+T) \in \mathcal{N}_f$$

where $\bar{\eta}_t$ are the measured or estimated states at time t, T the prediction horizon, $\mathcal{Y}$ and $\mathcal{N}$ the sets of admissible outputs and internal states, respectively, and $\mathcal{N}_f \subset \mathcal{N}$ a closed set that contains the origin. The input w is updated every δ sec, where time δ is greater than or equal to the sampling time.

3.2 Stability Analysis

The stability of the cascade-control scheme is discussed in this section. The key idea is to introduce a time-scale separation in order to be able to use results from singular-perturbation theory, which is enforced here by the presence of the small parameter ϵ. The results from (Guemghar *et al.*, 2002; Guemghar *et al.*, 2004) will be recalled here for completeness.

Theorem 1. Consider System (11)-(12) with ϕ_{ref}, $\dot{\phi}_{ref}$ obtained using (17), and the input u computed using (10) and (14). Let the following assumptions be satisfied:

(1) System (15) is controllable considering w as input,
(2) $(\bar{\eta} = 0,\, w = 0)$ is an equilibrium point of (15)-(16),
(3) The function $\bar{Q}(\bar{\eta},w)$ in (16) is such that
$$\|\bar{Q}(\bar{\eta},0)\| \leq L\|\bar{\eta}\|,$$
(4) P, S, and R used in (17) are positive definite,
(5) $\exists\; w_k = k(\bar{\eta})$ defined in $\mathcal{N}_f$, for which $F(\bar{\eta}) = \frac{1}{2}\bar{\eta}^T P\bar{\eta}$ defined in $\mathcal{N}_f$ and $q(\bar{\eta},w_k) = \frac{1}{2}\{\bar{\eta}^T S\,\bar{\eta} + R\,w_k{}^2\}$ satisfies
$$\dot{F}(\bar{\eta}) + q(\bar{\eta},w_k) \leq 0$$
(6) The set $\mathcal{N}_f$ is positively invariant with respect to w_k,
(7) The prediction horizon T is chosen sufficiently large to ensure that $\bar{\eta}(t+T) \in \mathcal{N}_f$. where $\bar{\eta}^*(.)$ represents the internal states obtained under predictive control.

Then, there exists an $\epsilon^* > 0$ such that, for all $\epsilon < \epsilon^*$, the origin of System (3) is exponentially stable.

The above theorem indicates that, if ϵ is chosen smaller than a certain value, i.e. if the feedback gains of the inner loop are chosen sufficiently large, then the overall system is stable. In other words, this means that an effective time-scale separation needs to be created for the stability of the proposed cascade scheme to be guaranteed.

The assumptions of Theorem 1 can be easily verified for the pendubot example.

(1) System (15) is not affine in w. However, it is affine in $\dot{w}$. Therefore, the controllability of (15) is shown considering $\dot{w}$ as input.
    System (15) can be rewritten as follows:

$$\begin{bmatrix} \dot{\bar{\eta}} \\ \dot{w} \end{bmatrix} = \bar{f}(\bar{\eta},w) + \bar{g}(\bar{\eta},w)\dot{w} \qquad (18)$$
$$s.t. \quad \bar{\eta}(0) = \bar{\eta}_0,\; w(0) = w_0$$

with

$$\bar{f} = \begin{bmatrix} \bar{\eta}_2 - w\left(\sqrt{J^2 - \bar{\eta}_1^2} + J_2\right) \\ g_2\sin(\bar{\eta}_3) + \dfrac{\bar{\eta}_1 w}{\sqrt{J^2 - \bar{\eta}_1^2}}\left(\bar{\eta}_2 - wJ_2\right) \\ w \\ 0 \end{bmatrix}$$

