APPROXIMATE OBSERVER ERROR LINEARIZATION FOR MULTI-OUTPUT SYSTEMS

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Abstract: In this paper we discuss a generalization of the extended Luenberger observer for multi-output systems. Our approach can be interpreted as approximate error linearization. While the extended Luenberger observer results from a first order approximation of exact error dynamics, this paper provides an explicit formula for a second order approximation. The design procedure is formulated in terms of Lie derivatives and Lie brackets.

Keywords: Observers, differential-geometric methods, multi-output systems

1. INTRODUCTION

We consider observer design for nonlinear multi-output systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x}$ denotes the state and $\boldsymbol{y}$ the measured output. One approach to observer design is to find a nonlinear change of coordinates which transforms (1) into a system with linear output map and linear dynamics driven by a nonlinear output injection. For a system in this so-called nonlinear observer canonical form, observer design is a straightforward task. The resulting error dynamics are linear in the transformed coordinates (Krener and Respondek, 1985; Xia and Gao, 1988; Xia and Gao, 1989). This approach is often called observer error linearization. The concept has been extended to adaptive systems (Marino and Tomei, 1995).

However, the conditions of the error linearizability are rather restrictive. Even if a system is error linearizable, it is extremely difficult to compute the associated coordinate transformation since it requires the symbolic solution of certain differential equations. Although alternative computation methods have been developed in (Keller, 1987; Phelps, 1991), it is still difficult to obtain a symbolic solution.

In recent years the problem of an approximate observer error linearization attracted some attention. For example, the coordinate transformation can be approximated by polynomials using Poincaré's normal form theory (Krener *et al.*, 1991). Similarly, one could also employ splines (Lynch and Bortoff, 1997; Lynch and Bortoff, 2001). Other approaches can be found in (Banaszuk and Sluis, 1997; Nam, 1997).

A completely different way to bypass the symbolic solution of any differential equations is the use of the extended Luenberger observer (Bestle and Zeitz, 1983; Zeitz, 1987; Birk and Zeitz, 1988). This observer consists of a simulation part and an injected difference of measured and estimated output weighted by a state-dependent observer

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gain. The observer gain is determined by Taylor linearization of the error dynamics in transformed coordinates. The extended Luenberger observer can be considered as a first order approximation of exact observer error linearization.

A generalization of the extended Luenberger observer has been proposed in (Röbenack and Lynch, 2004), where a sequence of observer gains is constructed to successively remove nonlinear terms in the error dynamics. These gain vectors can be computed by means of Lie derivatives and Lie brackets, i.e., without an explicit symbolic solution of differential equations.

In this contribution, we will extend the work in (Röbenack and Lynch, 2004) to multi-output systems. For a first order approximation, the gain matrix that can be calculated according to (Birk and Zeitz, 1988). For higher order approximations, the observer gain becomes tensor-valued. Therefore, we restrict ourselves to a second order approximation of exact linear error dynamics.

2. NORMAL FORM OBSERVER DESIGN

Consider system (1) with smooth maps $\boldsymbol{f}:\mathbb{R}^n\to\mathbb{R}^n$ and $\boldsymbol{h}:\mathbb{R}^n\to\mathbb{R}^p$. Let $\boldsymbol{x}_0\in\mathbb{R}^n$ be a vector such that $\boldsymbol{f}(\boldsymbol{x}_0)=0$, $\boldsymbol{h}(\boldsymbol{x}_0)=0$. We assume that there exists a smooth change of coordinates $\boldsymbol{z}=T(\boldsymbol{x})$ with the inverse map $\boldsymbol{x}=S(\boldsymbol{z})$ defined in a neighbourhood of $\boldsymbol{x}_0\in\mathbb{R}^n$ such that (1) can be expressed in *nonlinear observer canonical form*

$$\dot{\boldsymbol{z}}=A\boldsymbol{z}+\boldsymbol{\alpha}(\boldsymbol{y}),\quad \boldsymbol{y}=C\boldsymbol{z} \tag{2}$$

