LMI APPROACH TO ROBUST STABILITY ANALYSIS OF HOPFIELD NEURAL NETWORKS

Ce Ji, Hua-guang Zhang

Key Laboratory of Process Industry Automation,
Ministry of Education, China,

Abstract: The robust stability of a class of Hopfield neural networks with multiple delays and parameter perturbations is analysed. The sufficient conditions for the global robust stability of equilibrium point are given by way of constructing a suitable Lyapunov-Krasovskii functional. The conditions take the form of linear matrix inequality (LMI), so they are computationally efficient. In addition, the results are independent of delays and established without assuming the differentiability and monotonicity of activation functions.

Keywords: neural networks; delay; perturbation; robust stability; Lyapunov functional

1. INTRODUCTION

The investigation and application of Hopfield neural networks with symmetric interconnecting structure have extended to many fields and gained abundant fruits in classification, parallel computing, associative memory, especially in solving some optimization problems (Hopfield, 1982; Zhang, 2003; Wang, *et al*., 2002). However, it is very difficult to realize the absolute symmetry of interconnecting structure due to the influences of parameter perturbations and modelling errors. On the other hand, the finite switching speed of amplifiers and the inherent communication time of neurons inevitably induce time delays in the interaction between the neurons, and this may bring oscillation or network instability. Hence, it is important to consider the influences of time delays and parameter perturbations when the stability of Hopfield neural networks is analysed. So far, most of the existing results concerning with the robust stability of delayed Hopfield neural networks were done by intervalizing the self-feedback terms and interconnecting terms (Liao, et al., 2001; Liao, et al., 2003). Although the criteria established in those references are explicit and easy to verify generally, they often neglect the sign of the terms in the interconnecting matrices, and thus, the different effects between the neuronal excitatory and inhibitory might be ignored.

In this paper, some criteria for the global robust stability of delayed Hopfield neural networks will be derived without use of the intervalizing the self-feedback terms and interconnecting terms. These criteria are in the form of linear matrix inequality and hence are convenient to verify using the interior-point method. More importantly, they take into account the difference between the neuronal excitatory and inhibitory effects.

The paper is organized as follows. In section 2, the network model will be described and several lemmas will be given as the basis of later sections. By the method of constructing a suitable Lyapunov-Krasovskii functional and sector condition, the sufficient conditions for the robust stability of this model will be established in section 3. Simulation examples will be given in section 4 to demonstrate the effectiveness of the conclusions. Finally, section 5 will summarize the results and future work.

2. NETWORK MODEL

The Hopfield neural network model with multiple delays can be described by equation

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{T}_0\boldsymbol{S}(\boldsymbol{x}(t)) + \sum_{k=1}^{K}\boldsymbol{T}_k\boldsymbol{S}(\boldsymbol{x}(t-\tau_k)) + \boldsymbol{I} \quad (1)$$

where $\boldsymbol{x} = (x_1, \cdots, x_n)^T \in \Re^n$ denotes the state variables associated with the neurons, $\boldsymbol{C} = diag\left[c_1, \cdots, c_n\right]$ with $c_i > 0$, $i = 1, \cdots, n$ denotes self-feedback matrix of the neurons. $\boldsymbol{T}_0 \in \Re^{n\times n}$ denotes that part of the interconnecting structure which is not associated with delay, $\boldsymbol{T}_k \in \Re^{n\times n}$ denotes that part of the interconnecting structure which is associated with delay τ_k , where τ_k denotes *kth* delay, $k = 1, \cdots, K$ and $0 < \tau_1 < \cdots < \tau_K < +\infty$. $\boldsymbol{T} = \boldsymbol{T}_0 + \sum_{k=1}^{K}\boldsymbol{T}_k$ is a symmetric matrix. $\boldsymbol{I} = (I_1, \cdots, I_n)^T \in \Re^n$ is a constant vector representing bias terms. $\boldsymbol{S}(\boldsymbol{x}) = [s_1(x_1), \cdots, s_n(x_n)]^T$ denotes the activation functions, where $s_i(x_i)$, $i = 1, \cdots, n$ satisfies the later assumption H_1 and $s_i(0) = 0$.

The initial condition is $\boldsymbol{x}(s) = \varphi(s)$, for $s \in [-\tau_K, 0]$, where $\varphi \in C\left(\left[-\tau_K, 0\right], \Re^n\right)$. Here, $C([-\tau_K, 0], \Re^n)$ denotes the Banach space of continuous vector-valued functions mapping the interval $\left[-\tau_K, 0\right]$ into $\Re^n$ with a topology of uniform convergence.

Assumption H_1 For $i = 1, \cdots, n$, $s_i(x_i)$ is bounded and satisfies the following sector condition

$$0 \le \frac{s_i(x_i)}{x_i} \le \sigma_i^M \quad (2)$$

For the activation function $s_i(x_i)$, $i = 1, \cdots, n$, it is typically assumed to be sigmoid which implies that it is monotone and smooth. However, just as Van der Driessche and Zou pointed (1998), for some purpose of networks, non-monotonic and not necessarily smooth functions might be better candidates for neuron activation function in designing and implementing an artificial neural network. Note that in many electronic circuits, amplifiers that have neither monotonically increasing nor continuously differentiable input–output functions are frequently adopted. Hence, we relax the restriction that $s_i(x_i)$ is a Sigmoid function and only require that $s_i(x_i)$ satisfy H_1.

