THE STABLE EMBEDDING PROBLEM

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Abstract: We compute a stable polynomial matrix embedding a stabilizable one. The algorithm resembles the one described by Beelen and Van Dooren for the unimodular embedding problem. We also desribe the numerical problems associated with this kind of algorithms, and point out why the stable embedding algorithm has better prospects than its unimodular counterpart.

Keywords: Polynomial matrices, stabilization, behavior, numerical properties

INTRODUCTION

Let a system behavior be given. This system is called stabilizable if a controller, defined on the whole set of system variables, can be found such that the intersection of the system behavior and the controller behavior is stable and the interconnection is regular, see Willems (1997). A stabilizable system is characterized by the fact that it can be described by a kernel representation associated to a polynomial matrix, having full row rank in the right half plane. Stabilization by regular full interconnection is equivalent to adding rows to this polynomial in such a way that the resulting square matrix is *Hurwitz*, i.e. has full rank in the right half of the complex plane.

So the problem we solve here is embedding a non square polynomial matrix of full row rank in the right half plane into a square polynomial matrix with the same property. The method we use is closely related to the algorithm described by Beelen and Van Dooren (1988) to embed a polynomial matrix having full row rank in the whole plane into a unimodular one. For the unimodular embedding problem the algorithm leads to serious numerical problems, while in our problem the prospects are much better. We will come back to this issue in section 8.

A related problem is stabilization by partial interconnection, where the controller behavior is only defined on a subset of the system variables. This occurs for instance when only a part of the variables can be influenced by the controller. In the solution of the stabilization problem with regular partial interconnection the solution to the problem we treat here is an essential ingredient. We will report on that elsewhere.

1. PRELIMINARIES

Let us start with an easy example:

Example. Let $P(\xi) = (\xi 2-1 \;\; \xi+1)$. This matrix has clearly rank one for any $\lambda \in \overline{\mathbb{C}^+}$, the closed right half plane, that we substitute for ξ. We try to construct a constant vector Q such that the matrix $W(\xi)$ resulting from stacking $P(\xi)$ and Q is stable. Let $Q = (a \;\; b)$, then

$$\det(W(\xi)) = (\xi+1)(b\xi - (b+a)).$$

Choosing $a/b < -1$ yields a stable W.

□

This example shows that the solution is not unique, which is not really surprising, but it also shows that the set of solutions is big, contrary to the unimodular embedding problem, where the solution set generally is small.

Clearly we will need to stack matrices frequently. To save space we will denote that by a semicolon:

$$(P;Q) := \begin{pmatrix} P \\ Q \end{pmatrix}.$$

For notational convenience we denote the class of non constant polynomial matrices $P(\xi)$ that have full row rank for all λ in the closed right half plane by $\mathcal{M}$. A *Hurwitz* matrix is a square matrix in $\mathcal{M}$.

In this paper a special role is played by polynomial matrices of degree one. With a slight abuse of terminology we call this a (matrix) pencil, and we will denote it by $A + \xi E$.

2. MATHEMATICAL PROBLEM FORMULATION

Let $P(\xi)$ be an $m \times n$ (with $n > m$) polynomial matrix of degree d:

$$P(\xi) = P_0 + P_1\xi + P_2\xi^2 + ... + P_d\xi^d \qquad (1)$$

where each P_i is a real $m \times n$ matrix.

The objective is to construct another polynomial matrix

$$Q(\xi) = Q_0 + Q_1\xi + Q_2\xi^2 + ... + Q_d\xi^{d_q}, \qquad (2)$$

of size $(n-m) \times n$ such that the matrix $W(\xi) = (P(\xi); Q(\xi)) \in \mathcal{M}$. If such a $Q(\xi)$ exists we say that $P(\xi)$ is embedded into the Hurwitz matrix $W(\xi)$. Since a Hurwitz matrix is invertible for all λ in $\overline{\mathbb{C}^+}$, the embedding problem can only have a solution if $P(\lambda)$ has full row rank m for all λ in $\overline{\mathbb{C}^+}$, so if P is in $\mathcal{M}$.

To construct a Q we follow the idea from Beelen and Van Dooren (1988) for the unimodular embedding problem. This means linearizing the problem, but instead of bringing the resulting matrix pencil in full generalized Schur form as they do, we restrict ourselves to the specific needs for this problem.

