AN ITERATIVE METHOD FOR ROBUST PERFORMANCE ANALYSIS OF SAMPLED-DATA SYSTEMS AGAINST PARAMETER UNCERTAINTIES

Tomomichi Hagiwara † **Kazuhiro Horiguchi** †

Department of Electrical Engineering, Kyoto University, Kyotodaigaku-Katsura, Nishikyo-ku, Kyoto 615-8510, Japan

Abstract: This paper is concerned with robust performance analysis of sampled-data systems. In particular, we consider the uncertainty of a parameter in the continuous-time plant and study the problem of determining the allowable range of the uncertainty around the origin over which the sampled-data system remains internally stable and a prescribed L_2-induced norm level is retained. We provide an effective iterative procedure with guaranteed convergence that gives an exact allowable parameter range, together with rigorous arguments for the convergence.

Keywords: sampled-data systems, robust stability, robust performance, parameter uncertainties, convergence analysis

1. INTRODUCTION

Robust stability and robust performance are very fundamental issues in control system analysis and synthesis, and the widespread use of digital controllers these days has led us to the treatment of control systems as sampled-data systems to study these issues, in which the intersample behavior is dealt with directly and exactly. Among such studies about sampled-data systems are (Chen and Francis, 1991; Bamieh and Pearson, 1992; Toivonen, 1992; Kabamba and Hara, 1993; Dullerud and Glover, 1993; Sivashankar and Khargonekar, 1993; Yamamoto, 1994), just to name a few. In particular, (Sivashankar and Khargonekar, 1993) is a pioneering study in robust performance analysis of sampled-data systems, which gave a necessary and sufficient condition for robust performance by extending the so-called main loop theorem into the context of sampled-data systems.

As for robust internal stability of sampled-data systems against parameter uncertainties, a novel and efficient iterative method was provided recently in (Hagiwara and Mugiuda, 2004), which is guaranteed to be convergent in the sense that the stability margin with respect to the parameter uncertainties can be determined exactly. This paper aims at extending that related study so that not only robust stability but also robust performance can be taken into account. To be more precise, we study to determine the allowable range of a parameter in the continuous-time plant for which the sampled-data system remains internally stable and at the same time a prescribed performance level is retained between the given disturbance input w and controlled output z under the measure of the L_2-induced norm. The method we provide is also an iterative method with guaranteed convergence that is free from the gridding of the parameter space, and thus is exact. In other words, the significance of this paper can be regarded as providing a specific and exact numerical computation procedure that achieves the conceptual robust performance test suggested by the main loop theorem for sampled-data systems (Sivashankar and Khargonekar, 1993) mentioned above.

The contents of this paper are as follows. In Section 2, we review some fundamental results on robust stability of sampled-data systems. Section 3 constitutes the main contributions of this paper. In Subsection 3.1, we first formulate the problem we study in this paper. Next in Subsection 3.2, we consider the scaling of sampled-data systems with two parameters, and give some fundamental results with rigorous proofs for robust performance analysis. Then in Subsection 3.3, we introduce an iterative method for robust performance analysis of sampled-data systems against parameter uncertainties, and provide a proof for the convergence of the iterative method. Numerical examples are studied in Section 4.

For notational simplicity, $\|\cdot\|$ is used throughout the paper to denote the L_2 norm of a signal and the L_2-induced norm of an input-output mapping; the distinction will be clear from the context.

2. ROBUST STABILITY OF SAMPLED-DATA SYSTEMS

This section is devoted to reviewing some fundamental results on robust stability of sampled-data systems.

Let us consider the sampled-data system Σ shown in Fig. 1, in which P denotes the continuous-time generalized plant, Ψ the discrete-time controller, $\mathcal{S}$ the ideal sampler, and $\mathcal{H}$ the zero-order hold. Also, solid lines and dashed lines represent continuous-time and discrete-time (vector) signals, respectively, and the underlying sampling period

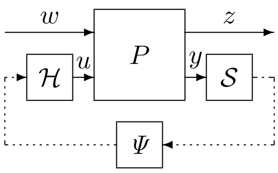

Fig. 1. Open-loop sampled-data system Σ.

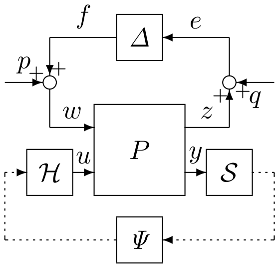

Fig. 2. Closed-loop sampled-data system Σ_Δ.

will be denoted by h. We assume that the state-space representations of P and Ψ are given respectively by

$$\frac{dx}{dt} = Ax + B_1 w + B_2 u,\ z = C_1 x + D_{11} w + D_{12} u,$$
$$y = C_2 x \tag{1}$$

and

$$\xi_{k+1} = A_\Psi \xi_k + B_\Psi y_k, \quad u_k = C_\Psi \xi_k + D_\Psi y_k \tag{2}$$

where $y_k = y(kh)$ and $u(t) = u_k$ $(kh \le t < (k+1)h)$. In other words, we assume that the direct feedthrough matrices D_{21} from w to y and D_{22} from u to y are both zero, which are standard assumptions in the study of sampled-data systems.

