SOME ISSUES IN COMMON QUADRATIC LYAPUNOV FUNCTION PROBLEM FOR A SET OF STABLE MATRICES IN COMPANION FORM

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Abstract: Two issues are addressed in common quadratic Lyapunov function(CQLF) problem for a set of stable matrices in companion form. It is first shown that an existence condition of a CQLF for a set of Schur stable companion matrices and that for Hurwitz stable counterparts are equivalent so far as the bilinear transformation connects them. The second issue is a sufficient condition for a diagonal-type CQLF to exist for a set of Schur companion matrices.

Keywords: common quadratic Lyapunov function, companion form, Schur stability, Stein inequality, Lyapunov inequality, diagonal solution

1. INTRODUCTION

The need for investigating existence conditions of a quadratic Lyapunov function common to a set of prescribed stable linear constant systems arises from various control schemes : stability problems of fuzzy control systems, quadratic stability analysis of uncertain systems, intelligent switching control approaches and so forth (Liberzon and Morse 1999; Narendra and Balakrishnan 1994; Liberzon 2003; Shorten and Narendra 2000). Should numerical data on the systems are given, one could easily obtain an answer to the existence of a common quadratic Lyapunov function(CQLF) with the help of existent conventional solution codes such as LMIs.

On the other hand, a general closed-form existence condition is hard to be attained and has escaped from extensive research efforts exerted thus far. Such an existence problem is currently solved only under some specific conditions or in certain restricted circumstances. For example, attempts are made to identify subclasses of systems which have a CQLF (Narendra and Balakrishnan 1994; Y. Mori *et al.* 2001) or to find a condition for a pair of systems having special structures in system matrices (Shorten and Narendra 2003; Shorten *et al.* 2004).

This brief also belongs to this last line of research and addresses two issues in the CQLF problem for a set of stable linear constant systems whose system matrices are in companion form. Both Hurwitz stability and Schur stability problems will be treated, yet weight will be given to the latter problem. The first result concerns a relation between the existence conditions of a CQLF for a set of Hurwitz companion matrices and of a CQLF for a set of Schur companion matrices. It is shown that these two conditions are equivalent so far as the two sets are connected by the bilinear transformation. This makes it possible to express an existence condition in one way or the other freely. The second result presents a sufficient condition for the existence of a diagonal-type CQLF for a set of Schur stable companion matrices. It also provides an explicit form of the CQLF in terms of the matrix entries of the given set.

The paper is organized as follows. In the next section, the problem is formulated and some preliminary results are collected. Section 3 establishes a relation between the existence conditions of a CQLF for a set of Hurwitz companion matrices and of its Schur counterparts connected by the bilinear transformation. As a consequence of this relation, an exact existence condition of a CQLF for a pair of Schur companion matrices is obtained through its existing Hurwitz counterpart. In section 4, retaining the companion form restriction, a sufficient condition is derived for the existence of a diagonal CQLF for a set of Schur matrices in companion form. A simple numerical example is also provided in this section to illustrate the obtained results. Section 5 concludes the paper. Standard symbols in linear algebra will be employed throughout. For an n by n real matrix $X \in R^{n\times n}$, X' denotes the transpose and $|X|$ the determinant. For $X = X'$, $X > 0(< 0)$ stands for positive(negative)-definiteness of X. While I represents a unit matrix as usual, J does the matrix whose second diagonal has all 1s and the rest 0s.

2. PROBLEM FORMULATION AND PRELIMINARIES

In what follows, we identify a real square matrix B with the continuous-time linear constant system $\dot{x} = Bx$ and refer to it with simply "system". A quadratic Lyapunov function $x'Px$ for a system and its coefficient matrix P will appear interchangeably. The similar convenience applies to a discrete-time constant system and the related quadratic Lyapunov function.

Assume that a set of Hurwitz stable systems(matrices) $\{B_i\}$, $B_i \in R^{n\times n}$ and a set of Schur stable ones $\{A_i\}$, $A_i \in R^{n\times n}$, $i \in \{1, \cdots, m\} \triangleq \bar{m}$ are given. Common quardatic Lyapunov function(CQLF) problems are formulated as follows.

