YOULA–KUČERA PARAMETERISATION APPROACH TO LQ TRACKING AND DISTURBANCE REJECTION PROBLEM

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Abstract: The contribution is focused on the design of optimal LQ (Linear Quadratic) controller when the YK (Youla–Kučera) controller parameterisation is used. We provide a procedure for computing a deterministic optimal SISO (single-input single-output) 2DoF controller from any stabilising 2DoF controller.

Keywords: LQ control, parameterisation, spectral factorisation

1. INTRODUCTION

In our earlier paper (Čirka *et al.* 2002) an algorithm for design of the deterministic LQ 1DoF (one-degree-of-freedom) controller was derived. The present paper gives the details for the tracking and disturbance rejection problem and for 2DoF (two-degrees-of-freedom) controller structure. The advantages of 2DoF controller structures are well known: feedback properties can be shaped independently of tracking properties (Youla and Bongiorno 1985, Grimble 1988).

As before, derivation is based on the class of all stabilising linear controllers for linear, time-invariant plant model. We use the fact that all stabilising controllers for the plant can be synthesised by conveniently parameterised augmentations to any stabilising controller, called a nominal controller. The augmentations are parameterised by an arbitrary stable parameter Q, called the Youla–Kučera parameter.

The approach presented in this paper translates the classical deterministic LQ tracking and disturbance rejection problem into an optimal control with emphasis on design of an optimal YK parameter. We provide a computational procedure for a deterministic optimal 2DoF controller from any nominal (stabilising) controller. This approach allows us to calculate a new optimal LQ deterministic controller from a previous one when the plant has changed. The nominal controller is based on algebraic approach developed by Kučera. The control design is performed in input-output formulation leading to Diophantine and spectral factorisation equations.

The choice of the LQ cost follows the ideas presented in (Dostál *et al.* 1994) where penalisation of the control signal derivative rather than the control signal itself is assumed. This choice of the LQ cost reflects more closely the practical needs of process control.

1.1 Notation

All systems in this work are assumed to be SISO and continuous-time. The systems are described by means of fractions of polynomials in complex

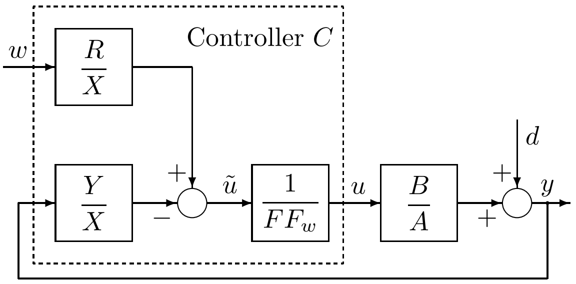

Fig. 1. Block diagram of the 2DoF closed-loop system

argument s, used in $\mathcal{L}$-transform. $\mathcal{R}_{ps}$ denotes the set of stable proper rational transfer functions and $\mathcal{S}$ denotes the set of stable polynomials.

For simplicity, the arguments of polynomials are omitted whenever possible - a polynomial $X(s)$ is denoted by X. We denote $X^*(s) = X(-s)$ for any function $X(s)$.

2. CLOSED-LOOP SYSTEM

2.1 System description

Consider a 2DoF controller feedback system with two exogenous inputs d and w illustrated in Fig. 1. A continuous-time linear time-invariant input-output representation of the plant to be controlled is considered

$$y = Gu + d, \qquad G = \frac{B}{A} \tag{1}$$

where y, u, d are process output, controller output, and disturbance signal, respectively. A and B are polynomials that describe the input-output properties of the plant. We assume that the condition deg $B \leq$ deg A holds (i.e. transfer function of the plant is proper) and A and B are coprime polynomials.

The reference w and the disturbance d are considered to be from a class of functions expressed in the form

$$F_w w = H_w, \qquad Fd = H \tag{2}$$

where H_w, F_w and H, F are pairs coprime polynomials and $\deg H_w \leq \deg F_w$ and $\deg H \leq \deg F$, respectively. For example the most common case of reference step changes implies $F_w = s$.