with the smooth output injection $\boldsymbol{\alpha}:\mathbb{R}^p\to\mathbb{R}^n$. The matrices $A\in\mathbb{R}^{n\times n}$ and $C\in\mathbb{R}^{p\times n}$ are block diagonal matrices $A=\operatorname{diag}(A_1,\ldots,A_p)$, $C=\operatorname{diag}(C_1,\ldots,C_p)$, where each pair

$$A_i=\begin{pmatrix}0&\cdots&0&0\\1&\cdots&0&0\\\vdots&\ddots&\vdots&\vdots\\0&\cdots&1&0\end{pmatrix}\in\mathbb{R}^{\kappa_i\times\kappa_i}$$
$$C_i=\begin{pmatrix}0&\cdots&0&1\end{pmatrix}\in\mathbb{R}^{1\times\kappa_i}$$

is in dual Brunovsky form. The synthesis of an observer is straightforward. An observer of the form

$$\dot{\hat{\boldsymbol{z}}}=A\hat{\boldsymbol{z}}+\boldsymbol{\alpha}(\boldsymbol{y})+L\,(\boldsymbol{y}-\hat{\boldsymbol{y}}),\quad \hat{\boldsymbol{y}}=C\hat{\boldsymbol{z}} \tag{3}$$

with $L\in\mathbb{R}^{n\times p}$ yields an observation error $\tilde{\boldsymbol{z}}=\boldsymbol{z}-\hat{\boldsymbol{z}}$ governed by the linear differential equation

$$\dot{\tilde{\boldsymbol{z}}}=(A-LC)\,\tilde{\boldsymbol{z}}. \tag{4}$$

The eigenvalues of $A-LC$ can be arbitrarily assigned. For a prescribed characteristic polynomial

$$\begin{aligned}\rho(\lambda)&=\det(\lambda I-A+LC)\\&=\prod_{i=1}^{p}\det(\lambda I-A_i+L_iC_i)\\&=\prod_{i=1}^{p}\left(p_{i0}+\cdots+p_{i\kappa_i-1}\lambda^{\kappa_i-1}+\lambda^{\kappa_i}\right)\end{aligned} \tag{5}$$

of the error dynamics (4) we have to set $L=\operatorname{diag}(L_1,\ldots,L_p)\in\mathbb{R}^{n\times p}$ with $L_i=(p_{i0},\ldots,p_{i\kappa_i-1})^T\in\mathbb{R}^{\kappa_i}$.

Estimates of $\boldsymbol{x}$ can be obtained by $\hat{\boldsymbol{x}}=S(\hat{\boldsymbol{z}})$. However, it is often more convenient to implement the observer in the original coordinates. With $\dot{\hat{\boldsymbol{z}}}=T'(\hat{\boldsymbol{x}})\dot{\hat{\boldsymbol{x}}}$ one obtains the observer

$$\dot{\hat{\boldsymbol{x}}}=\boldsymbol{f}(\hat{\boldsymbol{x}})+\boldsymbol{k}_\infty(\hat{\boldsymbol{x}},\boldsymbol{y})\ , \tag{6}$$

where $\boldsymbol{k}_\infty:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}^n$ is given by $\boldsymbol{k}_\infty(\hat{\boldsymbol{x}},\boldsymbol{y})=(T'(\hat{\boldsymbol{x}}))^{-1}\left(\boldsymbol{\alpha}(\boldsymbol{y})-\boldsymbol{\alpha}(\boldsymbol{h}(\hat{\boldsymbol{x}}))\,+\,L(\boldsymbol{y}-\boldsymbol{h}(\hat{\boldsymbol{x}}))\right)$. Because the design procedure is based on the nonlinear observer canonical form (2), the observers (3) and (6) are called *normal form observers.*

3. OBSERVER CANONICAL FORM

The observability matrix of (1) has the form

$$\begin{pmatrix}\mathrm{d}\boldsymbol{h}(\boldsymbol{x})\\\mathrm{d}L_{\boldsymbol{f}}\boldsymbol{h}(\boldsymbol{x})\\\vdots\\\mathrm{d}L_{\boldsymbol{f}}^{n-1}\boldsymbol{h}(\boldsymbol{x})\end{pmatrix}. \tag{7}$$