Considering the influences of parameter perturbations, then (1) can be described as

$$\dot{\boldsymbol{x}}(t) = -(\boldsymbol{C} + \varDelta\boldsymbol{C})\boldsymbol{x}(t) + (\boldsymbol{T}_0 + \varDelta\boldsymbol{T}_0)\boldsymbol{S}(\boldsymbol{x}(t)) + \sum_{k=1}^{K}(\boldsymbol{T}_k + \varDelta\boldsymbol{T}_k)\boldsymbol{S}(\boldsymbol{x}(t-\tau_k)) + \boldsymbol{I} \quad (3)$$

where $\varDelta\boldsymbol{C} = diag\left[\varDelta c_1, \cdots, \varDelta c_n\right] \in \Re^{n\times n}$, $\varDelta\boldsymbol{T}_k \in \Re^{n\times n}$, $k = 0, \cdots, K$.

Definition The equilibrium point of system (1) is said to be globally robustly stable with respect to the perturbations $\varDelta\boldsymbol{C}$ and $\varDelta\boldsymbol{T}_k$, $k = 0, \cdots, K$, if the equilibrium point of system (3) is globally asymptotically stable.

If letting $\tilde{x} = x - x_e$, where x_e is an equilibrium point of network (3), and defining

$$\tilde{\boldsymbol{S}}(\tilde{x}) = \boldsymbol{S}(\tilde{x} + \boldsymbol{x}_e) - \boldsymbol{S}(\boldsymbol{x}_e) \quad (4)$$

then the new description of the neural network (3) can be obtained

$$\dot{\tilde{\boldsymbol{x}}}(t) = -(\boldsymbol{C} + \varDelta\boldsymbol{C})\tilde{\boldsymbol{x}}(t) + (\boldsymbol{T}_0 + \varDelta\boldsymbol{T}_0)\tilde{\boldsymbol{S}}(\tilde{\boldsymbol{x}}(t)) + \sum_{k=1}^{K}(\boldsymbol{T}_k + \varDelta\boldsymbol{T}_k)\tilde{\boldsymbol{S}}(\tilde{\boldsymbol{x}}(t-\tau_k)) \quad (5)$$

From equation (4), it follows that $\tilde{\boldsymbol{S}}(0) = 0$ and the terms of $\tilde{\boldsymbol{S}}(\tilde{x})$ satisfy the assumption H_1, too. Distinctly, the origin is an equilibrium point of (5). For simplification, we neglect the sign ~ in (5), and then we have

$$\dot{\boldsymbol{x}}(t) = -(\boldsymbol{C} + \varDelta\boldsymbol{C})\boldsymbol{x}(t) + (\boldsymbol{T}_0 + \varDelta\boldsymbol{T}_0)\boldsymbol{S}(\boldsymbol{x}(t)) + \sum_{k=1}^{K}(\boldsymbol{T}_k + \varDelta\boldsymbol{T}_k)\boldsymbol{S}(\boldsymbol{x}(t-\tau_k)) \quad (6)$$

Thus, in order to study the globally robust stability of the equilibrium point of system (1) with respect to parametric uncertainties $\varDelta\boldsymbol{C}$ and $\varDelta\boldsymbol{T}_k$, $k = 0, \cdots, K$, it suffices to investigate the globally asymptotic stability of the zero solution of system (6). Now, the interconnecting matrix $\boldsymbol{T} = \boldsymbol{T}_0 + \varDelta\boldsymbol{T}_0 + \sum_{k=1}^{K}(\boldsymbol{T}_k + \varDelta\boldsymbol{T}_k)$ is nonsymmetric due to the influences of perturbations $\varDelta\boldsymbol{T}_k$, $k = 0, \cdots, K$.

The perturbations $\varDelta\boldsymbol{C}$ and $\varDelta\boldsymbol{T}_k$, $k = 0, \cdots, K$ are assumed to satisfy the following assumption.

Assumption H_2

$$\left[\varDelta\boldsymbol{C} \;\; \varDelta\boldsymbol{T}_0 \cdots \varDelta\boldsymbol{T}_K\right] = \boldsymbol{H}\boldsymbol{F}\left[\boldsymbol{A} \;\; \boldsymbol{B}_0 \cdots \boldsymbol{B}_K\right] \quad (7)$$

where $\boldsymbol{F}$ is an unknown matrix representing parametric uncertainty which satisfies

$$\boldsymbol{F}^T\boldsymbol{F} \le \boldsymbol{I} \quad (8)$$

and $\boldsymbol{H}, \boldsymbol{A}, \boldsymbol{B}_0, \cdots, \boldsymbol{B}_K$ can be regarded as the known structural matrices of uncertainty with appropriate dimensions.

The uncertainty model of (7) and (8) has been widely adopted in robust control and filtering for uncertain systems (Du, et al., 1999; Yang, et al., 2002; Xu, et al., 2002).

To proceed, the following lemmas are given.

Lemma 1 (Hale, 1977) For a functional differential equation with time delay $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}_t)$, if there exists a continuous functional $V(t,\varphi)$, such that there exist non-decreasing continuous functions $u, v, w: \Re^+ \to \Re^+$, which satisfy $u(0)=v(0)=0$, $u(\|\varphi(0)\|) \le V(t,\varphi) \le v(|\varphi|)$ and $\dot V(t,\varphi) \le -w(\|\varphi(0)\|)$, then the solution $x=0$ of the functional differential equation is asymptotically stable.