[I] Continuos-time case: find the existence condition of a solution $P_c = P_c' > 0$, $P_c \in R^{n\times n}$ to a set of Lyapunov inequalities,

$$B_i'P_c + P_cB_i < 0, \; i \in \bar{m}. \qquad (1)$$

[II] Discrete-time case: find the existence condition of a solution $P_d = P_d' > 0$, $P_d \in R^{n\times n}$ to a set of Stein inequlaities,

$$A_i'P_dA_i - P_d < 0, \; i \in \bar{m}. \qquad (2)$$

If solutions exist to these problems, a CQLF exsits for corresponding continuous-time systems in [I] and for discrete-time systems in [II]. A common feature characterizing the above problems is the following fact, which could be immediately checked.

[Lemma 1]

Both in [I] and [II], the conditions are invariant under the similarity transformation. For example in [I], the condition for the set $\{B_i\}$ is equivalent to that for $\{T^{-1}B_iT\}$ with T being any (common) nonsingular matrix. Thus, the problems are coordinate-free.

Another feature that connects the two problems is:

[Lemma 2] (Y. Mori *et al.* 2001)

There exists a $P_c > 0$ in [I], if and only if a $P_d > 0$ exists in [II], when A_i and B_i are related through the bilinear transformation:

$$B_i = (A_i + I)(A_i - I)^{-1} \text{ or}$$
$$A_i = (B_i - I)^{-1}(B_i + I), i \in \bar{m}. \qquad (3)$$

Furthermore, the problems share a solution, if any, i.e., $P_c = P_d = P > 0$.

The latter statement claims the solution sets for (1) and for (2) coincide with each other. We also note:

Remark 1 The bilinear transformation is a one-to-one onto mapping. Thus, once the desired condition is obtained in either [I] or [II], it can be readily translated to the other problem via (3), yielding conditions both for [I] and [II]. One more pivotal property of the bilinear transformation (3) is that it does not affect the similarity transformation carried out in its domain space and range space. For instance, putting $T^{-1}B_iT$ in place of B_i in (3) gives $T^{-1}A_iT$.

3. CQLF FOR A SET OF STABLE COMPANION MATRICES

In the problem [I], some results are recently obtained under two restrictions: the set consists of only two matrices, i.e., $m = 2$ and they are in companion form (Shorten and Narendra 2003; Shorten *et al.* 2004). In this section, we put only the latter restriction to [I] and [II] and consider relations between these two problems. In view of Lemma 2, assuming the relation (3), one is tempted to presume that they are equivalent in the sense of the lemma even if such a restriction is imposed. This would be obviously validated, if the bilinear transformation (3) preserves the companion form. Unfortunately, this is not the case and we need to look into the problem little more closely. It turns out, however, that the statements in Lemma 2 mostly hold true as shown in the following theorem.

[Theorem 1]

An existence condition of a CQLF in [I] with companion coefficient matrices leads to that in [II] with companion form of the coefficient matrices which are connected by (3) and *vice versa*. In this case, the CQLFs in both problems do not necessarily coincide but have a one-to-one correspondence.

The key to the proof of this result is the following fact.

[Lemma 3] (Barnett 1983)

Assume a Hurwitz matrix $B \in R^{n \times n}$ has the form of

$$B = \begin{bmatrix} b_1 & b_2 & \cdot & \cdot & b_n \\ 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & & & \vdots \\ & & \ddots & & \\ 0 & & \cdots & 1 & 0 \end{bmatrix} = \left[\begin{array}{c} b \\ \hline I \quad 0 \end{array}\right] \qquad (4)$$

where $b = (b_1, b_2, \cdots, b_n)$, and let A be the matrix obtained from B by the bilinear transformation, $A = (B - I)^{-1}(B + I)$. Then, the similarity transfomation $J\Gamma(-A)(J\Gamma)^{-1}$ puts $(-A)$ into the companion form which conforms to (4). Here, Γ is a constant matrix determined solely by the matrix order n.