Starting from the top we select the first linearly independent rows of (7). This rows are sorted into a so-called *selection matrix*

$$Q(\boldsymbol{x})=\left(\begin{array}{c}\mathrm{d}h_1(\boldsymbol{x})\\\vdots\\\mathrm{d}L_{\boldsymbol{f}}^{\kappa_1-1}h_1(\boldsymbol{x})\\\hline\vdots\\\hline\mathrm{d}h_p(\boldsymbol{x})\\\vdots\\\mathrm{d}L_{\boldsymbol{f}}^{\kappa_p-1}h_p(\boldsymbol{x})\end{array}\right). \tag{8}$$

The positive integers $\kappa_1,\ldots,\kappa_p$ are called *observability indices* (Nijmeijer, 1981). We say the system (1) is *observable* if $\kappa_1+\cdots+\kappa_p=n$. Then, the selection matrix (8) is regular. Next, we introduce the cumulative indices $\nu_1=\kappa_1$, $\nu_2=\kappa_1+\kappa_2$, $\ldots$, $\nu_p=\kappa_1+\cdots+\kappa_p$. Smooth vector fields $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_p:\mathbb{R}^n\to\mathbb{R}^n$ are called *starting vectors* if

$$Q(\boldsymbol{x})\cdot\boldsymbol{v}_i(\boldsymbol{x})=\boldsymbol{e}_{\nu_i},\quad 1\le i\le p, \tag{9}$$

where $\boldsymbol{e}_j$ denotes the jth unit vector. Sufficient conditions for the existence of (2) are given by the following theorem similar to (Krener and Respondek, 1985). These conditions are used by the observer design procedure. Necessary conditions are given in (Xia and Gao, 1988; Xia and Gao, 1989).

Theorem 1. There exists a local diffeomorphism $\boldsymbol{z}=T(\boldsymbol{x})$, $\boldsymbol{x}=S(\boldsymbol{z})$, $T(\boldsymbol{x}_0)=\boldsymbol{0}$ in a neighbourhood of $\boldsymbol{x}_0\in\mathbb{R}^n$, transforming (1) into (2) if

C1 $\operatorname{rank}Q(\boldsymbol{x}_0)=n$,
C2 $[ad_{-\boldsymbol{f}}^{\ell}\boldsymbol{v}_i,ad_{-\boldsymbol{f}}^{k}\boldsymbol{v}_j](\boldsymbol{x})=0$ for $0\le\ell\le\kappa_i-1$, $0\le k\le\kappa_j-1$, $1\le i,j\le p$,

C3 $\mathrm{d}\boldsymbol{h}(\boldsymbol{x}) \cdot (ad_{-\boldsymbol{f}}^{\kappa_1-1}\boldsymbol{v}_1(\boldsymbol{x}), \ldots, ad_{-\boldsymbol{f}}^{\kappa_p-1}\boldsymbol{v}_p(\boldsymbol{x})) = I$ in some neighbourhood of $\boldsymbol{x}_0$.

Sketch of the proof. The linear system (9) is solvable because of the rank condition C1, i.e., there exists smooth vector fields $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ that fulfill (9). From these starting vectors we can compute the Lie brackets $ad_{-\boldsymbol{f}}^{\ell}\boldsymbol{v}_i$ with $\ell = 0, \ldots, \kappa_i - 1$, $i = 1, \ldots, p$. It can be shown that these vector fields are linearly independent. Due to the integrability condition C2 we can apply the simultaneous rectification theorem (Nijmeijer and van der Schaft, 1990, Theorem 2.36). More precisely, in a neighbourhood of $\boldsymbol{x}_0$ there exists a local diffeomorphism $\boldsymbol{z} = T(\boldsymbol{x})$ such that

$$T'(\boldsymbol{x})\, ad_{-\boldsymbol{f}}^{\ell}\boldsymbol{v}_i(\boldsymbol{x}) = \frac{\partial}{\partial z_{\ell+1+\nu_{i-1}}} \qquad (10)$$

for $\ell = 0, \ldots, \kappa_i - 1$, $i = 1, \ldots, p$. Careful calculation reveals that in this coordinate system the Jacobian matrix of the vector field $\boldsymbol{f}$ from (1) has the following structure (Krener and Respondek, 1985, Theorem 5.1):