For lack of better terminologies, we call Σ an open-loop sampled-data system, while if w is given as $w = \Delta z$ with some causal mapping Δ, then we call the resulting system a closed-loop sampled-data system, which we denote by Σ_Δ. Also, the associated input-output mapping from $[p^T,\ q^T]^T$ to $[f^T,\ z^T]^T$ in Fig. 2 will be denoted by $\mathcal{G}_\Delta$ if it is well-defined. If $\mathcal{G}_\Delta$ maps L_2 into L_2, and if its L_2-induced norm is bounded, then $\mathcal{G}_\Delta$ is said to be L_2-stable. Here, we have the following result about stability of closed-loop sampled-data systems (Hagiwara, 2002).

Proposition 1. Suppose that Δ is a finite-dimensional linear time-invariant or h-periodic system and that both Σ and Δ are internally stable. Then, Σ_Δ is internally stable if and only if $\mathcal{G}_\Delta$ is L_2-stable.

For the sake of notational simplicity, we denote by $\boldsymbol{\Delta}(\gamma; m, n)$ the set of all linear causal h-periodic system Δ with input dimension n and output dimension m such that the L_2-induced norm $\|\Delta\|$ is strictly less than γ. Then, we have the following result, which is essentially the necessity assertion of Theorem 5.1 in (Sivashankar and Khargonekar, 1993) but is stated in a specialized form that is useful in the following arguments [†].

Proposition 2. Suppose that Σ is internally stable. If for some $\gamma > 0$, the input-output mapping $\mathcal{G}_\Delta$ associated with the closed-loop sampled-data system Σ_Δ in Fig. 2 is L_2-stable for all $\Delta \in \boldsymbol{\Delta}(\gamma; \dim(w), \dim(z))$, then $\|\Sigma\| \le 1/\gamma$.

3. ROBUST PERFORMANCE ANALYSIS OF SAMPLED-DATA SYSTEMS

3.1 Problem Formulation

In this section, we consider the (semi)closed-loop sampled-data system Φ_Δ shown in Fig. 3. Here, the causal mapping Δ represents the uncertainty in the continuous-time plant, while w and z denote the disturbance input and the controlled output, respectively. In other words, we suppose that the generalized plant Π with η set to $\Delta\zeta$, which we denote by Π_Δ, corresponds to the generalized plant P in Σ, and thus Φ_Δ corresponds to Σ with the continuous-time generalized plant being subject to some uncertainty represented by Δ; by dealing with Φ_Δ, we are interested in analyzing the disturbance rejection performance $\|\Phi_\Delta\|$ attained by the discrete-time controller Ψ under the presence of the plant uncertainty Δ. To facilitate the following arguments, we also consider the open-loop sampled-data system shown in Fig. 4, which we denote by Φ. Note carefully the difference between Φ and Φ_0 (i.e., Φ_Δ with Δ set to 0); the dimension of the input of Φ_Δ is $\dim(w)$ (even if $\Delta = 0$) while that of Φ is $\dim(w) + \dim(\eta)$, and similarly for the output dimensions. Further note that the same comments apply also to Π and Π_Δ.

To conform to the assumptions about the direct feedthrough matrices D_{21} and D_{22} stated in the preceding section, we assume that the generalized plant Π satisfies the condition that the direct feedthrough matrices to y from w, η and u are all zero. Also, we assume that Δ is a real matrix, which corresponds to considering the uncertainties in the real parameters of the plant. For the sake of simplicity, we further assume that $\Delta = kI$, assuming that $\dim(\eta) = \dim(\zeta)$, where k is a real scalar. In spite of this restrictive assumption,

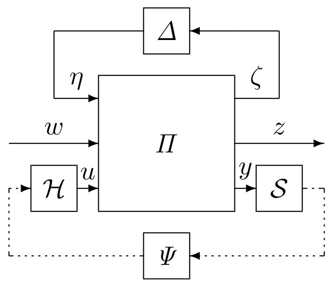

Fig. 3. Sampled-data system Φ_Δ.

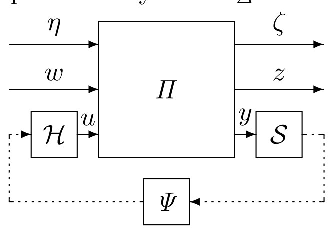

Fig. 4. Sampled-data system Φ.