In the above Lemma, $\|\cdot\|$ denotes the Euclidean vector norm on $\Re^n$. $x_t(\cdot)$ denotes the restriction of $x(\cdot)$ to the interval $[t-\tau_K, t]$ translated to $[-\tau_K, 0]$. For $s\in[-\tau_K,0]$, we have $\boldsymbol{x}_t(s)=\boldsymbol{x}(t+s)$, where $t>0$. For any $\varphi\in C([-\tau_K,0],\Re^n)$, we define $|\varphi| = \max\ \{\|\varphi(t)\| : t\in[-\tau_K,0]\}$.

Lemma 2 (Boyd, et al., 1994) The following LMI

$$\begin{bmatrix} \boldsymbol{Q}(\boldsymbol{x}) & \boldsymbol{S}(\boldsymbol{x}) \\ \boldsymbol{S}^T(\boldsymbol{x}) & \boldsymbol{R}(\boldsymbol{x}) \end{bmatrix} < 0 \tag{9}$$

is equivalent to $\boldsymbol{Q}(\boldsymbol{x}) - \boldsymbol{S}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{S}^T(\boldsymbol{x}) < 0$ and $\boldsymbol{R}(\boldsymbol{x})<0$, where $\boldsymbol{Q}(\boldsymbol{x})=\boldsymbol{Q}^T(\boldsymbol{x})$, $\boldsymbol{R}(\boldsymbol{x})=\boldsymbol{R}^T(\boldsymbol{x})$, and $\boldsymbol{S}(\boldsymbol{x})$ depend affinely on $\boldsymbol{x}$.

Lemma 3 (Singh, 2004) If $\boldsymbol{U},\boldsymbol{V}$ and $\boldsymbol{W}$ are real matrices of appropriate dimension with $\boldsymbol{M}$ satisfying $\boldsymbol{M}=\boldsymbol{M}^T$, then

$$\boldsymbol{M} + \boldsymbol{UVW} + \boldsymbol{W}^T\boldsymbol{V}^T\boldsymbol{U}^T < 0 \tag{10}$$

for all $\boldsymbol{V}^T\boldsymbol{V} \le \boldsymbol{I}$, if and only if there exists a positive constant ε such that

$$\boldsymbol{M} + \varepsilon^{-1}\boldsymbol{UU}^T + \varepsilon\boldsymbol{W}^T\boldsymbol{W} < 0 \tag{11}$$

In the following section, we will give the sufficient conditions for the globally asymptotic stability of equilibrium point $\boldsymbol{x}=0$ of system (6).

3. ROBUST STABILITY

Theorem The equilibrium point $\boldsymbol{x}=0$ of system (6) is globally asymptotically stable for arbitrarily bounded delay τ_k if there exists a positive definite matrix $\boldsymbol{P}$, positive constant ε and positive diagonal matrixes $\boldsymbol{\varLambda}_k = diag[\lambda_{k1},\cdots,\lambda_{kn}]$, where $\lambda_{ki}>0$, $i=1,\cdots,n$, $k=0,\cdots,K$, such that the following linear matrix inequality (LMI) holds

$$\begin{bmatrix}
-\boldsymbol{C}^T\boldsymbol{P}-\boldsymbol{PC}+\sum_{k=0}^{K}\boldsymbol{\varLambda}_k+\varepsilon\boldsymbol{A}^T\boldsymbol{A} & \boldsymbol{PT}_K\boldsymbol{E}^M-\varepsilon\boldsymbol{A}^T\boldsymbol{B}_K\boldsymbol{E}^M & \cdots & \cdots & \boldsymbol{TP}_0\boldsymbol{E}^M-\varepsilon\boldsymbol{A}^T\boldsymbol{B}_0\boldsymbol{E}^M & \boldsymbol{PH} \\
\boldsymbol{E}^M\boldsymbol{T}_K^T\boldsymbol{P}-\varepsilon\boldsymbol{E}^M\boldsymbol{B}_K^T\boldsymbol{A} & -\boldsymbol{\varLambda}_K+\varepsilon\boldsymbol{E}^M\boldsymbol{B}_K^T\boldsymbol{B}_K\boldsymbol{E}^M & \cdots & \cdots & \varepsilon\boldsymbol{E}^M\boldsymbol{B}_K^T\boldsymbol{B}_0\boldsymbol{E}^M & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\
\boldsymbol{E}^M\boldsymbol{T}_0^T\boldsymbol{P}-\varepsilon\boldsymbol{E}^M\boldsymbol{B}_0^T\boldsymbol{A} & \varepsilon\boldsymbol{E}^M\boldsymbol{B}_0^T\boldsymbol{B}_K\boldsymbol{E}^M & \cdots & \cdots & -\boldsymbol{\varLambda}_0+\varepsilon\boldsymbol{E}^M\boldsymbol{B}_0^T\boldsymbol{B}_0\boldsymbol{E}^M & 0 \\
\boldsymbol{H}^T\boldsymbol{P} & 0 & \cdots & \cdots & 0 & -\varepsilon I
\end{bmatrix} < 0 \tag{12}$$

where $\boldsymbol{E}^M = diag[\sigma_1^M,\cdots,\sigma_n^M]$.