Remark 2 The first row of Γ consists of the binomial coefficients of $(-1 + \mu)^n$ whereas the last column of all 1s and the other entries are determined successively with a recurrent formula. For details, see Barnett(1983).

Now, we are in position to verify Theorem 1.

Proof of Theorem 1

Suppose that $B_i(A_i), i \in \bar{m}$ are Hurwitz(Schur) companion matrices of the form as shown in (4) and A_i and B_i are linked by (3). Let $\bar{A}_i(\bar{B}_i), i \in \bar{m}$ be the companion matrices, each of which is derived from $A_i(B_i)$. It will be proven that the existence of a common solution to the set of Lyapunov inequalities (1) assures a solution to the corresponding Stein inequalities. Due to Lemma 2, the common solution to

$$B_i' P_c + P_c B_i < 0, \; i \in \bar{m} \qquad (5)$$

also satisfies

$$A_i' P_c A_i - P_c < 0 \text{ or } (-A_i)' P_c (-A_i) - P_c < 0. \quad (6)$$

Denoting

$$J\Gamma(-A_i)(J\Gamma)^{-1} = \bar{A}_i, \qquad (7)$$

$$((J\Gamma)')^{-1} P_c (J\Gamma)^{-1} = \bar{P}_d, \qquad (8)$$

we see from Lemma 3 that $\bar{A}_i$ is in fact in the companion form and from Lemma 1 along with (6) satisfies

$$\bar{A}_i{}' \bar{P}_d \bar{A}_i - \bar{P}_d < 0. \qquad (9)$$

We have thus proven that the existence of a common solution to Hurwitz inequalities (5) implies that to Stein inequalities (9). Since the transformation matrix $J\Gamma$ is constant but still not an identity, the solutions P_c and $\bar{P}_d$ do not in general coincide but still maintain a one-to-one correspondence. The converse process, starting from (2) to arrive at (1) with $B_i = \bar{B}_i$, can proceed in the similar manner using the properties of the bilinear transformation noted in Remark 1. This completes the proof. *Q.E.D.*

Theorem 1 indicates Remark 1 on Lemma 2 still applies in the case of companion matrices. Namely, we have only to know the existence condition for either [I] or [II] to obain the both, because one of them yields the other through the bilinear transformation. To illustrate this point, we note a recent result mentioned in the beginning of this section, which assumes $m = 2$ and the companion form restriction.

[Lemma 4] (Shorten and Narendra 2003)

Let B_1 and B_2 be Hurwitz companion matrices. Then, a CQLF exists in the problem [I] if and only if the product matrix B_1B_2 has no real negative eigenvalues.

Remark 3 Apparently, the above condition is coordinate-free. This implies the pair in question is not necessarily confined to comapnion form, but such pairs are allowed that can be transformed to the companion form by a similarity transformation with a common transformation matrix (Shorten *et al.* 2004).

As a consequence of Theorem 1, Lemma 4 immediately produces the discrete-time counterpart.

[Corollary 1]

Let A_1 and A_2 be Schur companion matrices. Then, a CQLF exits in the problem [II] if and only if the marix

$$S \triangleq (A_1 - I)^{-1}(A_1 + I)(A_2 + I)(A_2 - I)^{-1} \quad (10)$$

has no real negative eigenvalues.

It is stressed that Theorem 1 enables one to relate the two problems not only for $m = 2$ case as above but for any sets of companion systems.