[†] In Theorem 5.1 of (Sivashankar and Khargonekar, 1993), it was proved that $\|\Sigma\| \le 1/\gamma$ follows under a robust stability condition that is slightly stronger than the robust stability condition of $\mathcal{G}_\Delta$ assumed in Proposition 2 (i.e., discrete-time external inputs were also taken into account there). Fortunately, it is not hard to see that the necessity proof therein remains valid even under the relaxed condition in Proposition 2.

the significance of this paper lies in establishing that under such a situation, we can perform *exact* and effective robust performance analysis via an iterative method, including rigorous arguments for the convergence proof about the iteration.

In the sequel, we assume $k > 0$ without loss of generality. Also, when $\Delta = kI$, we simply denote Φ_Δ by Φ_k (rather than Φ_{kI}) for simplicity; similar shorthand notations are used throughout the paper.

We are now in a position to state our standing assumption, as well as the formulation of the problem studied in this paper.

Assumption 1. The discrete-time controller Ψ is internally stabilizing. That is, Φ (or equivalently, Φ_0) is internally stable. Furthermore, $\|\Phi_0\| < 1$. That is, the L_2-induced norm of the subsystem from w to z is strictly less than 1 in Φ.

Problem 1. Find the number $k_{\max}$, which is defined as the largest $\bar{k} > 0$ such that Φ_k is internally stable and $\|\Phi_k\| < 1$ for all $k \in [0, \bar{k})$.

A similar problem can readily be considered for $k < 0$ by slightly modifying the generalized plant Π (i.e., by "replacing η by $-\eta$"), and the pair of these two problems corresponds to finding the range of the real parametric uncertainty k for which Φ_k remains internally stable and the performance level $\|\Phi_k\| < 1$ is retained. It is obvious that for any $\gamma > 0$, a more general performance level $\|\Phi_k\| < \gamma$ can also be dealt with by appropriately scaling the generalized plant Π.

3.2 Robust Performance Analysis via Two-Parameter Scaling

We consider the system $\Phi(\alpha, \beta)$ shown in Fig. 5, which is obtained by scaling Φ with nonnegative numbers α and β. When $\beta = 1$, we denote $\Phi(\alpha, \beta)$ simply by $\Phi(\alpha)$, and when $\alpha = 1$, we denote $\Phi(\alpha, \beta)$ simply by $\Phi(\beta)$; in spite of the abuse of the same notation, the input-weighted system $\Phi(\alpha)$ and the output-weighted system $\Phi(\beta)$ could be distinguished from the context (or the argument α or β). It is not hard to show the following fundamental result from the definition of the L_2-induced norm and linearity of the input-output mapping of Φ.

Lemma 1. Suppose that Φ is internally stable. Then, $\|\Phi(\alpha, \beta)\|$ is continuous in $\{(\alpha, \beta) \,|\, \alpha \geq 0,\ \beta \geq 0\}$. Furthermore, $\|\Phi(\alpha)\|$ and $\|\Phi(\beta)\|$ are nondecreasing with respect to $\alpha \geq 0$ and $\beta \geq 0$, respectively.

Proof. It is straightforward to show that $\|\Phi(\beta)\|$ is nondecreasing if we note that $\|[\tilde{\beta}\zeta^T, z^T]^T\| \geq \|[\beta\zeta^T, z^T]^T\|$ whenever $\tilde{\beta} \geq \beta$. To establish that $\|\Phi(\alpha)\|$ is nondecreasing is also straightforward if we use the same property on η and w, but it takes space to describe the full details, and hence the proof is omitted. The first assertion is also easy

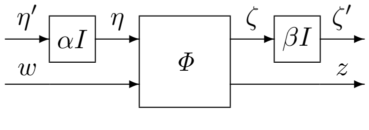

Fig. 5. Scaled system $\Phi(\alpha, \beta)$.

to establish, but since giving its proof is helpful to some extent for the subsequent arguments, we devote some space for the proof. For notational simplicity, let us partition Φ into $(\Phi)_{ij}$ $(i = 1, 2; j = 1, 2)$ conformably to the input $(\eta^T, w^T)^T$ and the output $(\zeta^T, z^T)^T$ (hence, Φ_{22} is nothing but Φ_0). Also, let us denote by (ζ', z) and $(\tilde{\zeta}', \tilde{z})$ the output of $\Phi(\alpha, \beta)$ and $\Phi(\tilde{\alpha}, \tilde{\beta})$, respectively, to the common input (η', w). Then, by linearity of Φ, it readily follows that

$$
\begin{aligned}
\left| \left\| \begin{bmatrix} \tilde{\zeta}' \\ \tilde{z} \end{bmatrix} \right\| - \left\| \begin{bmatrix} \zeta' \\ z \end{bmatrix} \right\| \right| &\leq \left\| \begin{bmatrix} \tilde{\zeta}' \\ \tilde{z} \end{bmatrix} - \begin{bmatrix} \zeta' \\ z \end{bmatrix} \right\| \\
&\leq (\tilde{\alpha}\tilde{\beta} - \alpha\beta)\|\Phi_{11}\| \cdot \|\eta'\| + (\tilde{\alpha} - \alpha)\|\Phi_{21}\| \cdot \|\eta'\| \\
&\qquad + (\tilde{\beta} - \beta)\|\Phi_{12}\| \cdot \|w\| \\
&\leq M \left\| \begin{bmatrix} \eta' \\ w \end{bmatrix} \right\| \qquad (3)
\end{aligned}
$$