The 2DoF controller is described by the equation

$$X\tilde{u} = Rw - Yy \tag{3}$$

where the pairs X, Y and X, R are coprime polynomials and $X(0)$ is nonzero.

Note 1. It is clear that by putting $R = Y$ we get the traditional 1DoF controller (Fig. 2).

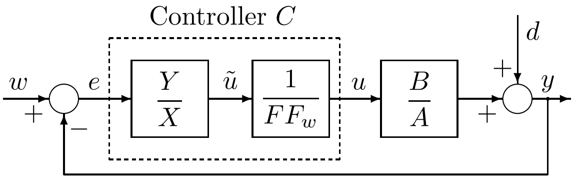

Fig. 2. Block diagram of the 1DoF closed-loop system

In order to track asymptotically the desired reference class and to reject disturbances, a precompensator is inserted into the closed-loop system of the form

$$FF_w u = \tilde{u} \tag{4}$$

3. NOMINAL CONTROLLER

The general conditions posed on the control system properties are

- stability of the control system
- asymptotic tracking of the reference
- disturbance rejection

3.1 Feedback system, stability and disturbance rejection

Consider the nominal plant and the nominal controller transfer functions with fractional representations

$$G = \frac{N_G}{D_G}, \quad C_y = \frac{N_{C_y}}{D_C}, \quad C_r = \frac{N_{C_r}}{D_C} \tag{5}$$

where

$$N_G = \frac{B}{M_1},\ D_G = \frac{A}{M_1} \in \mathcal{R}_{ps} \tag{6}$$

$$N_{C_y} = \frac{Y}{M_2},\ N_{C_r} = \frac{R}{M_2},$$

$$D_C = \frac{FF_w X}{M_2} \in \mathcal{R}_{ps} \tag{7}$$

$$\left(w = \frac{N_w}{D_w},\ N_w = \frac{H_w}{M_1},\ D_w = \frac{F_w}{M_1}\right)$$

and M_1, $M_2 \in \mathcal{S}$ with degrees

$$\deg(M_1) = \max(\deg A, \deg F_w)$$
$$\deg(M_2) = \deg(FF_w X)$$

Stabilising nominal controllers are then given by solution of the Diophantine equation

$$D_G D_C + N_G N_{C_y} = 1 \tag{8}$$

Substituting equations (6) and (7) into (8), the stability condition in $\mathcal{S}$ is of the form

$$AFF_w X + BY = M_1 M_2 = M \tag{9}$$

3.2 Asymptoting tracking

The aim of the nominal system is not only to achieve stability and disturbance rejection but also asymptotic tracking of the reference.

First, we will formulate the conditions for asymptotic tracking. We consider the feedback system (1)–(4). The objective is to design controller C_r such that output y asymptotically tracks the reference signal w. From the elementary algebra follows that the tracking error (for $d = 0$) is given as

$$w - y = \left(1 - \frac{N_G N_{C_r}}{D_G D_C + N_G N_{C_y}}\right) \frac{N_w}{D_w} \tag{10}$$

Since $D_G D_C + N_G N_{C_y} = 1$, it follows for asymptotic tracking that D_w has to divide $(1 - N_G N_{C_r})$ in $\mathcal{R}_{ps}$, which leads to the second Diophantine equation

$$D_w W_w + N_G N_{C_r} = 1 \tag{11}$$

Substituting equations (6) and (7) into (11), the tracking Diophantine equation is finally of the form

$$F_w W + BR = M_1 M_2 = M \tag{12}$$

4. LQ CONTROLLER DESIGN

The goal of optimal deterministic LQ tracking is to design a controller that enables the control system to satisfy the above basic requirements and in addition the control law that minimises the cost function

$$J = \int_0^\infty \left(\varphi \tilde{u}^2(t) + \psi e^2(t)\right) dt \tag{13}$$

where $e = w - y$ denotes the control error and $\varphi > 0$, $\psi \geq 0$ are weighting coefficients. The cost function (13) can be rewritten using Parseval's theorem, to obtain an expression in the complex domain

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left(\tilde{u}^*(s) \varphi \tilde{u}(s) + e^*(s) \psi e(s)\right) ds \tag{14}$$

4.1 The YK parameterisation

Suppose that a nominal stabilising controller that gives rise to the closed-loop polynomial M (not necessarily LQ optimal or minimum degree) has been found as a solution of Diophantine equations (9) and (12), respectively.