$$\left(\begin{array}{cccc|cccc|c|cccc}
0 & \cdots & 0 & * & & & & * & & & & & * \\
1 & \ddots & \vdots & * & & & & * & & & & & * \\
 & \ddots & 0 & * & & & & * & & & & & * \\
0 & & 1 & * & & & & * & & & & & * \\
\hline
 & & & * & 0 & \cdots & 0 & * & & & & & * \\
 & & & * & 1 & \ddots & \vdots & * & & & & & * \\
 & & & * & & \ddots & 0 & * & & & & & * \\
 & & & * & 0 & & 1 & * & & & & & * \\
\hline
 & & & \vdots & & & & \vdots & \ddots & & & & \vdots \\
\hline
 & & & * & & & & * & & 0 & \cdots & 0 & * \\
 & & & * & & & & * & & 1 & \ddots & \vdots & * \\
 & & & * & & & & * & & & \ddots & 0 & * \\
 & & & * & & & & * & & 0 & & 1 & *
\end{array}\right)$$

From this we conclude that in these coordinates the vector field has the form described in (2).

Using the inverse map $\boldsymbol{x} = S(\boldsymbol{z})$ of $\boldsymbol{z} = T(\boldsymbol{z})$ we can rewrite (10) as

$$ad_{-\boldsymbol{f}}^{\ell}\boldsymbol{v}_i(\boldsymbol{x})\big|_{\boldsymbol{x}=S(\boldsymbol{z})} = S'(\boldsymbol{z})\,\frac{\partial}{\partial z_{\ell+1+\nu_{i-1}}} \quad . \qquad (11)$$

If we collect the vector fields (11) for $\ell = 0, \ldots, \kappa_i - 1$ and $i = 1, \ldots, p$ we obtain

$$S'(\boldsymbol{z}) = \Pi(\boldsymbol{x})\big|_{\boldsymbol{x}=S(\boldsymbol{z})} \qquad (12)$$

with

$$\Pi = (\boldsymbol{v}_1, ..., ad_{-\boldsymbol{f}}^{\kappa_1-1}\boldsymbol{v}_1, ..., \boldsymbol{v}_p, ..., ad_{-\boldsymbol{f}}^{\kappa_p-1}\boldsymbol{v}_p). \quad (13)$$

Now, we will consider the output map under the action of the diffeomorphism constructed above. In $\boldsymbol{z}$-coordinates, the Jacobian matrix of $\boldsymbol{h}$ has the form

$$\frac{\partial \boldsymbol{h}(S(\boldsymbol{z}))}{\partial \boldsymbol{z}} = \frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial \boldsymbol{x}} S'(\boldsymbol{z}) = \boldsymbol{h}'(\boldsymbol{x})\,\Pi(\boldsymbol{x}) \quad . \qquad (14)$$

Eq. (9) can be written as

$$\langle\, \mathrm{d}L_{\boldsymbol{f}}^{\ell} h_i(\boldsymbol{x}), \boldsymbol{v}_i(\boldsymbol{x})\rangle = \begin{cases} 0 \text{ for } \ell = 0, \ldots, \kappa_i - 2, \\ 1 \text{ for } \ell = \kappa_i - 1. \end{cases}$$

This implies

$$\langle\, \mathrm{d}h_i(\boldsymbol{x}), ad_{-\boldsymbol{f}}^{\ell}\boldsymbol{v}_i(\boldsymbol{x})\rangle = \begin{cases} 0 \text{ for } \ell = 0, \ldots, \kappa_i - 2, \\ 1 \text{ for } \ell = \kappa_i - 1. \end{cases}$$