where $M := (\tilde{\alpha}\tilde{\beta} - \alpha\beta)\|\Phi_{11}\| + (\tilde{\alpha} - \alpha)\|\Phi_{21}\| + (\tilde{\beta} - \beta)\|\Phi_{12}\|$. Since $\|\Phi_{ij}\|$ $(i = 1, 2; j = 1, 2)$ is finite and independent of (α, β), it is easy to see that M can be made arbitrarily small by letting $|(\tilde{\alpha}, \tilde{\beta}) - (\alpha, \beta)|$ small enough. Hence, the assertion follows immediately. ■

We now state the following result, which forms a basis of our robust performance analysis of sampled-data systems.

Theorem 1. Suppose that Φ is internally stable and $\|\Phi(\alpha, \beta)\| = 1$ for some $\alpha > 0$ and $\beta > 0$. Further suppose that either (or both) of the following two conditions holds:

(i) $\|\Phi(\alpha', \beta)\| < 1$ whenever $0 \leq \alpha' < \alpha$.
(ii) $\|\Phi(\alpha, \beta')\| < 1$ whenever $0 \leq \beta' < \beta$.

Then, Φ_k is internally stable and $\|\Phi_k\| < 1$ whenever $0 \leq k < \alpha\beta$.

Remark 1. The result may not be surprising in the sense that a similar result could be obtained by applying the small-gain theorem in terms of the L_2 norm, together with Theorem 6.1 (a kind of the so-called main loop theorem) in (Sivashankar and Khargonekar, 1993), which is a pioneering work in the robust stability and performance analysis of sampled-data systems. However, such arguments will not immediately and explicitly lead to the internal stability assertion of Φ_k $(0 \leq k < \alpha\beta)$, as long as the statement of Theorem 6.1 in (Sivashankar and Khargonekar, 1993) and related definitions and arguments are scrutinized. This issue could be resolved if we apply Proposition 1, but another difficulty in the arguments based on Theorem 6.1 of (Sivashankar and Khargonekar, 1993) is that an immediate direct consequence will be merely $\|\Phi_k\| \leq 1$ $(0 \leq k < \alpha\beta)$, which is a slightly weaker assertion because the inequality is not strict; the strict inequality $\|\Phi_k\| < 1$ in the above theorem will turn out to be a key that assists our rigorous discussions about an iterative method developed in the following subsection for the robust performance analysis. Even though we do not mean to degrade Theorem 6.1 of (Sivashankar and Khargonekar, 1993) by these comments since it could be modified to alleviate the difficulties, here we take a somewhat different approach that is independent of Theorem 6.1 in (Sivashankar and Khargonekar, 1993) to establish this theorem. That is, we give a proof by means of Propositions 1 and 2, in which the condition about α' and β' plays a crucial role.

Remark 2. One might claim that the properties (i) and (ii) of this theorem are direct consequences from $\|\Phi(\alpha,\beta)\| = 1$ and thus (i) and (ii) always hold. However, since $\Phi(\alpha,\beta)$ fails to be *strictly* increasing with respect to α and/or β, in general (too see this, consider the case $\Phi = \mathrm{diag}[0.1,\ 1]$, for example), it is not straightforward to see if the claim suggested above is indeed correct. We do not go into the details of this issue, since the "additional" condition (i)/(ii) does not in fact lead to any difficulty in finding appropriate α and β, as will be shown shortly.

Proof of Theorem 1. We only consider the case in which $\|\Phi(\alpha',\beta)\| < 1$ for all $0 \le \alpha' < \alpha$ (parallel arguments can be applied to the case in which $\|\Phi(\alpha,\beta')\| < 1$ for all $0 \le \beta' < \beta$). If $0 \le k < \alpha\beta$, then such k can be represented as $k = \alpha'\beta$ for some $0 \le \alpha' < \alpha$. By the assumption stated just above, it follows that $\|\Phi(\alpha',\beta)\| < 1 - \varepsilon$ for some $\varepsilon > 0$. Let us introduce $\Delta_0 := \mathrm{diag}[I_{\dim(\zeta)}, 0_{\dim(w),\dim(z)}]$, which satisfies $\|\Delta_0\| = 1$. Then, by the small-gain theorem, the closed-loop system consisting of $\Phi(\alpha',\beta)$ and Δ_0 is L_2-stable, and thus internally stable by Proposition 1. Recalling that $k = \alpha'\beta$, however, it is straightforward to see that the internal stability of this closed-loop system is equivalent to that of Φ_k. This completes the proof of the first assertion.