There are infinitely many solutions of (9) and (12) that stabilise the plant. The nominal solution (X, Y, R) will serve only as a starting point. It is possible to search among general solutions to minimise the cost (14). In our case, all such controllers are given by the following theorem:

Theorem 1. Let the nominal model plant $G = N_G/D_G = B/A$, with N_G, D_G, B and A defined by (6), be stabilised by a 2DoF controller $C = [N_{C_r}\ N_{C_y}]/D_C = [R\ \ Y]/FF_wX$, with N_{C_r}, N_{C_y}, D_C, R, Y, and FF_wX defined by (7). Then the set of all feedback stabilising controllers for the plant G is given by

$$C_y(Q_y) = \frac{Y_m + A_m FF_w Q_y}{X_m - B_m Q_y} \frac{1}{FF_w} \tag{15}$$

All feedforward parts are given by

$$C_r(Q_r, Q_y) = \frac{R_m + M_2 F_w Q_r}{X_m - B_m Q_y} \frac{1}{FF_w} \tag{16}$$

where

$$Q_r, Q_y \in \mathcal{R}_{ps},\ A_m = AM_2,\ B_m = BM_2,$$
$$X_m = XM_1,\ Y_m = YM_1,\ \text{and}\ R_m = RM_1$$

Proof 1. (Vidyasagar 1985)

We now present a solution to the deterministic LQ controller design problem in the Youla-Kučera parameterisation framework starting from the plant model B/A and any stabilising 2DoF controller $[R\ \ Y]/FF_wX$, using the set of all stabilising controllers for the plant, i.e. we show how to compute optimal parameters Q_r and Q_y that minimise the cost function (13).

Note 2. In fact, Q_r and Q_y are not parameters such as a time constant or gain, but are stable filters built into a stabilising controller. This theory has been developed in a continuous-time setting by (Youla *et al.* 1976) and in a discrete-time setting by (Kučera 1979). Moreover, all the relevant input/output operators of the associated closed-loop system turn out to be linear, or more precisely affine in the operators Q_r and Q_y.

Theorem 2. Consider the minimisation of the cost function (13) with respect to the Youla-Kučera parameters Q_r and Q_y that are specified as transfer functions. Solve spectral factorisation equations

$$D_c^* D_c = \varphi A^* F^* F_w^* A F F_w + \psi B^* B \tag{17}$$

$$D_f^* D_f = A^* A H_d^* H_d \tag{18}$$

$$D_r^* D_r = H_w^* H_w \tag{19}$$

for stable D_c, D_f, and D_r and bilateral Diophantine equations with unknown Q_{yn}, Q_{rn}, V^*, and T^*

$$\psi D_f B^* F_w X - \varphi D_f A^* F^* F_w^* F_w Y = Q_{yn} D_c^* F_w + M V^* \tag{20}$$

$$\psi D_r B^* M - D_r D_c D_c^* R = Q_{rn} D_c^* F_w^- + M F_w^+ T^* \tag{21}$$

The optimal Youla-Kučera parameters are then given as

$$Q_y = \frac{Q_{yn}}{D_c D_f} \frac{M_1}{M_2} \in \mathcal{R}_{ps} \tag{22}$$

$$Q_r = \frac{Q_{rn}}{D_c D_r F_w^+} \frac{M_1}{M_2} \in \mathcal{R}_{ps} \tag{23}$$

As D_c, D_f, D_r, and M_2 are stable, it follows that Q_r and Q_y are stable transfer functions and fulfil the condition from the Youla-Kučera parameterisation.