Hence, the Jacobian matrix (14) has the form

$$\left(\begin{array}{cccc|cccc|c|cccc}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & * & \cdots & 0 & \cdots & 0 & * \\
\hline
0 & \cdots & 0 & * & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & * \\
\hline
 & & \vdots & & & & \vdots & & \ddots & & & & \\
\hline
0 & \cdots & 0 & * & 0 & \cdots & 0 & * & \cdots & 0 & \cdots & 0 & 1
\end{array}\right)$$

with p non-zero columns. These non-zero columns can be expressed by $\mathrm{d}\boldsymbol{h}(\boldsymbol{x}) \cdot (ad_{-\boldsymbol{f}}^{\kappa_1-1}\boldsymbol{v}_1(\boldsymbol{x}), \ldots, ad_{-\boldsymbol{f}}^{\kappa_p-1}\boldsymbol{v}_p(\boldsymbol{x}))$. This matrix is the $p \times p$ identity matrix by condition C3. Therefore, the Jacobian matrix (14) has the form $\frac{\partial \boldsymbol{h}(S(\boldsymbol{z}))}{\partial \boldsymbol{z}} = C$. Since this Jacobian matrix is constant, the output map is linear with $\boldsymbol{y} = C\boldsymbol{z}$ as in (2). □

In principle, the change of coordinates can be obtained as follows: The computation of the selection matrix (8) is straight forward. Due to C1 we can compute the starting vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ from the linear system (9). Eq. (12) is solvable due to C2. Solving (12) yields S, inverting S results in T. However, it is very difficult to obtain a symbolic solution of (12) since it requires the flows of the vector fields occurring in (13), see (Nijmeijer and van der Schaft, 1990, Theorem 2.36). Condition C3 holds if $\kappa_1 = \cdots = \kappa_p$. Otherwise, condition C3 can always be ensured by an appropriate output transformation.

4. EXTENDED LUENBERGER OBSERVER

One interesting approach to avoid the symbolic solution of (12) has been proposed in (Bestle and Zeitz, 1983; Zeitz, 1987; Birk and Zeitz, 1988). The observer

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{f}(\hat{\boldsymbol{x}}) + \sum_{i=1}^{p} \boldsymbol{k}_{1i}(\hat{\boldsymbol{x}})(y_i - \hat{y}_i), \quad \hat{\boldsymbol{y}} = \boldsymbol{h}(\hat{\boldsymbol{x}}) \quad (15)$$

with the vectors fields $\boldsymbol{k}_{11}, \ldots, \boldsymbol{k}_{1p} : \mathbb{R}^n \to \mathbb{R}^n$ has the classical Luenberger structure. Assume that the conditions of Theorem 1 are fulfilled. Then there exists a transformation of (1) into observer canonical form (2). We apply this transformation to (15) and obtain

$$\begin{aligned} \dot{\hat{\boldsymbol{z}}} &= A\hat{\boldsymbol{z}} + \boldsymbol{\alpha}(\hat{\boldsymbol{y}}) \\ &\quad + (S'(\hat{\boldsymbol{z}}))^{-1} \sum_{i=1}^{p} \boldsymbol{k}_{1i}(S(\hat{\boldsymbol{z}}))\,(y_i - \hat{y}_i) \quad . \end{aligned}$$

The observation error $\tilde{\boldsymbol{z}} = \boldsymbol{z} - \hat{\boldsymbol{z}}$ is governed by

$$\begin{aligned}\dot{\tilde{\boldsymbol{z}}} &= A\tilde{\boldsymbol{z}} + \boldsymbol{\alpha}(\boldsymbol{y}) - \boldsymbol{\alpha}(\hat{\boldsymbol{y}}) \\ &\quad - (S'(\hat{\boldsymbol{z}}))^{-1} \sum_{i=1}^{p} \boldsymbol{k}_{1i}(S(\hat{\boldsymbol{z}}))\, \tilde{y}_i\end{aligned} \tag{16}$$

with $\tilde{\boldsymbol{y}} = \boldsymbol{y} - \hat{\boldsymbol{y}}$. Without the knowledge of the transformation, the term $\boldsymbol{\alpha}(\boldsymbol{y})$ must be regarded as unknown even though we have $\boldsymbol{y} = C\boldsymbol{z}$. We expand $\boldsymbol{\alpha}(\boldsymbol{y})$ along the reference output curve $\hat{\boldsymbol{y}}$ of the observer, i.e.,

$$\boldsymbol{\alpha}(\boldsymbol{y}) = \boldsymbol{\alpha}(\hat{\boldsymbol{y}}) + \sum_{i=1}^{p} \frac{\partial}{\partial \hat{y}_i} \boldsymbol{\alpha}(\hat{\boldsymbol{y}})\, \tilde{y}_i + o(\|\tilde{\boldsymbol{y}}\|) \ .$$