To prove the second assertion, we fix k and thus ε, as in the above discussions. Let us take $\Delta_1^{wz} \in \boldsymbol{\Delta}((1-\varepsilon)^{-1}; \dim(w), \dim(z))$ and consider $\Delta_1 := \mathrm{diag}[I_{\dim(\zeta)}, \Delta_1^{wz}]$, which satisfies $\|\Delta_1\| < (1-\varepsilon)^{-1}$. Recalling that $\|\Phi(\alpha',\beta)\| < 1-\varepsilon$ by the assumption, it follows from the small-gain theorem that the closed-loop system consisting of $\Phi(\alpha',\beta)$ and Δ_1 is L_2-stable for any $\Delta_1^{wz} \in \boldsymbol{\Delta}((1-\varepsilon)^{-1}; \dim(w), \dim(z))$. However, it is straightforward to see that the L_2-stability of this closed-loop system in particular implies that of the closed-loop system consisting of Φ_k and Δ_1^{wz} because $k = \alpha'\beta$. Since we have already established that Φ_k is internally stable, we can apply Proposition 2 to arrive at the conclusion that $\|\Phi_k\| \le 1 - \varepsilon < 1$. ∎

Theorem 1 clearly indicates that for the robust performance analysis or the computation of $k_{\max}$, it is important to find $\alpha > 0$ and $\beta > 0$ satisfying $\|\Phi(\alpha,\beta)\| = 1$ together with the property (i) or (ii), while Lemma 1 suggests that such α and β can be found with a bisection method. Note, however, that $\|\Phi(0,0)\| = \|\Phi_0\| < 1$ by Assumption 1 while, in general, $\|\Phi(\alpha)\| = \|\Phi(\alpha,1)\| \not< 1$ for $\alpha = 0$ and $\|\Phi(\beta)\| = \|\Phi(1,\beta)\| \not< 1$ for $\beta = 0$, which is why we introduce both α and β at the same time. Here we give one such "two-parameter bisection procedure" for the computation of α and β satisfying the required condition; even though it is a straightforward extension of well-known bisection methods, its details will be described explicitly below since they will be a matter of concern in the convergence arguments that we bring forward about the iterative algorithm we provide for the computation of $k_{\max}$ defined in Problem 1. In the bisection procedure, we assume that the initial values of α and β are positive and *coincide with each other*; this assumption will turn out to be quite important in the convergence arguments.

For simplicity, the procedure given below is described only for the case when $\|\Phi(\alpha,\beta)\| < 1$ for the initial values; if $\|\Phi(\alpha,\beta)\| \ge 1$, on the other hand, then Step 0 and Step 1 should be modified in the following fashion: 2α and 2β are replaced by $\alpha/2$ and $\beta/2$, respectively; the conditions $\|\Phi(\alpha,\beta)\| < 1$ and $\|\Phi(\alpha,\beta)\| \ge 1$ are mutually interchanged; the upper bounds $\bar\alpha$ and $\bar\beta$ are interchanged with the lower bounds $\underline{\alpha}$ and $\underline{\beta}$ and vice versa.

Procedure 1. (Bisection procedure for computing appropriate weights α and β)

(Step 0) Set the initial values and initial lower bounds $\underline{\alpha} = \alpha = \underline{\beta} = \beta$ (such that $\|\Phi(\alpha,\beta)\| < 1$).

(Step 1) Redefine α by 2α, and execute either (a) or (b) below that is appropriate.

(a) If $\|\Phi(\alpha,\beta)\| \ge 1$, then let $\bar\alpha := \alpha$ and go to Step 2.

(b) If $\|\Phi(\alpha,\beta)\| < 1$, then redefine β by 2β. If $\|\Phi(\alpha,\beta)\| < 1$ still holds, then return to Step 1. If $\|\Phi(\alpha,\beta)\| \ge 1$, then let $\bar\beta := \beta$ and go to Step 3.

(Step 2) Let $\alpha := (\bar\alpha + \underline{\alpha})/2$, and execute either (a) or (b) below that is appropriate.

(a) If $\|\Phi(\alpha,\beta)\| < 1$, then let $\underline{\alpha} := \alpha$.

(b) If $\|\Phi(\alpha,\beta)\| \ge 1$, then let $\bar\alpha := \alpha$.

If $\bar\alpha - \underline{\alpha}$ is sufficiently small, then stop after letting $\alpha = \underline{\alpha}$. Return to Step 2, otherwise.

(Step 3) Let $\beta := (\bar\beta + \underline{\beta})/2$, and execute either (a) or (b) below that is appropriate.

(a) If $\|\Phi(\alpha,\beta)\| < 1$, then let $\underline{\beta} := \beta$.

(b) If $\|\Phi(\alpha,\beta)\| \ge 1$, then let $\bar\beta := \beta$.

If $\bar\beta - \underline{\beta}$ is sufficiently small, then stop after letting $\beta = \underline{\beta}$. Return to Step 3, otherwise.