Proof 2. The proof of Theorem 2 is given in the Appendix.

5. ILLUSTRATIVE EXAMPLE

In this section, an example is presented to show all steps of the calculation in case of LQ design. Let us consider the controlled system described by the following transfer function

$$G = \frac{B}{A} = \frac{3}{5s+1}$$

The reference has been chosen as step change $w(t) = 1(t)$ and disturbance $d(t) = 0.1\sin(t)$. From this follows that $F_w = s$ and $F = s^2 + 1$. The weighting coefficients φ and ψ in the cost function (13) have been selected as $\varphi = 0.1$, $\psi = 1$. Using MATLAB/Polynomial[1] function `spf` (spectral polynomial factorisation) for equations (17), (18), and (19), the following stable polynomials D_c, D_f, and D_r were obtained

$$D_c = 1.581s^4 + 3.961s^3 + 6.511s^2 + 6.258s + 3$$
$$D_f = 0.5s + 0.1$$
$$D_r = 1$$

For the YK parameterised LQ controller a nominal 2DoF controller that stabilises the closed-loop is chosen as

$$C_y = \frac{Y}{FF_wX} = \frac{13.4s^3 + 8.333s^2 + 6.733s + 1.667}{s^4 + 4.8s^3 + s^2 + 4.8s}$$

$$C_r = \frac{R}{FF_wX} = \frac{1.667}{s^4 + 4.8s^3 + s^2 + 4.8s}$$

and yields the closed-loop pole polynomial of the form

$$D = M_1 M_2$$

where $M_1 = (s+1)$ and $M_2 = (s+1)^4$

The polynomials Q_{yn} and Q_{rn} are calculated from (20) and (21). This gives the optimal YK transfer functions Q_y and Q_r as

$$Q_y = \frac{Q_{yn}}{D_c D_f} \frac{M_1}{M_2}$$

$$Q_r = \frac{Q_{rn}}{D_c D_r F_w^+} \frac{M_1}{M_2}$$

where

$$Q_{yn} = -0.2594s^4 - 0.4269s^3 - 0.2325s^2 - 0.01259s - 0.03603$$

$$Q_{rn} = 12.53s^4 + 41.29s^3 + 53.08s^2 + 30.99s + 6.958$$

Finally, calculation of the 2DoF LQ controller C yields

$$C_y = \frac{Y_m + A_m F F_w Q_y}{FF_w(X_m - B_m Q_y)} = \frac{5.196s^3 + 3.461s^2 + 3.647s + 0.6325}{s^4 + 2.505s^3 + s^2 + 2.505s}$$

$$C_r = \frac{R_m + F_w M_2 Q_r}{FF_w(X_m - B_m Q_y)} = \frac{3.162s + 0.6325}{s^4 + 2.505s^3 + s^2 + 2.505s}$$

6. CONCLUSIONS

In this paper, we have presented a procedure to compute deterministic LQ (2DoF) controller from a stabilising controller using the Youla-Kučera parameterisation. The presented controller design procedure ensures stability of the controlled system, asymptotic tracking of the references and disturbance rejection. The proposed approach can be applied in adaptive control framework.

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Appendix A. PROOF OF THEOREM 2

From the 2DoF system structure shown in Fig. 1 the control input $\tilde{u}$ and tracking error e can be written as

$$\begin{aligned}\tilde{u} &= \frac{R_m + M_2F_wQ_r}{M_1M}AFH_w \\ &\quad -\frac{Y_m + A_mFF_wQ_y}{M_1M}AF_wH \\ e &= \left[1 - \frac{B(R_m + M_2F_wQ_r)}{M_1M}\right]\frac{H_w}{F_w} \\ &\quad +\frac{X_m - B_mQ_y}{M_1M}AF_wH\end{aligned}$$