Proof By (2), we can rewrite (6) as

$$\dot{\boldsymbol{x}}(t) = -(\boldsymbol{C}+\boldsymbol{\varDelta C})\boldsymbol{x}(t) + (\boldsymbol{T}_0+\boldsymbol{\varDelta T}_0)\boldsymbol{E}(\boldsymbol{x}(t))\boldsymbol{x}(t) + \sum_{k=1}^{K}(\boldsymbol{T}_k+\boldsymbol{\varDelta T}_k)\boldsymbol{E}(\boldsymbol{x}(t-\tau_k))\boldsymbol{x}(t-\tau_k) \tag{13}$$

where

$$\boldsymbol{E}(\boldsymbol{x}) = diag[\sigma_1(x_1),\cdots,\sigma_n(x_n)],$$
$$\sigma_i(x_i) = s_i(x_i)/x_i,\ i=1,\cdots,n.$$

Here, we introduce the following Lyapunov-Krasovskii functional

$$V(\boldsymbol{x}_t) = \boldsymbol{x}^T(t)\boldsymbol{Px}(t) + \sum_{k=1}^{K}\int_{-\tau_k}^{0}\boldsymbol{x}_t^T(\theta)\boldsymbol{\varLambda}_k\boldsymbol{x}_t(\theta)d\theta \tag{14}$$

Clearly, we have

$$\lambda_{\min}(\boldsymbol{P})\|\boldsymbol{x}_t(0)\|^2 \le V(\boldsymbol{x}_t) \le \left(\lambda_{\max}(\boldsymbol{P}) + \sum_{k=1}^{K}\tau_k\lambda_{\max}(\boldsymbol{\varLambda}_k)\right)|\boldsymbol{x}_t|^2$$

The derivative of $V(\boldsymbol{x}_t)$ with respect to t along any trajectory of system (13) is given by

$$\begin{aligned}
\dot V(\boldsymbol{x}_t) =&\ \dot{\boldsymbol{x}}^T(t)\boldsymbol{Px}(t) + \boldsymbol{x}^T(t)\boldsymbol{P}\dot{\boldsymbol{x}}(t) \\
&+\sum_{k=1}^{K}\boldsymbol{x}^T(t)\boldsymbol{\varLambda}_k\boldsymbol{x}(t) - \sum_{k=1}^{K}\boldsymbol{x}^T(t-\tau_k)\boldsymbol{\varLambda}_k\boldsymbol{x}(t-\tau_k) \\
=& -\boldsymbol{x}^T(t)[(\boldsymbol{C}+\boldsymbol{\varDelta C})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{C}+\boldsymbol{\varDelta C})]\boldsymbol{x}(t) \\
&+\sum_{k=1}^{K}\boldsymbol{x}^T(t)\boldsymbol{\varLambda}_k\boldsymbol{x}(t) + \boldsymbol{x}^T(t)\boldsymbol{\varLambda}_0\boldsymbol{x}(t) \\
&+\boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{T}_0+\boldsymbol{\varDelta T}_0)\boldsymbol{E}(\boldsymbol{x}(t))\boldsymbol{\varLambda}_0^{-1}\boldsymbol{E}^T(\boldsymbol{x}(t))(\boldsymbol{T}_0+\boldsymbol{\varDelta T}_0)^T\boldsymbol{Px}(t) \\
&-\left[\boldsymbol{\varLambda}_0^{1/2}\boldsymbol{x}(t) - \boldsymbol{\varLambda}_0^{-1/2}\boldsymbol{E}^T(\boldsymbol{x}(t))(\boldsymbol{T}_0+\boldsymbol{\varDelta T}_0)^T\boldsymbol{Px}(t)\right]^T \\
&\times\left[\boldsymbol{\varLambda}_0^{1/2}\boldsymbol{x}(t) - \boldsymbol{\varLambda}_0^{-1/2}\boldsymbol{E}^T(\boldsymbol{x}(t))(\boldsymbol{T}_0+\boldsymbol{\varDelta T}_0)^T\boldsymbol{Px}(t)\right] \\
&+\sum_{k=1}^{K}\boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{T}_k+\boldsymbol{\varDelta T}_k)\boldsymbol{E}(\boldsymbol{x}(t-\tau_k)) \\
&\times\boldsymbol{\varLambda}_k^{-1}\boldsymbol{E}^T(\boldsymbol{x}(t-\tau_k))(\boldsymbol{T}_k+\boldsymbol{\varDelta T}_k)^T\boldsymbol{Px}(t) \\
&-\sum_{k=1}^{K}\left[\boldsymbol{\varLambda}_k^{1/2}\boldsymbol{x}(t-\tau_k) - \boldsymbol{\varLambda}_k^{-1/2}\boldsymbol{E}^T(\boldsymbol{x}(t-\tau_k))(\boldsymbol{T}_k+\boldsymbol{\varDelta T}_k)^T\boldsymbol{Px}(t)\right]^T \\
&\times\left[\boldsymbol{\varLambda}_k^{1/2}\boldsymbol{x}(t-\tau_k) - \boldsymbol{\varLambda}_k^{-1/2}\boldsymbol{E}^T(\boldsymbol{x}(t-\tau_k))(\boldsymbol{T}_k+\boldsymbol{\varDelta T}_k)^T\boldsymbol{Px}(t)\right] \\
\le& -\boldsymbol{x}^T(t)[(\boldsymbol{C}+\boldsymbol{\varDelta C})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{C}+\boldsymbol{\varDelta C})]\boldsymbol{x}(t) + \sum_{k=0}^{K}\boldsymbol{x}^T(t)\boldsymbol{\varLambda}_k\boldsymbol{x}(t) \\
&+\boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{T}_0+\boldsymbol{\varDelta T}_0)\boldsymbol{E}(\boldsymbol{x}(t))\boldsymbol{\varLambda}_0^{-1}\boldsymbol{E}^T(\boldsymbol{x}(t))(\boldsymbol{T}_0+\boldsymbol{\varDelta T}_0)^T\boldsymbol{Px}(t) \\
&+\sum_{k=1}^{K}\boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{T}_k+\boldsymbol{\varDelta T}_k)\boldsymbol{E}(\boldsymbol{x}(t-\tau_k)) \\
&\times\boldsymbol{\varLambda}_k^{-1}\boldsymbol{E}^T(\boldsymbol{x}(t-\tau_k))(\boldsymbol{T}_k+\boldsymbol{\varDelta T}_k)^T\boldsymbol{Px}(t)
\end{aligned} \tag{15}$$