Remark 3. Some tolerance is allowed for the errors between the upper and lower bounds of α and β in Steps 2 and 3 for numerical reasons, which ensures that the procedure will terminate unless there exist no α and β such that $\|\Phi(\alpha,\beta)\| = 1$, or equivalently[‡] $k_{\max} = \infty$. On the other hand, however, the tolerance leads to $\|\Phi(\alpha,\beta)\| < 1$ rather than $\|\Phi(\alpha,\beta)\| = 1$ for the resulting weights, strictly speaking (even though the limiting values of α and β, if such tolerance is not introduced, indeed satisfy all the conditions of Theorem 1). Fortunately, we can see by the inspection of the proof of Theorem 1 that we can still assert for such α and β that Φ_k is internally stable and $\|\Phi_k\| < 1$ whenever $0 \le k < \alpha\beta$, at the sacrifice of the bound $\alpha\beta$ becoming a little more conservative. Furthermore, we can readily see from the construction of this procedure that

$$\frac{1}{2} \le \frac{\beta}{\alpha} \le 2 \tag{4}$$

since the initial values of α and β coincide by the assumption. The above inequality plays a crucial role in the convergence arguments to follow.

3.3 Iterative Procedure with Convergence for Robust Performance Analysis

Suppose we apply Procedure 1 and get α and β, which we denote by α_0 and β_0, respectively. Then, it is immediate from Theorem 1 that Φ_k is internally stable and $\|\Phi_k\| < 1$ whenever $0 \le$

[‡] If $\|\Phi(\alpha,\beta)\| \ne 1$, then it follows from Lemma 1 and Assumption 1 that $\|\Phi(\alpha,\beta)\| < 1$ for all $\alpha \ge 0$ and $\beta \ge 0$. Hence, by following similar arguments to the proof of Theorem 1, we are led to $k_{\max} = \infty$.

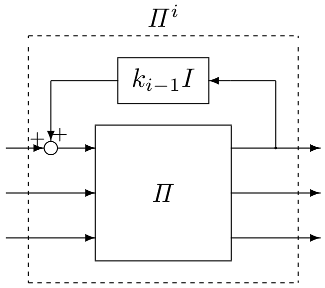

Fig. 6. Modified generalized plant Π^i.

$k < k_{\max(0)} := \alpha_0\beta_0$. Obviously, however, $k_{\max(0)}$ is not equal to $k_{\max}$, in general; that is, we only have $k_{\max(0)} \le k_{\max}$. In this subsection, we aim at exact computation of $k_{\max}$ via an iterative method.

To this end, let us introduce $\Pi^0 := \Pi$, $\Phi^0 := \Phi$, and $k_0 := k_{\max(0)}$ for notational convenience. Also, let us consider the generalized plant Π^1 shown in Fig. 6 (with i set to 1), and let us denote by Φ^1 the sampled-data system Φ shown in Fig. 4 with the generalized plant Π replaced by Π^1. The idea here is that if Φ^1 is internally stable and if $\|\Phi^1_0\| < 1$ [††], or equivalently, if Assumption 1 holds with $\Phi = \Phi^0$ replaced by Φ^1, then we can apply Procedure 1 to Φ^1 to get α and β, which we denote by α_1 and β_1, respectively. Then, it immediately follows that Φ^1_k is internally stable and satisfies $\|\Phi^1_k\| < 1$ whenever $0 \le k < k_{\max(1)} := \alpha_1\beta_1$. However, noting that Φ^1_k is nothing but $\Phi_{k_{\max(0)}+k}$ by the structure of Π^1, we are led to the consequence that Φ_k is internally stable and satisfies $\|\Phi_k\| < 1$ whenever $0 \le k < k_1 := k_{\max(0)} + k_{\max(1)}$.

Obviously, the above idea can be repeated for each $i = 0, 1, 2, \cdots$ recursively to get α_i, β_i, $k_{\max(i)} := \alpha_i\beta_i$,

$$k_i := \sum_{l=0}^{i} k_{\max(l)} \tag{5}$$

Π^{i+1} and Φ^{i+1} (which is defined as Φ with Π replaced by Π^{i+1}), provided that Φ^i is internally stable and $\|\Phi^i_0\| < 1$.

Here, it is obvious from the definition of $k_{\max}$ that k_{i-1} gives exactly $k_{\max}$ if either Φ^i is internally unstable or $\|\Phi^i_0\| \ge 1$. Thus, let us suppose that Φ^i is indeed internally stable and satisfies $\|\Phi^i_0\| < 1$ for all $i = 0, 1, 2, \cdots$, and thus the above iteration repeats itself infinitely many times. In this case, we have the following important result on the convergence of the strictly increasing sequence $\{k_i\}_{i=0}^{\infty}$.

Theorem 2. $\lim_{i\to\infty} k_i = k_{\max}$. In particular, if $k_{\max} = \infty$, then $\{k_i\}_{i=0}^{\infty}$ diverges to ∞.