Minimising equation (13) with respect to all stable Q_r and Q_y corresponds to minimisation of the following cost function in complex domain

$$J(Q_r, Q_y) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}(\varphi S_{\tilde{u}} + \psi S_e)\,ds \quad \text{(A.1)}$$

where $S_{\tilde{u}}$ and S_e are spectral functions of the form

$$\begin{aligned}S_{\tilde{u}} &= \tilde{u}^*\tilde{u} = S_{\tilde{u}1}^*S_{\tilde{u}1} = S_{\tilde{u}2}S_{\tilde{u}3} + S_{\tilde{u}4}S_{\tilde{u}5} \\ S_{\tilde{u}1} &= \frac{R_m + M_2F_wQ_r}{M_1M}AFH_w \\ &\quad -\frac{Y_m + A_mFF_wQ_y}{M_1M}AF_wH \\ S_{\tilde{u}2} &= \frac{A^*F^*H_w^*AFH_w}{M_1^*M_1M^*M} \\ S_{\tilde{u}3} &= M_2^*M_2F_w^*F_wQ_r^*Q_r + R_mM_2^*F_w^*Q_r^* \\ &\quad +R_m^*M_2F_wQ_r + R_m^*R_m \\ S_{\tilde{u}4} &= \frac{A^*F_w^*H^*AF_wH}{M_1^*M_1M^*M} \\ S_{\tilde{u}5} &= A_m^*F^*F_w^*A_mFF_wQ_y^*Q_y + Y_mA_m^*F^*F_w^*Q_y^* \\ &\quad +Y_m^*A_mFF_wQ_y + Y_m^*Y_m\end{aligned}$$

$$\begin{aligned}S_e &= e^*e = S_{e1}^*S_{e1} = S_{e2}S_{e3} - S_{e4}S_{e5} + S_{e6}S_{e7} \\ S_{e1} &= \left[1 - \frac{B(R_m + M_2F_wQ_r)}{M_1M}\right]\frac{H_w}{F_w} \\ &\quad -\frac{X_m - B_mQ_y}{M_1M}AF_wH \\ S_{e2} &= \frac{B^*BH_w^*H_w}{M_1^*M_1M^*MF_w^*F_w} \\ S_{e3} &= M_2^*M_2F_w^*F_wQ_r^*Q_r + R_mM_2^*F_w^*Q_r^* \\ &\quad +R_m^*M_2F_wQ_r + R_m^*R_m \\ S_{e4} &= \frac{H_w^*H_w}{M_1^*M_1M^*MF_w^*F_w} \\ S_{e5} &= M_1MB^*M_2^*F_w^*Q_r^* + M_1^*M^*BM_2F_wQ_r \\ &\quad +M_1MB^*R_m^* + M_1^*M^*BR_m \\ &\quad +M_1^*M_1M^*M \\ S_{e6} &= \frac{A^*F_w^*H^*AF_wH}{M_1^*M_1M^*M} \\ S_{e7} &= B_m^*B_mQ_y^*Q_y - X_mB_m^*Q_y^* - X_m^*B_mQ_y \\ &\quad +X_m^*X_m\end{aligned}$$

The integrand may now be split into terms that depend on each part of the controller and terms that do not depend on the controller at all. Completing the squares in (A.1) the integrand can be expressed as

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}(Q_1^*Q_1 + Q_2^*Q_2 + y_d)\,ds \quad \text{(A.2)}$$

$$\begin{aligned}Q_1 &= \frac{D_fD_cF_w}{M_1M_1}Q_y + \frac{\varphi D_fA^*F^*F_w^*F_wY}{MD_c^*} \\ &\quad -\frac{\psi D_fB^*F_wX}{MD_c^*} \\ Q_2 &= \frac{D_rD_c}{M_1M_1}Q_r + \frac{D_rD_cR}{MF_w} - \frac{\psi D_rB^*}{F_wD_c^*}\end{aligned}$$

where the term y_d does not depend on the controller and does not, therefore, enter into the following cost minimisation procedure. The first two terms in (A.2) depend on the feedback (Q_y) and feedforward (Q_r) parts of the controller, respectively. The stable polynomials D_c, D_f and D_r are defined from three spectral factorisation equations (17), (18), and (19).