3. THE STATE SPACE REPRESENTATION

Any behavior, defined by a kernel representation, we can also represent by an observable state space representation (?Rapisarda and Willems (1997)). Let the behavior be given by $\mathcal{B} = \{w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^n | P(\frac{d}{dt})w = 0\}$ (we consider solutions to the differential equations in distributional sense in the space of locally integrable functions). Let the truncation of P, $T(P)$ be defined by $T(P) = (-P_d\xi; -P_d\xi^2 - P_{d-1}\xi; \dots; -P_d\xi^{d-1} - \dots - P_2\xi)$.

Let $\mathcal{B}_s = \{(x, w) \ |(E\frac{d}{dt} + A)x = 0, w = Cx\}$ where the $dm \times (d-1)m + n$ matrices A and E are defined by

$$A = \begin{pmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & P_0 \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & & & P_d \\ -I & \ddots & & P_{d-1} \\ & \ddots & \ddots & \vdots \\ & & -I & P_1 \end{pmatrix},$$

and $C = (0 \ 0 \ \dots \ 0 \ I)$. $\mathcal{B}_s$ is an observable state space representation for $\mathcal{B}$, in fact $(x, w) \in \mathcal{B}_s$ if and only if $x = (T(P)(\frac{d}{dt})w; w)$ and $P(\frac{d}{dt})w = 0$, as can be seen by premultiplying $A + \xi E$ with the unimodular matrix

$$U(\xi) = \begin{pmatrix} I & 0 & \dots & 0 \\ \xi I & I & \dots ; 0 & \\ \vdots & & \ddots & \\ \xi^{d-1} I \dots I & & & \end{pmatrix}:$$

$(x, w) \in \mathcal{B}_s$ if and only if

$$\begin{pmatrix} I & -T(P)(\dfrac{d}{dt}) \\ 0 & P(\dfrac{d}{dt}) \end{pmatrix} x = 0$$

and $w = Cx$, so if and only if $x = (T(P)w; w)$, with $w \in \mathcal{B}$. So $\mathcal{B}_s$ is a latent variable representation of $\mathcal{B}$, and since the equations (in the original formulation) are of degree one in x and of degree zero in w, it is a state space representation.

This also shows that $\mathcal{B}$ is stabilizable if and only if $\mathcal{B}_s$ is, or stated differently that the rank of $A + \lambda E$ equals the rank of $P(\lambda) + (d-1)m$, so in particular we have

Corollary. $P(\xi) \in \mathcal{M}$ if and only if $A + \xi E \in \mathcal{M}$.

Suppose we find a $K(\xi)$ such that $(A + \xi E; K(\xi))$ is Hurwitz. Premultiplying with $\text{diag}(U, I)$, we see that this amounts to adding an equation $K(\frac{d}{dt})x = 0$ to the state space representation. By plugging in $x = (T(P)(\frac{d}{dt})w; w)$, this means adding $Q(\frac{d}{dt})w := (K_1 T(P) + K_2)(\frac{d}{dt})w = 0$ to the equations. Since the augmented pencil is Hurwitz, so is $(P; Q)$.

4. STABLE EMBEDDING FOR A PENCIL

If the pencil $A+\xi E$ has a special form: upper block diagonal with full row rank constant or Hurwitz matrices on the diagonal, then finding K is easy.

Lemma 1. Let the pencil $\hat{A}+\xi\hat{E}$ be defined by

$$\left(\begin{array}{c|c|c|c|c} A_{11} & A_{12}+\xi E_{12} & \dots & & A_{1\ell+1}+\xi E_{1\ell+1} \\ \hline 0 & A_{22} & \dots & & A_{2\ell+1}+\xi E_{2\ell+1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \dots & A_{\ell\ell} & A_{\ell\ell+1}+\xi E_{\ell\ell+1} \\ \hline 0 & 0 & \dots & 0 & A_{\ell+1\ell+1}+\xi E_{\ell+1\ell+1} \end{array}\right) \tag{3}$$

such that each $A_{i,i}$ has full row rank for $i=1\dots\ell$ and $A_{\ell+1\ell+1}+\xi E_{\ell+1\ell+1}$ is Hurwitz. Let $\hat{K} = (\mathrm{diag}(K_{1,1}(\xi),\dots,K_{\ell,\ell}(\xi),\ 0)$, be such that each block $(A_{i,i}; K_{i,i}(\xi))$ is Hurwitz. Then $(\hat{A}+\xi\hat{E};\hat{K})$ is Hurwitz.