4. DIAGONAL CQLF FOR A SET OF SCHUR COMPANION MATRICES

In this section, we will focus on a set of Schur stable companion systems which have a diagonal common quadratic Lyapunov function(diagonal CQFL) and derive a sufficient condition for the existence of such a function. For a single Schur stable companion system, a diagonal Lyapunov function ensures Schur stability of the system whose state is computed through a finite precision arithmetic (Regalia 1992). Considering a set of such systems amounts, for example, to studying stability of a discretized switching system under the above arithmetic scheme. The reason why we exclusively investigate the problem for Stein inequalities rather than Lyapunov ones is simply that for $n \geq 2$ no diagonal solution exists to Lyapunov inequalty with companion form coefficient matrix (Wimmer 1998; Kaszkurewicz and Bhaya 2000). An advantage of dealing with an $n \times n$ companion matrix is that it includes only n significant(other than 0 and 1) entries in contrast to n^2 for matrices without specific forms. This, along with a diagonal solution which also contains only n unknowns, makes the existence condition of such a diagonal-type solution to Stein inequality obtainable in terms of the entries of the coefficient companion matrix (Wimmer 1998). Moreover, in this case, the matrix inequality can be reduced to a scalar inequaility, yielding a solution in a closed-form.

Consider a Stein inequality,

$$A'PA - P < 0,\ A = \left[\begin{array}{cc} \multicolumn{2}{c}{a} \\ \hline I & 0 \end{array}\right] :\ \text{Schur stable,} \quad (11)$$

where $a = (a_1, a_2, \cdots, a_n)$.

The following known result (Wimmer 1998) will be central to the later argument.

[Lemma 5]

A diagonal solution to (11), $P = \mathrm{diag}(p_1, p_2, \cdots, p_n)$, $P > 0$ exists, if and only if

$$s_0 < 1,\ s_0 \triangleq \sum_{\nu=1}^{n} |a_\nu|. \quad (12)$$

If (12) is satisfied, a solution is given as in the following form:

For $s_0 = 0$, $P = I$ fulfills (11). If $s_0 \neq 0$, then, with l being the integer such that

$$a_n = \cdots = a_{l+1} = 0,\ a_l \neq 0, \quad (13)$$

P is given by

$$P = \mathrm{diag}(p_1, \cdots, p_l, \delta_0, \cdots, \delta_0), \quad (14)$$

where

$$p_\nu = \frac{1}{s_0}(|a_\nu| + \cdots + |a_l|),\ \nu = 1, \cdots, l \quad (15)$$

and δ_0 is a positive constant depending upon a_l and s_0 and satisfying

$$\frac{|a_1|^2}{p_1 - p_2} + \cdots + \frac{|a_{l-1}|^2}{p_{l-1} - p_l} + \frac{|a_l|^2}{p_l - \delta_0} < 1. \quad (16)$$

The existence of such a δ_0 is always guaranteed under the condition (12). The convention on each of the fractions in the left hand side(LHS) of (16) is : when the denominator is zero, so are the numerator and the value of the fraction as well. This means when some $a_\nu = 0$ corresponding fraction is disregarded in the LHS of (16).

Remark 4 Rewriting the Stein inequality (11) with a diagonal solution, we arrive at the scalar inequality (16).

It is noted that the condition (12) is a necessary and sufficient condition for robust Schur stability of a polynomial with varying coefficient vector a (Mori and Kokame 1986). On the basis of this lemma, we now consider the diagonal CQLF problem in [II] where the coefficient matrices have the form,

$$A_i = \left[\begin{array}{c} a^i \\ \hline I \quad 0 \end{array}\right],\ i \in \bar{m}, \tag{17}$$

with $a^i = (a_1^i, \cdots, a_n^i)$. In contrast to the single system case(Lemma 5), however, an exact existence condition for a diagonal CQLF is still hard to obtain, yet a simple sufficient one which gives the explicit form of a CQLF is obtainable.

[Theorem 2]

Let Schur matrices A_i in the problem [II] be all in the form of (17). Then under the condition,

$$s := \sum_{\nu=1}^{n} b_\nu < 1,\ b_\nu \triangleq \max_{i\in\bar{m}} |a_\nu^i|,\ \nu = 1, \cdots, n, \tag{18}$$

there exists a diagonal CQLF or a diagonal solution $P_D > 0$ to (2).