For any given t, if letting

$$\boldsymbol{y}_k^T(t) = (y_{k1}, \cdots, y_{kn}) = \boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{T}_k + \boldsymbol{\Delta T}_k), \quad (16)$$

then the last term of (15) assumes the form

$$\sum_{k=1}^{K} \boldsymbol{y}_k^T(t)\boldsymbol{E}(\boldsymbol{x}(t-\tau_k))\boldsymbol{\Lambda}_k^{-1}\boldsymbol{E}^T(\boldsymbol{x}(t-\tau_k))\boldsymbol{y}_k(t)$$
$$= \sum_{k=1}^{K}\sum_{i=1}^{n} y_{ki}^2 \lambda_{ki}^{-1}\sigma_i^2(x_i(t-\tau_k))$$
$$\le \sum_{k=1}^{K}\sum_{i=1}^{n} y_{ki}^2 \lambda_{ki}^{-1}(\sigma_i^M)^2$$
$$= \sum_{k=1}^{K} \boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{T}_k + \boldsymbol{\Delta T}_k)\boldsymbol{E}^M\boldsymbol{\Lambda}_k^{-1}\boldsymbol{E}^M(\boldsymbol{T}_k + \boldsymbol{\Delta T}_k)^T\boldsymbol{P}\boldsymbol{x}(t) \quad (17)$$

Similarly,

$$\boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{T}_0 + \boldsymbol{\Delta T}_0)\boldsymbol{E}(\boldsymbol{x}(t))\boldsymbol{\Lambda}_0^{-1}\boldsymbol{E}^T(\boldsymbol{x}(t))(\boldsymbol{T}_0 + \boldsymbol{\Delta T}_0)^T\boldsymbol{P}\boldsymbol{x}(t)$$
$$\le \boldsymbol{x}^T(t)\boldsymbol{P}(\boldsymbol{T}_0 + \boldsymbol{\Delta T}_0)\boldsymbol{E}^M\boldsymbol{\Lambda}_0^{-1}\boldsymbol{E}^M(\boldsymbol{T}_0 + \boldsymbol{\Delta T}_0)^T\boldsymbol{P}\boldsymbol{x}(t) \quad (18)$$

From (17) and (18), (15) can be expressed as

$$\dot{V}(\boldsymbol{x}_t) \le \boldsymbol{x}^T(t)\Big\{-\left[(\boldsymbol{C}+\boldsymbol{\Delta C})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{C}+\boldsymbol{\Delta C})\right] + \sum_{k=1}^{K}\boldsymbol{\Lambda}_k$$
$$+\sum_{k=0}^{K}\boldsymbol{P}(\boldsymbol{T}_k + \boldsymbol{\Delta T}_k)\boldsymbol{E}^M\boldsymbol{\Lambda}_k^{-1}\boldsymbol{E}^M(\boldsymbol{T}_k + \boldsymbol{\Delta T}_k)^T\boldsymbol{P}\Big\}\boldsymbol{x}(t) \quad (19)$$

Now, we define

$$\boldsymbol{S}^M = -\left[(\boldsymbol{C}+\boldsymbol{\Delta C})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{C}+\boldsymbol{\Delta C})\right] + \sum_{k=0}^{K}\boldsymbol{\Lambda}_k$$
$$+\sum_{k=0}^{K}\boldsymbol{P}(\boldsymbol{T}_k + \boldsymbol{\Delta T}_k)\boldsymbol{E}^M\boldsymbol{\Lambda}_k^{-1}\boldsymbol{E}^M(\boldsymbol{T}_k + \boldsymbol{\Delta T}_k)^T\boldsymbol{P} \quad (20)$$