Proof. It is straightforward. Using row permutations we bring the resulting pencil in an upper block triangular form with the blocks $(A_{ii}; K_{ii}(\xi))$ and $A_{\ell+1\ell+1}+\xi E_{\ell+1\ell+1}$ on the diagonal, proving that the determinant of the pencil has its roots in the left half plane.

The idea of the construction is to find orthogonal constant matrices M and N such that $M(A+\xi E)N$ has the structure displayed in equation (3). Then $\hat{K}(\xi)N^T$ is the polynomial matrix that we are looking for:

Lemma 2. Let $A+\xi E$ be an arbitrary matrix pencil in $\mathcal{M}$, and let M, N be orthogonal, constant matrices such that $M(A+\xi E)N$ has the structure (3) of lemma 1. Then, there exists a polynomial matrix $K(\xi)$ such that $(A+\xi E; K(\xi))$ is Hurwitz.

Proof. Choose $\hat{K}(\xi)$ such that the embedding $(M(A+\xi E)N; \hat{K})$ is Hurwitz. Since

$$\begin{pmatrix} A+\xi E \\ \hat{K}(\xi)N^T \end{pmatrix} = \begin{pmatrix} M^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M(A+\xi E)N \\ \hat{K}(\xi))N^T \end{pmatrix}$$

it is also Hurwitz. So we can take $K(\xi) = \hat{K}(\xi)N^T$.

So the remaining task is to find for an arbitrary pencil in $\mathcal{M}$ the matrices M and N.

5. TRANSFORMING THE PENCIL

The next theorem proves that we can find M and N.

Theorem 1. Let $A+\xi E$ be a pencil in $\mathcal{M}$. Then there exist orthogonal matrices M and N such that $M(A+\xi E)N =$

$$\left(\begin{array}{c|c|c|c|c} A_{11} & A_{12}+\xi E_{12} & \dots & & A_{1\ell+1}+\xi E_{1\ell+1} \\ \hline 0 & A_{22} & \dots & & A_{2\ell+1}+\xi E_{2\ell+1} \\ \hline \vdots & \ddots & \ddots & \vdots & \vdots \\ \hline 0 & \dots & 0 & A_{\ell\ell} & A_{\ell\ell+1}+\xi E_{\ell\ell+1} \\ \hline 0 & & \dots & 0 & A_{\ell+1\ell+1}+\xi E_{\ell+1\ell+1} \end{array}\right)$$

where A_{ii} has full row rank for $i=1\dots\ell$, and the pencil $A_{\ell+1\ell+1}+\xi E_{\ell+1\ell+1}$ is either Hurwitz or missing.

Remark. By missing we mean that it has size 0×0, so then the whole last block column is missing.

Proof. The theorem is proved by induction on the size s of the pencil, the sum of the number of rows and columns.

For $s=2$ the pencil is scalar, and being in $\mathcal{M}$ it has to be a nonzero constant, so $\ell=1$, and the lower right pencil is missing (or $\ell=0$, and the lower right pencil is constant).

Now suppose that $s>2$. Let the pencil have m rows and n columns (because it is in $\mathcal{M}$ we have that $m\le n$). If $m=n$ then the pencil is Hurwitz and it is the desired form already with $\ell=0$, so assume that $m>n$.

Let N_1 be an orthogonal matrix such that $EN_1 = (0\ E_2)$, where E_2 has full column rank. Decompose AN_1 in the same way: $AN_1 = (A_1\ A_2)$.

If $A_1=0$, then $A_2+\xi E_2$ has to be Hurwitz, so we are finished with $N_1=N, M=I$, and $\ell=0$.

If $A_1\neq 0$, then let M_1 be an orthogonal matrix such that $M_1A_1 = (A_{11};\ 0)$, with A_{11} having full row rank. Decompose the products of M_1 with the other blocks likewise: $M_1A_2 =: (A_{12};\ A_{22})$, $M_1E_2 =: (E_{12};\ E_{22})$.

The lower right pencil $A_{22}+\xi E_{22}$ is in $\mathcal{M}$ and has size smaller than s, so by induction there exist M_2, N_2 such that $M_2(A_{22}+\xi E_{22})N_2$ has the desired structure.