Applying this first order series expansion to (16) results in the linearized error equation

$$\begin{aligned}\dot{\tilde{\boldsymbol{z}}} = A\tilde{\boldsymbol{z}} &- \sum_{i=1}^{p} \left((S'(\hat{\boldsymbol{z}}))^{-1} \boldsymbol{k}_{1i}(\hat{\boldsymbol{x}}) - \frac{\partial}{\partial \hat{y}_i} \boldsymbol{\alpha}(\hat{\boldsymbol{y}}) \right) \bar{C}_i \tilde{\boldsymbol{z}} \\ &+ o(\|\tilde{\boldsymbol{z}}\|),\end{aligned} \tag{17}$$

where $\bar{C}_i$ denotes the ith row of the matrix C. Let $\bar{L}_i \in \mathbb{R}^n$ denote the ith column of L. If we set

$$\boldsymbol{k}_{1i}(\hat{\boldsymbol{x}}) = S'(\hat{\boldsymbol{z}})\bar{L}_i + S'(\hat{\boldsymbol{z}}) \frac{\partial}{\partial \hat{y}_i} \boldsymbol{\alpha}(\hat{\boldsymbol{y}}), \tag{18}$$

the linearized error dynamics (17) becomes

$$\begin{aligned}\dot{\tilde{\boldsymbol{z}}} &= A\tilde{\boldsymbol{z}} - \sum_{i=1}^{p} \bar{L}_i \bar{C}_i \tilde{\boldsymbol{z}} + o(\|\tilde{\boldsymbol{z}}\|) \\ &= (A - LC)\tilde{\boldsymbol{z}} + o(\|\tilde{\boldsymbol{z}}\|) \ .\end{aligned} \tag{19}$$

This is a first order approximation of exactly linear error dynamics (4).

Now, we want to express the observer gain (18) in $\boldsymbol{x}$-coordinates. Due to (12) and (13) we obtain

$$S'(\hat{\boldsymbol{z}})\bar{L}_i = p_{i0}\boldsymbol{v}_i(\hat{\boldsymbol{x}}) + \cdots + p_{i\kappa_i} ad_{-\boldsymbol{f}}^{\kappa_i - 1} \boldsymbol{v}_i(\hat{\boldsymbol{x}}) \ ,$$

where $p_{i0}, \ldots, p_{i\kappa_i}$ are the coefficients of the characteristic polynomial (5). Further (but rather lengthy) calculations show that

$$S'(\hat{\boldsymbol{z}}) \frac{\partial}{\partial \hat{y}_i} \boldsymbol{\alpha}(\hat{\boldsymbol{y}}) = ad_{-\boldsymbol{f}}^{\kappa_i} \boldsymbol{v}_i(\hat{\boldsymbol{x}}) \ . \tag{20}$$

We finally obtain

$$\boldsymbol{k}_{1i}(\hat{\boldsymbol{x}}) = \sum_{j=0}^{\kappa_i} p_{ij} ad_{-\boldsymbol{f}}^{j} \boldsymbol{v}_i(\hat{\boldsymbol{x}}) \text{ with } p_{i\kappa_i} := 1, \tag{21}$$

see (Birk and Zeitz, 1988). Eq. (21) can be regarded as a generalization of Ackermann's formula for nonlinear multi-output systems (Ackermann, 1977). Since the observer gain is computed by an extended Taylor linearization technique, the observer (15) is called *extended Luenberger observer*.