Hence, $\dot{V}(\boldsymbol{x}_t) < 0$ if $\boldsymbol{S}^M < 0$. By (19), $\dot{V}(x_t) \le -\lambda_{\max} \times(-S^M)\|x(t)\|^2 = -\lambda_{\max}(-S^M)\|x_t(0)\|^2$ can be derived. From lemma 1, we know that the equilibrium point $\boldsymbol{x} = 0$ of system (6) is globally asymptotically stable. Then, according to lemma 2, $\boldsymbol{S}^M < 0$ can be expressed by the following linear matrix inequality

$$\begin{bmatrix} -\left[(\boldsymbol{C}+\boldsymbol{\Delta C})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{C}+\boldsymbol{\Delta C})\right] + \sum_{k=0}^{K}\boldsymbol{\Lambda}_k & \boldsymbol{P}(\boldsymbol{T}_K+\boldsymbol{\Delta T}_K)\boldsymbol{E}^M & \cdots & \cdots & \boldsymbol{P}(\boldsymbol{T}_0+\boldsymbol{\Delta T}_0)\boldsymbol{E}^M \\ \boldsymbol{E}^M(\boldsymbol{T}_K+\boldsymbol{\Delta T}_K)^T\boldsymbol{P} & -\boldsymbol{\Lambda}_K & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \boldsymbol{E}^M(\boldsymbol{T}_0+\boldsymbol{\Delta T}_0)^T\boldsymbol{P} & 0 & \cdots & 0 & -\boldsymbol{\Lambda}_0 \end{bmatrix} < 0 \quad (21)$$

In fact, (21) is exactly

$$\begin{bmatrix} -\boldsymbol{C}^T\boldsymbol{P}-\boldsymbol{PC}+\sum_{k=0}^{K}\boldsymbol{\Lambda}_k & \boldsymbol{PT}_K\boldsymbol{E}^M & \cdots & \cdots & \boldsymbol{PT}_0\boldsymbol{E}^M \\ \boldsymbol{E}^M\boldsymbol{T}_K^T\boldsymbol{P} & -\boldsymbol{\Lambda}_K & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \boldsymbol{E}^M\boldsymbol{T}_0^T\boldsymbol{P} & 0 & \cdots & 0 & -\boldsymbol{\Lambda}_0 \end{bmatrix}$$
$$+\begin{bmatrix} -\boldsymbol{\Delta C}^T\boldsymbol{P}-\boldsymbol{P\Delta C} & \boldsymbol{P\Delta T}_K\boldsymbol{E}^M & \cdots & \cdots & \boldsymbol{P\Delta T}_0\boldsymbol{E}^M \\ \boldsymbol{E}^M\boldsymbol{\Delta T}_K^T\boldsymbol{P} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \boldsymbol{E}^M\boldsymbol{\Delta T}_0^T\boldsymbol{P} & 0 & \cdots & \cdots & 0 \end{bmatrix} < 0 \quad (22)$$

Because of $[\boldsymbol{\Delta C} \;\; \boldsymbol{\Delta T}_0 \cdots \boldsymbol{\Delta T}_K] = \boldsymbol{HF}[\boldsymbol{A} \;\; \boldsymbol{B}_0 \cdots \boldsymbol{B}_K]$, (22) can be expressed as

$$\begin{bmatrix} -\boldsymbol{C}^T\boldsymbol{P}-\boldsymbol{PC}+\sum_{k=0}^{K}\boldsymbol{\Lambda}_k & \boldsymbol{PT}_K\boldsymbol{E}^M & \cdots & \cdots & \boldsymbol{PT}_0\boldsymbol{E}^M \\ \boldsymbol{E}^M\boldsymbol{T}_K^T\boldsymbol{P} & -\boldsymbol{\Lambda}_K & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \boldsymbol{E}^M\boldsymbol{T}_0^T\boldsymbol{P} & 0 & \cdots & 0 & -\boldsymbol{\Lambda}_0 \end{bmatrix}$$
$$+\begin{bmatrix} \boldsymbol{PH} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}\boldsymbol{F}\left[-\boldsymbol{A} \quad \boldsymbol{B}_K\boldsymbol{E}^M \quad \cdots \quad \cdots \quad \boldsymbol{B}_0\boldsymbol{E}^M\right]$$
$$+\begin{bmatrix} -\boldsymbol{A}^T \\ \boldsymbol{E}^M\boldsymbol{B}_K^T \\ \vdots \\ \vdots \\ \boldsymbol{E}^M\boldsymbol{B}_0^T \end{bmatrix}\boldsymbol{F}^T\left[\boldsymbol{H}^T\boldsymbol{P} \quad 0 \quad \cdots \quad \cdots \quad 0\right] < 0 \quad (23)$$

Using Lemma 3, we know (23) holds for all $\boldsymbol{F}^T\boldsymbol{F} \le \boldsymbol{I}$ if and only if there exists a constant $\varepsilon > 0$ such that

$$\begin{bmatrix} -\boldsymbol{C}^T\boldsymbol{P}-\boldsymbol{PC}+\sum_{k=0}^{K}\boldsymbol{\Lambda}_k & \boldsymbol{PT}_K\boldsymbol{E}^M & \cdots & \cdots & \boldsymbol{PT}_0\boldsymbol{E}^M \\ \boldsymbol{E}^M\boldsymbol{T}_K^T\boldsymbol{P} & -\boldsymbol{\Lambda}_K & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \boldsymbol{E}^M\boldsymbol{T}_0^T\boldsymbol{P} & 0 & \cdots & 0 & -\boldsymbol{\Lambda}_0 \end{bmatrix}$$
$$+\frac{1}{\varepsilon}\begin{bmatrix} \boldsymbol{PHH}^T\boldsymbol{P} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$
$$+\varepsilon\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{A} & -\boldsymbol{A}^T\boldsymbol{B}_K\boldsymbol{E}^M & \cdots & \cdots & -\boldsymbol{A}^T\boldsymbol{B}_0\boldsymbol{E}^M \\ -\boldsymbol{E}^M\boldsymbol{B}_K^T\boldsymbol{A} & \boldsymbol{E}^M\boldsymbol{B}_K^T\boldsymbol{B}_K\boldsymbol{E}^M & \cdots & \cdots & \boldsymbol{E}^M\boldsymbol{B}_K^T\boldsymbol{B}_0\boldsymbol{E}^M \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ -\boldsymbol{E}^M\boldsymbol{B}_0^T\boldsymbol{A} & \boldsymbol{E}^M\boldsymbol{B}_0^T\boldsymbol{B}_K\boldsymbol{E}^M & \cdots & \cdots & \boldsymbol{E}^M\boldsymbol{B}_0^T\boldsymbol{B}_0\boldsymbol{E}^M \end{bmatrix} < 0 \quad (24)$$