Proof. Let us denote $k^\star := \lim_{i\to\infty} k_i$. It is obvious from the construction of $\{k_i\}_{i=0}^{\infty}$ that $k^\star \le k_{\max}$. Hence it is enough to establish that assuming $k^\star < \infty$ and $k^\star < k_{\max}$ leads to contradiction.

Now, since $k^\star < \infty$ and since $\{k_i\}_{i=0}^{\infty}$ is strictly increasing, we can take a number $K > 0$ and an integer $i_0 > 0$ such that $k_i \in [K, k^\star] =: \mathbf{K}$ $(\forall i \ge i_0)$. Since $k^\star < k_{\max}$ by the assumption, it follows that for all $k \in \mathbf{K}$, we have $k < k_{\max}$, or equivalently, Φ_k is internally stable and $\|\Phi_k\| < 1$. Here, let us denote by $\Pi_{[k]}$ the modified generalized plant Π^i in Fig. 6 *with $k_{i-1}I$ replaced by kI* (which is similar to Π_k but still retains the input and output corresponding to η and ζ), and let us denote by $\Phi_{[k]}$ the corresponding sampled-data system Φ with Π replaced by $\Pi_{[k]}$. Then, we can restate the preceding consequence as follows: $\Phi_{[k]}$ is internally stable and $\|\Phi_{[k]}(0,0)\| < 1$ whenever $k \in \mathbf{K}$. Hence, it follows from Lemma 1 that for each $k \in \mathbf{K}$, there exists some $a > 0$ such that $\|\Phi_{[k]}(\alpha,\beta)\| < 1$ $(0 \le \forall\alpha \le a,\ 0 \le \forall\beta \le a)$. More precisely, we can show that such a can be taken independently of $k \in \mathbf{K}$ by establishing (with a technique similar to (3), together with the continuity of $\|\Phi_k\|$ in $k \in \mathbf{K}$) that $\|\Phi_{[k]}(\alpha,\beta)\|$ is continuous in $\{(\alpha,\beta,k)\,|\,\alpha \ge 0, \beta \ge 0, k \in \mathbf{K}\}$.

We are now ready to complete the proof. Let us recall the inequality (4) to see that (α_i, β_i) is always located in the sector shown in Fig. 7, while for $i \ge i_0$, it is located outside the square shown in the figure, so that it is located within the shaded area. This in particular implies that $\alpha_i > a/2$ and $\beta_i > a/2$ $(\forall i \ge i_0)$. Hence, it follows that $k_{\max(i)} = \alpha_i\beta_i > a^2/4$ $(\forall i \ge i_0)$, which clearly contradicts the assumption that $\{k_i\}_{i=0}^{\infty}$ converges to $k^\star < \infty$. This completes the proof. ■

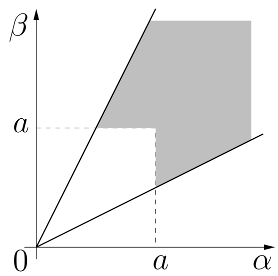

Fig. 7. Possible locations of (α_i, β_i).

4. NUMERICAL EXAMPLES

In this section, we study numerical examples to illustrate the proposed iterative method for robust performance analysis.

Example 1. Let us consider the generalized plant Π with the structure shown in Fig. 8. We assume that this Π (or equivalently, Π') is constructed in such a way that when η is set to $k\zeta$ with $k = 0$, the transfer function from u to y coincides with the nominal transfer function

$$\frac{c}{s^2 + as + b} \quad (a = 1.5,\ b = 0.5,\ c = 2) \tag{6}$$

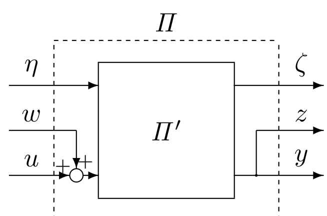

Fig. 8. Generalized plant Π for the example.

[††] Recall that Φ^1_0 denotes Φ^1_Δ with Δ set to 0, where Φ^1_Δ is defined as the system Φ_Δ shown in Fig. 3 with the generalized plant Π replaced by Π^1. Hence, this inequality implies that the L_2-induced norm of the subsystem from w to z in Φ^1 is strictly less than 1.

while when $k \neq 0$, it coincides with the actual transfer function represented by

$$\frac{c_k}{s^2+as+b}, \quad c_k = (1-k)c \tag{7}$$

(i.e., k corresponds to the uncertainty in the parameter c). Let the sampling period be $h = 0.5$ and the stabilizing controller Ψ for the nominal transfer function be

$$\Psi(z) = \frac{\psi_{n0}z^2 + \psi_{n1}z + \psi_{n2}}{z^2 + \psi_{d1}z + \psi_{d2}},$$
$$\psi_{n0} = -7.5079, \psi_{n1} = 2.2897, \psi_{n2} = -4.7746 \times 10^{-12}$$
$$\psi_{d1} = 0.74353,\ \psi_{d2} = -1.5506 \times 10^{-12} \tag{8}$$

with which we have the nominal performance $\|\Phi_0\| = 0.309 < 1$.