Let $M = (\mathrm{diag}(I, M_2))M_1$, $N = N_1(\mathrm{diag}(I, N_2))$, then

$$\begin{aligned} &M(A+\xi E)N \\ &= \begin{pmatrix} I & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12}+\xi E_{12} \\ 0 & A_{22}+\xi E_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & N_2 \end{pmatrix} \\ &= \begin{pmatrix} A_{11} & A_{12}+\xi E_{12}N_2 \\ 0 & M_2(A_{22}+\xi E_{22})N_2 \end{pmatrix} \end{aligned}$$

which proves the theorem.

6. IMPLEMENTATION

Summing up: Starting with a polynomial matrix $P(\xi) \in \mathcal{M}$ we construct the associated pencil $A+\xi E \in \mathcal{M}$. We find M and N such that $M(A+\xi E)N$ has the structure of equation (3), and construct a $\hat{K}(\xi)$ which embeds this structured pencil into a Hurwitz pencil. $K = \hat{K}N^T$ then embeds the original $A+\xi E$. Decomposing $K = (K_1 \;\; K_2)$, then yields $Q(\xi)$ as $K_1(\xi)T(P)(\xi) + K_2(\xi)$.

Note that the associated pencil and $T(P)$ are defined directly in terms of the original matrix P, so the only elements that we really calculate are the matrices $K_i(\xi)$. Starting from the associated pencil this is accomplished by QR decompositions alone.

7. EXAMPLE

Let us consider the following polynomial matrix $P(\xi)$:

$$P(\xi) = \begin{pmatrix} 11\xi+1 & 9.5\xi+2 & 3\xi+3 \\ 1.4\xi+2.5 & 3\xi+1.7 & 2.7\xi+7.6 \end{pmatrix}$$

Running the algorithm, we get the following (non-reduced) pencil:

$$A+\xi E = \xi \begin{pmatrix} -11.0 & -9.5 & -3.0 \\ -1.4 & -3.0 & -2.7 \end{pmatrix} - \begin{pmatrix} 1.0 & 2.0 & 3.0 \\ 2.5 & 1.7 & 7.6 \end{pmatrix}$$

The matrices $\widehat{A}$ and $\widehat{E}$, respectively are

$$\widehat{A} = \begin{pmatrix} 4.1370 & -5.9817 & -4.6801 \\ 0 & -1.6853 & -1.8058 \end{pmatrix}$$

$$\widehat{E} = \begin{pmatrix} 0 & 2.5787 & 5.8323 \\ 0 & 0 & 14.0654 \end{pmatrix}$$

For the matrix $\widehat{K}_F$ we take

$$\widehat{K}_F = [0 \quad 0 \quad -1]$$

In consequence, $K_F = \widehat{K}_F Q^T$ is

$$K_F = [-0.7549 \quad -0.6310 \quad -0.1787]$$

Since $d = 1$, in this case $Q(\xi) = K_F$. Then we see that the embedings for the non reduced pencil $A+\xi E$ and its corresponding staircase form $\hat{A}+\xi\hat{E}$, are, respectively

$$\begin{pmatrix} A+\xi E \\ \hline K_F \end{pmatrix} =$$

$$\begin{pmatrix} -11.\xi-1. & -9.500\xi-2. & -3.\xi-3. \\ -1.400\xi-2.500 & -3.\xi-1.700 & -2.700\xi-7.600 \\ \hline -0.7549 & -0.6310 & -0.1787 \end{pmatrix}$$

with determinant $-.8854\cdot 10^{-15}\xi^2 - .4066\cdot 10^{-15}\xi - 6.972 \approx -6.972$.
and

$$\begin{pmatrix} \hat{A}+\xi\hat{E} \\ \hline \widehat{K}_F \end{pmatrix} =$$

$$\begin{pmatrix} -4.137 & 2.579\xi+5.982 & 5.832\xi+4.680 \\ 0 & 1.685 & 14.07\xi+1.806 \\ \hline 0 & 0 & -1 \end{pmatrix}$$

with determinant 6.971.

The program finishes showing

$$W(\xi) = \begin{pmatrix} P(\xi) \\ \hline Q(\xi) \end{pmatrix} =$$

$$\begin{pmatrix} 11.\xi+1. & 9.500\xi+2. & 3.\xi+3. \\ 1.400\xi+2.500 & 3.\xi+1.700 & 2.700\xi+7.600 \\ \hline -.7549 & -.6310 & -.1787 \end{pmatrix}$$

with $\det(W(\xi)) = -.8854 \cdot 10^{-15}\xi^2 - .4066 \cdot 10^{-15}\xi - 6.972 \approx -6.972$.