Rearrange (24), we get

$$\begin{bmatrix} -C^TP-PC+\sum_{k=0}^{K}\Lambda_k+\frac{1}{\varepsilon}PHH^TP+\varepsilon A^TA & PT_KE^M-\varepsilon A^TB_KE^M & \cdots & \cdots & PT_0E^M-\varepsilon A^TB_0E^M \\ E^MT_K^{\,T}P-\varepsilon E^MB_K^TA & -\Lambda_K+\varepsilon E^MB_K^TB_KE^M & \cdots & \cdots & \varepsilon E^MB_K^TB_0E^M \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ E^MT_0^{\,T}P-\varepsilon E^MB_0^TA & \varepsilon E^MB_0^TB_KE^M & \cdots & \cdots & -\Lambda_0+\varepsilon E^MB_0^TB_0E^M \end{bmatrix}<0 \quad (25)$$

By use of Lemma 3, (25) is equivalent to condition (12). This proves the globally asymptotic stability of the equilibrium point $x=0$ for system (6). Thus the equilibrium point of system (1) is globally robustly stable with respect to the perturbations ΔC and ΔT_k , $k=0,\cdots,K$.

When $\Delta C=\Delta T_k=0,\ \ k=0,\cdots,K$, the following corollary can be derived.

Corollary The equilibrium point $x=0$ of system (6) is globally asymptotically stable for arbitrarily bounded delay τ_k if there exists a positive definite matrix P and positive diagonal matrixes $\Lambda_k=diag[\lambda_{k1},\cdots,\lambda_{kn}]$, where $\lambda_{ki}>0,\ i=1,\cdots,n,$ $k=0,\cdots,K$, such that the following liner matrix inequality (LMI) holds

$$\begin{bmatrix} -(C^TP+PC)+\sum_{k=0}^{K}\Lambda_k & PT_KE^M & \cdots & \cdots & PT_0E^M \\ E^MT_K^{\,T}P & -\Lambda_K & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ E^MT_0^TP & 0 & \cdots & 0 & -\Lambda_0 \end{bmatrix}<0 \quad (26)$$

where $E^M=diag[\sigma_1^M,\cdots,\sigma_n^M]$.

The corollary follows in a straightforward manner by choosing $\Delta C=\Delta T_k=0,\ \ k=0,\cdots,K$ in (21).

4. SIMULATION RESULTS

Now, we use the under MATLAB's LMI toolbox and Simulink modules to accomplish a simulation for system (6) and verify the effectiveness of the theorem. Considering the case that there exit two delays τ_1,τ_2, system (6) can be rewritten as

$$\dot{x}(t)=-(C+\Delta C)x(t)+(T_0+\Delta T_0)S(x(t)) + \sum_{k=1}^{2}(T_k+\Delta T_k)S(x(t-\tau_k)) \quad (27)$$

For (27), we choose the parameters as

$$C=\begin{bmatrix}22 & 0\\ 0 & 6\end{bmatrix},\ T_0=\begin{bmatrix}2.5 & 3\\ 1 & 0.5\end{bmatrix}$$

$$T_1=\begin{bmatrix}1 & 0.8\\ 1 & 1\end{bmatrix},\ T_2=\begin{bmatrix}6.3 & 1.02\\ 12 & 3\end{bmatrix}$$

$$S(x(t))=[tanh(0.2x_1),tanh(0.4x_2)]^T$$

By linearizing $S(x(t))$ at the origin, we have

$$E^M=\begin{bmatrix}0.2 & 0\\ 0 & 0.4\end{bmatrix}.$$

From assumption H_2, we have

$$[\Delta C\ \ \Delta T_0\ \Delta T_1\ \Delta T_2]=HF[A\ \ B_0\ B_1\ B_2],$$

where

$$A=\begin{bmatrix}0.1 & 0\\ 0 & 0.05\end{bmatrix},\ B_0=\begin{bmatrix}0.2 & 0.3\\ 0.1 & 0.1\end{bmatrix},$$

$$B_1=\begin{bmatrix}0.2 & -0.03\\ 0.1 & 0.1\end{bmatrix},B_2=\begin{bmatrix}0.2 & -0.04\\ 0.01 & 0.01\end{bmatrix}$$

$$H=\begin{bmatrix}1 & 0.4\\ 0.2 & 0\end{bmatrix},\ F(t)=\begin{bmatrix}\sin t & 0\\ 0 & \sin t\end{bmatrix}$$

By solving the associated linear matrix inequality using LMI toolbox, we know that there exists a positive definite matrix

$$P=\begin{bmatrix}0.1021 & -0.0021\\ -0.0021 & 0.3077\end{bmatrix},$$

three positive diagonal matrixes

$$\Lambda_0=\begin{bmatrix}1.1200 & 0\\ 0 & 0.9684\end{bmatrix},$$

$$\Lambda_1=\begin{bmatrix}1.1200 & 0\\ 0 & 0.9605\end{bmatrix},$$

$$\Lambda_2=\begin{bmatrix}1.1198 & 0\\ 0 & 0.9597\end{bmatrix},$$

and a positive constant $\varepsilon=1.1180$, such that the linear matrix inequality (12) holds, hence the equilibrium point $x=0$ of system (27) is globally asymptotically stable for arbitrarily bounded delay τ_k , $k=1,2$.