$$\bar{g} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

System (18) is controllable if the distribution $\Delta(\bar{\eta},w) = span\{g, ad_f g, ad_{ff} g, ad_{fff} g\}$, with $ad_N g = L_g N - L_N g$, has dimension 4, for all $(\bar{\eta},w)$, and is locally controllable since $\Delta(0,0)$ has dimension 4 (Vidyasagar, 2002). Computing Δ gives a nonlinear function of $\bar{\eta}$ and w, which is not of dimension 4 everywhere. However $\Delta(0,0)$ has dimension 4, and (18) is locally controllable using $\dot{w}$ as input, and therefore System (15) is locally controllable using w as input.
(2) By replacing $(\bar{\eta},w)$ by the value $(0,0)$ in (16), it can be easily verified that $(\bar{\eta},w) = (0,0)$ is an equilibrium point.
(3) Replacing $w = 0$ in (16), and computing the norm of $\bar{Q}(\bar{\eta},0)$ gives $\|\bar{Q}(\bar{\eta},0)\| < L\|\bar{\eta}\|$, with $L = 1$.
(4) P, Q and R are chosen positive definite.
(5) and (6) depend on item (7). Here, the prediction horizon $T = 0.6$ sec, for which the pendubot is stable. Therefore, there exists $\mathcal{N}_f$ and w_k which satisfy points (5) and (6).

The above verification shows that the pendubot can be controlled with guaranteed local stability with the cascade scheme.

It is important to note that only local stability of the cascade control of the pendubot can be verified. In fact, an important drawback of the
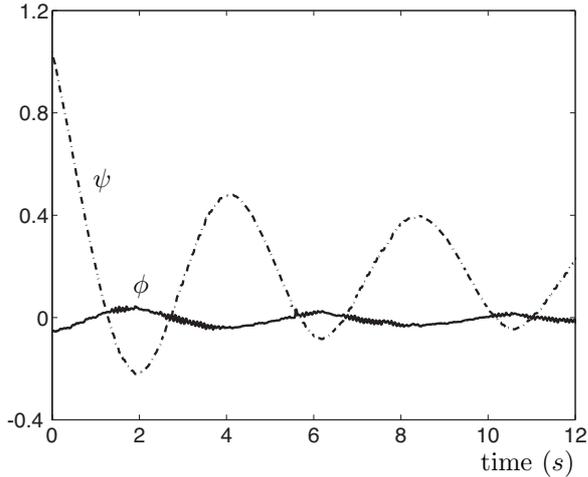
Fig. 3. Performance of the cascade-control scheme
- Angular position ϕ (-), ψ (- -)

Fig. 4. Performance of the cascade-control scheme
- Input u

methodology used here is that input-output feedback linearization of the pendubot causes a singularity when $\cos(\psi - \phi) = 0$. At the singularity, the feedback linearizing input (10) is infinite!

4. EXPERIMENTAL RESULTS

4.1 Experimental Results with the Cascade Scheme

In this section, experimental results of applying the cascade-control scheme to the pendubot are discussed. The parameter chosen for the inner-loop controller (14) is $\epsilon = 0.05$. The parameters chosen for the outer-loop controller (17) are

$$R = 1, \ Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ P = \begin{bmatrix} 4.67 & 5.91 & 1.17 \\ 5.91 & 24.32 & 4.89 \\ 1.17 & 4.89 & 2.17 \end{bmatrix}.$$

The matrix P is computed by solving the algebraic Riccati equation of the linear quadratic regulation problem of the linearized version of Subsystem (15)-(16) (Morari and Lee, 1999). The choice of the re-optimization time δ is taken from an implementation perspective, $\delta = 0.3$ sec, which corresponds to 60 times the sampling period h = 0.005 sec.

The experimental results for the cascade-control scheme are presented in Figures 3-4, where Figure 3 shows the evolution of the pendubot angles ϕ and ψ, and Figure 4 the input u. Although the pendubot is stable, the angle ψ oscillates considerably. This is due to the fact that model predictive control is applied open loop between two re-optimizations. Especially, ψ is fed back only once every δ time units. This is the main disadvantage of using a low re-optimization frequency in presence of disturbances.