We apply the iterative method given in the preceding section, in which we set the initial values of the weights α and β in the ith iteration by

$$\alpha_i = \beta_i = (\alpha_{i-1}\beta_{i-1})^{1/2} \tag{9}$$

to conform to the crucial assumption for the iterative method that the initial values of α and β should coincide with each other[‡‡], and at the same time, to accelerate the convergence of (α_i, β_i). In this case, the proposed iterative method gives $k_{\max} = 1.0626$, which implies that internal stability and robust performance (i.e., the L_2-induced norm from w to z being less than 1) of the sampled-data system are retained for $((1-k_{\max})c, c] = (-0.1252, 2]$. By slightly modifying Π (i.e., putting the gain -1 at the input η) and repeating similar computations, we are led to $k'_{\max} = 0.2942$ and thus $[c, (1+k'_{\max})c) = [2, 2.5884)$. Hence, the allowable range for the parameter c is determined to be $(-0.1252, 2.5884)$. Indeed, we can verify that the sampled-data system is internally stable and its L_2-induced norm from w to z is indeed equal to 1 at $c = -0.1252$ and 2.5884, which proves that our analysis is exact. It should be noted that while c decreases from 2 to -0.1252, it takes the value $c = 0$, at which the L_2-induced norm is obviously 0. This in particular implies that the L_2-induced norm is not monotonic with respect to the parameter c.

By similar techniques, we can also obtain the exact allowable ranges for the parameters a and b, which are given by $(-11.9487, 20.7727) \ni 1.5$ and $(0.1029, 0.8432) \ni 0.5$, respectively.

Example 2. Let us consider the continuous-time plant (Anderson and Moore, 1990) whose transfer function is given by

$$\frac{1}{4s^2}\cdot\frac{(s/a+1)\prod_{i=0}^{1}\{(s/\omega_i)^2 + 2\zeta_i(s/\omega_i)+1\}}{\prod_{i=2}^{4}\{(s/\omega_i)^2 + 2\zeta_i(s/\omega_i)+1\}} \tag{10}$$

with $a = 4.84$, $\zeta_0 = 0.02$, $\zeta_1 = -0.4$, $\zeta_2 = \zeta_3 = \zeta_4 = 0.02$, $\omega_0 = 1$, $\omega_1 = 5.65$, $\omega_2 = 0.765$, $\omega_3 = 1.41$, $\omega_4 = 1.85$, and the discrete-time controller obtained by applying the Tustin transformation at $h = 8$ to the continuous-time controller (Anderson and Moore, 1990) whose transfer function is given by

$$-\frac{0.0513s^3 + 0.00424s^2 + 0.0296s + 0.00157}{s^4 + 0.693s^3 + 0.779s^2 + 0.293s + 0.0739} \tag{11}$$

[‡‡] Indeed, our numerical studies show that failing to follow this rule and setting the initial values of α and β at the ith iteration simply by α_{i-1} and β_{i-1}, respectively, lead to the lack of the convergence of $\{k_i\}_{i=0}^{\infty}$ to $k_{\max}$, in general.

Suppose that the above nominal plant is embedded in the generalized plant Π with the structure shown in Fig. 8 exactly in the same manner as in Example 1 (i.e., as the subsystem from u to y), and let us analyze the allowable ranges of parameters, e.g., ζ_i $(i = 0, \cdots, 4)$, by suitably "completing" the generalized plant Π accordingly. More specifically, since the nominal performance is $\|\Phi_0\| = 111.8$, we aim at analyzing the ranges of these parameters under which the L_2-induced norm from w to z is less than 120.

Applying the proposed iterative method, we obtain the allowable ranges given respectively by

$$\mathbf{Z}_0 = \{-2.2334\times10^{-2} < \zeta_0 < 2.6788)\}\ (\ni 0.02) \tag{12}$$
$$\mathbf{Z}_1 = \{-0.67646 < \zeta_1 < 15.522)\}\ (\ni -0.4) \tag{13}$$
$$\mathbf{Z}_2 = \{1.9490\times10^{-3} < \zeta_2 < 0.045436)\}\ (\ni 0.02) \tag{14}$$
$$\mathbf{Z}_3 = \{7.7280\times10^{-4} < \zeta_3 < 0.10699)\}\ (\ni 0.02) \tag{15}$$
$$\mathbf{Z}_4 = \{4.6319\times10^{-4} < \zeta_4 < 0.13432)\}\ (\ni 0.02) \tag{16}$$

Again, we can verify that these ranges are exact in the same fashion as in Example 1.

5. CONCLUSION

We gave an exact iterative method with guaranteed convergence for robust performance analysis of sampled-data systems. Dealing with multiple uncertain parameters and synthesis problems is an open future topic.

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