8. NUMERICAL CONSIDERATIONS

In the preceding section we gave an example in which the final matrix W is Hurwitz, with two poles with a real part of approximately $-4/9$ and a modulus of magnitude 107. These poles stem from rounding errors: The reduced pencil is embedded in a unimodular matrix (because the starting matrix P was not only stabilizable, but in fact controllable), but after transformation the non reduced pencil is embedded in a Hurwitz one, although in theory we have only applied constant orthogonal transformations. If we run higher order examples more problems occur, and the resulting matrix W is often not even Hurwitz.

Although we do not exactly understand the relation we have noticed that there is a relation with geometry of the space of pencils. This geometry has been described by Elmroth (1995). From a global point of view we can say that generically every wide polynomial matrix will be controllable, while we are trying to construct an embedding in a unimodular matrix, a highly non generic form. So the existence of numerical difficulties is to be expected. In the case of degree higher than one, the problem is even trickier: in the calculation of the generalized Schur form we only employ orthogonal matrices, which leave the singular values of E and A unchanged (in theory). But note that that $A = \mathrm{diag}(I, P_0)$ has many singular values equal to one, which makes the pencil $A+\xi E$ highly non generic. After calculation of $\hat{K}$ the calculation of K is supposedly done by the inverse transformations. If we would carry out the actual inverse transformation not only on $\hat{K}$ but also on $\hat{A} + \xi \hat{E}$ we are probably not exactly back in our original pencil $A + \xi E$. So the calculated Q is not the Q that we need.

The proposed algorithm by Beelen and Van Dooren for the unimodular embedding problem calculates a Q of lower degree than P. Allowing the degree of Q to be higher gives more freedom in the construction ($(A_{ii}; K_i)$ has to be unimodular instead of invertible). For the unimodular embedding problem this does not give a lot of freedom.

For the stable embedding problem the situation looks more promising. Not only is the set of stable matrices an open subset of all square polynomial matrices of the same degree, but also the increased freedom in choosing $\hat{K}$ is bigger. At this moment we have not worked out this in full detail, but we will address both the underlying structure causing the difficulties and the proposed improvements of the algorithm in more detail in the near future.

9. MORE EXAMPLES

In table 9 we show the result of twelve examples of polynomial matrices with the corresponding pencils. The last two examples correspond to real physical systems, a cement kiln, Ex_K and and an electric motor, Ex_M. Note: The sizes of the matrices below are indicated as (m, n),etc. $\sigma_1(A)$ denotes the number of singular values of A equal to 1. We see that only in case of degree 1 we end up with W that is Hurwitz.

$d = 1$	P	$A + \xi E$	$\sigma_1(A)$	$W(\xi)$
Ex_T	$(3, 6)$	$(3, 6)$	3	ok
Ex_0	$(2, 3)$	$(2, 3)$	0	ok
Ex_1	$(5, 14)$	$(5, 14)$	0	ok
Ex_2	$(7, 9)$	$(7, 9)$	0	not ok
$d = 2$				
Ex_1	$(5, 24)$	$(10, 29)$	6	not ok
Ex_2	$(4, 6)$	$(8, 10)$	4	not ok
$d = 3$				
Ex_1	$(3, 4)$	$(9, 10)$	6	not ok
Ex_2	$(3, 5)$	$(9, 11)$	6	not ok
Ex_3	$(4, 6)$	$(12, 14)$	8	not ok
$d = 4$				
Ex_1	$(4, 5)$	$(16, 17)$	12	not ok
$d = 6$				
Ex_K	$(3, 5)$	$(18, 20)$	15	not ok
$d = 7$				
Ex_M	$(2, 5)$	$(14, 17)$	12	not ok

Table 1. *12 examples*

The first column includes the set of examples for different degrees d. Next to it, we give the sizes of the corresponding polynomial matrices $P(\xi)$ under study as well as the size of their associated pencils in the third column (notice the way the size increases). Columns 4 contains the number of singular values of that equal 1, and the last column indicates whether the final W is Hurwitz or not. Note that only in a few examples the algorithm finds a Hurwitz matrix.