4.2 Experimental Results with the Cascade Scheme and Neighboring Extremals

As a solution to this problem, the results from (Ronco et al., 2001) are used. This work suggests the use of an additional linear feedback whenever the numerical optimizer (nonlinear model predictive control) is unable to compute the optimal input. The feedback is computed using neighboring-extremal theory, based on the analytical law derived from the linearized problem:

- Linear model predictive control is applied on the linearized internal dynamics under quasi-steady-state assumption (15)-(16), which leads to the linear state feedback $\bar{w}$:

$$\begin{aligned} \bar{w} &= -\bar{\eta}_1 - 4.67\bar{\eta}_2 - 2.12\phi_{ref} \\ \dot{\phi}_{ref} &= \bar{w}, \quad \phi_{ref}(0) = \phi_0 \end{aligned} \quad (19)$$

- Then, this linear state feedback is used in combination with the nonlinear model predictive control (w^* is given by (17)), leading to the new reference $w = \bar{w} + w^*$ for the inner loop of the cascade-control scheme. Although w^* is updated every 0.3 sec, the reference $\bar{w}$ is updated at each sampling period.

The experimental results for the cascade-control scheme using the neighboring-extremal theory are presented in Figures 5-6, where Figure 5 shows the evolution of the pendubot angles ϕ and ψ, and Figure 6 the input u. With the neighboring-extremal approach, the angles are much smoother than using only nonlinear model predictive control. Also, the angles converge faster to the origin and the input energy is smaller. Due to the feedback of ψ that is present every sampling time, the linear state feedback helps reject the effect of measurement noise.
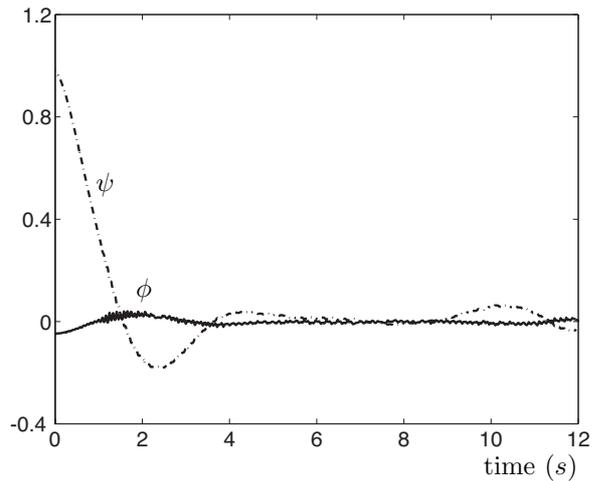
Fig. 5. Performance of the cascade-control scheme using neighboring extremals - Angular position ϕ (-), ψ (- -)
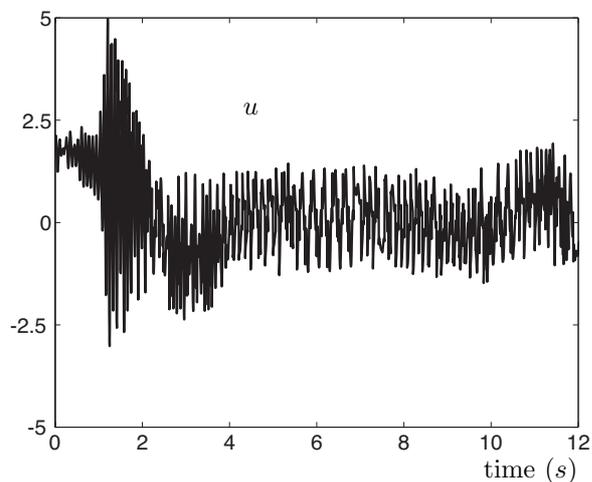
Fig. 6. Performance of the cascade-control scheme using neighboring extremals - Input u

5. CONCLUSION

A cascade structure that allows the use of nonlinear model predictive control on fast systems such as the pendubot has been proposed. First, input-output feedback linearization is applied to separate the fast input-output system dynamics from the slow internal dynamics. Model predictive control is then used to stabilize the internal dynamics and can be implemented at a lower frequency. Experimental results have been obtained showing excellent performance.

However, the issue of constraints, which has been one of the main advantages of predictive control techniques, has not been addressed in this paper. The presence of constraints would prevent the separation between the input-output dynamics and the internal dynamics.

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