Proof When $s = 0$, $P_D = I$ apparently serves as a solution due to Lemma 5. Assume $s \neq 0$. Let k be the maximum integer such that b_k does not vanish and define

$$\hat{p}_\nu = \frac{1}{s}(b_\nu + \cdots + b_k),\ \nu = 1, \cdots, k. \tag{19}$$

From Lemma 5, we can find a δ satisfying

$$\frac{b_1^2}{\hat{p}_1 - \hat{p}_2} + \cdots + \frac{b_{k-1}^2}{\hat{p}_{k-1} - \hat{p}_k} + \frac{b_k^2}{\hat{p}_k - \delta} < 1. \tag{20}$$

With these $\hat{p}_i$s and δ, a desired solution will be given by

$$P_D = \mathrm{diag}(\hat{p}_1, \cdots, \hat{p}_k, \delta, \cdots, \delta). \tag{21}$$

To see this, fix any superfix $i \in \bar{m}$ and write a^i as simply a by dropping i for brevity. Now, with this omission we regard Stein inequality (11) as the i-th one. To achieve the goal, it suffices to show that the LHS of (16) with P of (14) being replaced by P_D remains less than unity. Letting l be the largest integer such that a_ν, the element of a, is not vanishing, we have $l \leq k$. Now we compare the LHS of (16) where P_D substitutes P with the LHS of (20). Note first that the number of the non-zero terms in LHS of (16) does not exceed that of (20). This is because $l \leq k$ and whenever $b_\mu = 0$ the corresponding a_ν also disappears. Furthermore, due to (18), in any pair of the corresponding non-zero fractional terms the numerator value in (20) is larger than or equal to the counterpart in (16). This observation leads to the fact that the solution P_D given in (21) can replace P in (16). In other words, b_νs in (20) can be reduced to $|a_\nu|$s in the LHS of (20) without violating the inequality, thus leading to (16) with P_D. By Remark 4, P_D is a desired common diagonal solution. This proves the claimed result. *Q.E.D.*

We finally give a simple example to illustrate the obtained results of this brief. Consider a pair of companion systems,

$$A_1 = \begin{bmatrix} \alpha & 0 \\ 1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & \beta \\ 1 & 0 \end{bmatrix}. \tag{22}$$

Schur stability condition of these matrices are $|\alpha| < 1$ and $|\beta| < 1$, respectively. Because of Lemma 5 they are as well the necessary and sufficient condition for the existence of a diagonal solution to each of Stein inequalities in (2). For this pair, Theorem 2 gives a sufficient condition for a common diagonal solution as $|\alpha| + |\beta| < 1$, while the exact existence condition for such a solution can be readily calculated as $\alpha^2 + \beta^2 < 1$. These three inequalities show gaps among the respective conditions for this example. Putting $\alpha = 0.6$ and $\beta = 0.8$, we see that each Stein inequality has a diagonal solution by Lemma 5, nevertheless the last inequality indicates that no common diagonal solution exists. We can, however, assure the existence of a common(not diagonal) solution for this companion matrix case owing to Corollary 1, because the matrix S of the corollary is given by

$$S = \begin{bmatrix} 36 & 32 \\ 55 & 49 \end{bmatrix},$$

which is a positive matrix known to have a positive real eigenvalue (Barman and Plemmons 1979). Since $|S| = 4$, a positive value, so is the other eigenvalue, which concludes the existence of a common solution by virtue of Corollary 1.

5. CONCLUDING REMARKS

Two issues are addressed in CQLF problem where system matrices are in companion form. The relation is made clear between the CQLF problems for discrete-time and continuous-time cases when the system matrices are in companion form. It is shown that once an existence condition of a CQLF is established for either of the two cases it can be readily carried over to the other by the bilinear transformation. The second issue concerns with an existence problem of a diagonal CQLF for a set of Schur companion matrices. Using the feature of the companion form, *i.e.*, sparsity of its non-zero entries, a sufficient condition is obtained for the existence of a diagonal-type CQLF in terms of the entries of given matrices. The condition also gives rise to the desired CQLF. These two issues contrast with each other: a parallelism between the two cases enabled by the bilinear transformation and the diagonal CQLF problem which is specific only to the discrete-time case. Any transformation that preserves the eigen-structure as the bilinear transformation could afford such a parallelism among different-types of Lyapunov inequalities (Mori and Kokame 2002). This topic would be worthy of further exploration.

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