10. THE PSEUDOSPECTRUM OF A PENCIL

A pseudospectrum (or ϵ *pseudospectrum* Higham and Tisseur (2002), Wright and Trefethen (2002)) for a rectangular matrix is a generalization of the spectrum for square matrices. There also exists a generalization for non-square matrix pencils Boutry et al. (2004), Wright and Trefethen (2002)).

$$\Lambda_\varepsilon(E, A) = \{\lambda \in \mathbb{C} : ||(\lambda E - A)|| \leq \varepsilon\}$$

Linked to the latter definition is the following one (Boutry et al. (2004)) which is often used to construct the pseudospectra of square matrices:

$$f(\lambda) = \sigma_{min}(\lambda E - A)$$

$f(\lambda)$ was calculated for the twelve examples given above and the results obtained can be summarized as follows. Notice that $\sigma_{min}(\lambda E - A) \leq \sigma_{min}(A)$. Since $A + \xi E$ has full row rank for all $\lambda \in \mathbb{C}$ we can see that $f(\lambda)$ is strictly positive. Two graphs are displayed in figures 1 and 2.

Although the set of curves look very different at first sight, there is a pattern. For $d = 1$ the family of curves $f(\lambda)$ is - basically - up concave and it is floating above a *critical plane* , $f(\lambda) = 1$.

For $d = 2$, the curves are shifted down and two downward spikes appear (they are the local minima). Moreover, there appears a local maximum in $f(\lambda) = 1$ The up concavity is still present but both local minima have *crossed* the forbidden region, the plane $f(\lambda) = 1$. If we increase the value of d, this pattern becomes stronger, the local minima go down to almost zero, and the local maximum peak, starts to go up (with respect to lambdas of high magnitude, although $f(\lambda) = 1$ *is fixed for all the curves)* , until the roles of maximum - minimum are inverted. Then, $\sigma_{min}(A)$ becomes the global maximum of the the curves (for $d = 7$). If we are to embed $P(\xi)$ stably by $Q(\xi)$ then the resultant determinant, $det(W(\xi))$ will have many eigenvalues in the left hand side of the complex plane but also some on the right side of a big magnitude.

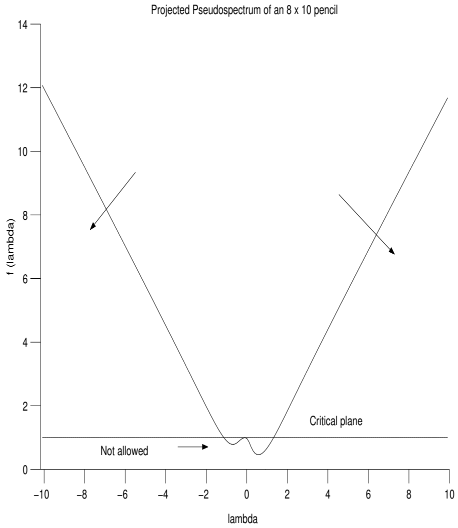

Fig. 1. *Projected pseudospectrum of an 8 x 10 pencil.*

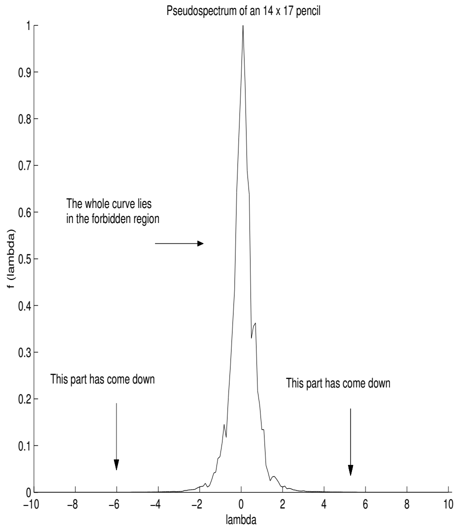

Fig. 2. *Projected pseudospectrum of an 14 x 17 pencil.*

11. COMMENTS

Unimodular and stable embeddings are quite meaningful in polynomial systems and control theory. Nevertheless, although such embeddings were found to be theoretically applicable, we found out that this is true but up to some extend. We realize that some other theoretic considerations had (and still have) to be taken into account, not only to complete the theory that already exist, but also to prevent the arising of numerical problems at the moment of actual implementation (lack of accuracy, ill conditioning and even ill posing). We shall report on it in the future.

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