Each of the controller-dependent terms in (A.2) may now be simplified separately as follows.

Q_y dependent term We can manipulate the second and third terms in the term Q_1. These can be separated in

$$\frac{\psi D_fB^*F_wX}{MD_c^*} - \frac{\varphi D_fA^*F^*F_w^*F_wY}{MD_c^*} = \frac{Q_{yn}F_w}{M} + \frac{V^*}{D_c^*} \quad \text{(A.3)}$$

Finally, the term Q_1 may be expressed as

$$Q_1 = \left(\frac{D_f D_c F_w}{M_1 M_1} Q_y - \frac{Q_{yn} F_w}{M}\right) - \frac{V^*}{D_c^*} = S_1^+ + S_1^- \quad \text{(A.4)}$$

where S_1^+ denotes the term in brackets and S_1^- denotes strictly unstable term.

Q_r dependent term We can manipulate the second and third terms in the term Q_2. These can be separated in

$$\frac{\psi D_r B^*}{F_w D_c^*} - \frac{D_r D_c R}{M F_w} = \frac{Q_{rn}}{M F_w^+} + \frac{T^*}{D_c^* F_w^-} \quad \text{(A.5)}$$

Finally, the term Q_2 may be expressed as

$$Q_2 = \left(\frac{D_r D_c}{M_1 M_1} Q_r - \frac{Q_{rn}}{M F_w^+}\right) - \frac{T^*}{D_c^* F_w^-} = S_2^+ + S_2^- \quad \text{(A.6)}$$

where S_2^+ denotes the term in brackets and S_2^- denotes strictly unstable term.

A.1 Minimisation

Substituting from (A.4) and (A.6) into (A.2), the cost function integrand may be written as

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (S_{11} S_{11}^* + S_{22} S_{22}^* + y_d)\, ds \quad \text{(A.7)}$$

$$S_{11} = S_1^+ + S_1^-, \quad S_{22} = S_2^+ + S_2^-$$

In (A.7) S_i^+, S_i^- denote strictly stable and unstable terms, respectively. Therefore, the terms $S_i^+ S_i^{-*}$ are analytic in $Re(s) \geq 0$. Thus, using the identity

$$\int S_i^- S_i^{+*} ds = \int S_i^+ S_i^{-*} ds \quad \text{(A.8)}$$

and invoking Cauchy's theorem, the integrals of the cross terms $S_i^- S_i^{+*}$, $S_i^+ S_i^{-*}$ in (A.7) are zero. The cost function therefore simplifies to

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[\sum_{i=1}^{2} (S_i^+ S_i^{+*} + S_i^- S_i^{-*}) + y_d\right] ds \quad \text{(A.9)}$$

Since the terms S_i^- and y_d are independent of the controller, the cost function J is minimised by setting

$$S_i^+ = 0, \qquad i = 1, 2 \quad \text{(A.10)}$$

Feedback controller From (A.4), setting $S_1^+ = 0$ results in

$$\frac{D_f D_c F_w}{M_1 M_1} Q_y - \frac{Q_{yn} F_w}{M_1 M_2} = 0 \quad \text{(A.11)}$$

or

$$Q_y = \frac{Q_{yn}}{D_c D_f} \frac{M_1}{M_2} \quad \text{(A.12)}$$

Feedforward controller From (A.6), setting $S_2^+ = 0$ gives

$$\frac{D_r D_c}{M_1 M_1} Q_r - \frac{Q_{rn}}{M_1 M_2 F_w F_w^+} = 0 \quad \text{(A.13)}$$

or

$$Q_r = \frac{Q_{rn}}{D_c D_r F_w^+} \frac{M_1}{M_2} \quad \text{(A.14)}$$

This concludes the proof.