During the process of simulation, we choose the initial function as a constant vector $\varphi=[0.4,-1.3]^T$, and $F(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. The simulation curves are presented below. When $\tau_1=0.3$, $\tau_2=1.2$, the simulation curves of system converging to asymptotically stable equilibrium point $x=0$ is presented in Figure 1; When $\tau_1=0.6$, $\tau_2=1$, the simulation curves are presented in Figure 2.

By the simulation results, we can know clearly that for any bounded delay τ_k , $k=1,2$, the state curves of system converge to asymptotically stable equilibrium point $x=0$.

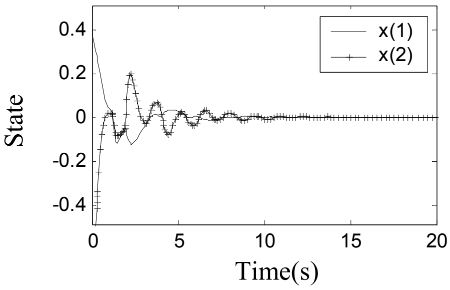

Fig. 1. State curves of system when $\tau_1 = 0.3$, $\tau_2 = 1.2$.

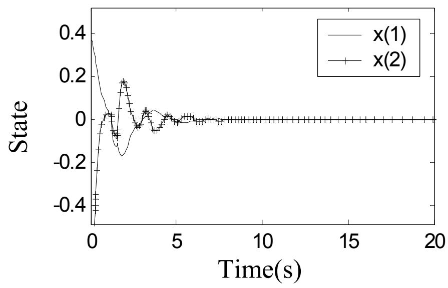

Fig. 2. State curves of system when $\tau_1 = 0.6$, $\tau_2 = 1$.

5. CONCLUSION

During the implementation process of Hopfield neural networks by electronic circuits, time delays and parameter perturbations are inevitable. Aiming at the cases, this paper analyses the robust stability of a class of Hopfield neural network model with multiple delays and parameter perturbations, and give the sufficient conditions for the globally robust stability of equilibrium point for arbitrarily bounded delay τ_k , $k = 1, \cdots, K$. The results of this paper take the form of linear matrix inequality (LMI) and are very practical to the analysis and design of Hopfield neural networks with delays.

In applications, the bound of the delays is frequently not very large and is usually known. Therefore, the next research work is to discuss further whether we can obtain the sufficient conditions for the globally robust stability of equilibrium point, which depend on time delay τ_k , $k = 1, \cdots, K$.

ACKNOWLEDGEMENTS

This work was supported in part by the National Natural Science Foundation of China under Grant #60274017 and #60325311 to H.G. Zhang.

REFERENCES

Boyd, S., L. EI Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear matrix inequalities in system and control theory*. PA:SIAM, Philadelphia.

Hopfield, J.J. (1982). Neural networks and physical systems with emergent collective computational abilities. In: *Proceedings of the National Academy of Sciences USA,* **79**, 2554-2558.

Liao, X.X., K. Wong, Z. Wu and G. Chen (2001). Novel robust stability criteria for interval delayed Hopfield neural networks. *IEEE Trans. on Circuits and Systems-I*, **48**, 1355-1358.

Liao, X.X., J. Wang and J. Cao (2003). Global and robust stability of interval Hopfield neuralnetworks with time-varying delays. *International Journal of Neural Systems*, **13**, 171-182.

Hale, J. (1977). *Theory of functional differential equations*. Germany: Springer-Verlag, Berlin.

Singh, V. (2004). Robust stability of cellular neural networks with delay: linear matrix inequality approach. *IEE Proc.-Control Theory Appl.*, **151**,125-129.

Van den Driessche, P. and X. Zou (1998). Global attractivity in delayed Hopfield neural network models. *SIAM J. Appl. Math*, **58**, 1878-1890.

Wang, R.L., Z. Tang and Q.P. Cao (2002). A learning method in Hopfield neural network for combinatorial optimization problem. *Neurocomputing*, **48**, 1021-1024.

Zhang, J.Y. (2003). Global stability analysis in hopfield neural network. *Applied Mathematics Letters,* **16**, 925-931.

Du, C. and L. Xie (1999). Stability analysis and stabilization of uncertain two-dimensional discrete systems: an LMI approach. *IEEE Trans. on Circuits and Systems-I,* **46**,1371-1374.

Yang, F. and Y. Hung (2002). Robust mixed H_2/H_∞ filtering with regional pole assignment for uncertain discrete-time systems. *IEEE Trans. on Circuits and Systems-I,* **49**,1236-1241.

Xu, S., J. Lam, Z. Lin and K. Galkowski (2002). Positive real control for uncertain twodimensional systems. *IEEE Trans. on Circuits and Systems-I,* **49**,1659-1666.