If we put the vector fields $\boldsymbol{k}_{11}, \ldots, \boldsymbol{k}_{1p}$ into a $n \times p$ matrix $K_1(\hat{\boldsymbol{x}}) = (\boldsymbol{k}_{11}(\hat{\boldsymbol{x}}), \ldots, \boldsymbol{k}_{1p}(\hat{\boldsymbol{x}}))$, the observer (15) can be written as

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{f}(\hat{\boldsymbol{x}}) + K_1(\hat{\boldsymbol{x}})(\boldsymbol{y} - \boldsymbol{h}(\hat{\boldsymbol{x}})). \tag{22}$$

5. SECOND ORDER APPROXIMATION

The extended Luenberger observer is based on a first order approximation of linear error dynamics. To achieve a second order approximation we consider an observer of the form

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}} &= \boldsymbol{f}(\hat{\boldsymbol{x}}) + \sum_{i=1}^{p} \boldsymbol{k}_{1i}(\hat{\boldsymbol{x}})(y_i - \hat{y}_i) \\ &\quad + \sum_{i=1}^{p} \sum_{j=1}^{p} \boldsymbol{k}_{2ij}(\hat{\boldsymbol{x}})(y_i - \hat{y}_i)(y_j - \hat{y}_j)\end{aligned} \tag{23}$$

with additional vector fields $\boldsymbol{k}_{211}, \boldsymbol{k}_{212}, \ldots, \boldsymbol{k}_{2pp} : \mathbb{R}^n \to \mathbb{R}^n$. The error dynamics is governed by

$$\begin{aligned}\dot{\tilde{\boldsymbol{z}}} &= A\tilde{\boldsymbol{z}} + \boldsymbol{\alpha}(\boldsymbol{y}) - \boldsymbol{\alpha}(\hat{\boldsymbol{y}}) \\ &\quad - (S'(\hat{\boldsymbol{z}}))^{-1} \sum_{i=1}^{p} \boldsymbol{k}_{1i}(S(\hat{\boldsymbol{z}}))\, \tilde{y}_i \\ &\quad - (S'(\hat{\boldsymbol{z}}))^{-1} \sum_{i=1}^{p} \sum_{j=1}^{p} \boldsymbol{k}_{2ij}(S(\hat{\boldsymbol{z}}))\, \tilde{y}_i \tilde{y}_j \ .\end{aligned} \tag{24}$$

A second order series expansion of the output injection yields

$$\begin{aligned}\boldsymbol{\alpha}(\boldsymbol{y}) &= \boldsymbol{\alpha}(\hat{\boldsymbol{y}}) + \sum_{i=1}^{p} \frac{\partial}{\partial \hat{y}_i} \boldsymbol{\alpha}(\hat{\boldsymbol{y}})\, \tilde{y}_i \\ &\quad + \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{1}{2} \frac{\partial^2}{\partial \hat{y}_i \partial \hat{y}_j} \boldsymbol{\alpha}(\hat{\boldsymbol{y}})\, \tilde{y}_i \tilde{y}_j + o(\|\tilde{\boldsymbol{y}}\|^2) \ .\end{aligned}$$

Choosing $\boldsymbol{k}_{1i}$ according to (18) results in

$$\begin{aligned}\dot{\tilde{\boldsymbol{z}}} &= (A - LC)\tilde{\boldsymbol{z}} \\ &\quad + \sum_{i=1}^{p} \sum_{j=1}^{p} \left(\frac{1}{2} \frac{\partial^2}{\partial \hat{y}_i \partial \hat{y}_j} \boldsymbol{\alpha}(\hat{\boldsymbol{y}}) \right. \\ &\quad \left. - (S'(\hat{z}))^{-1} \boldsymbol{k}_{2ij}(\hat{\boldsymbol{x}}) \right) \tilde{y}_i \tilde{y}_j + o(\|\tilde{\boldsymbol{y}}\|^2).\end{aligned} \tag{25}$$

The second order terms occurring in (25) vanish if and only if

$$\boldsymbol{k}_{2ij}(\hat{\boldsymbol{x}}) = \frac{1}{2} S'(\hat{\boldsymbol{z}}) \frac{\partial^2}{\partial \hat{y}_i \partial \hat{y}_j} \boldsymbol{\alpha}(\hat{\boldsymbol{y}}) \ . \tag{26}$$

In this case, Eq. (25) becomes $\dot{\tilde{\boldsymbol{z}}} = (A - LC)\, \tilde{\boldsymbol{z}} + o(\|\tilde{\boldsymbol{z}}\|^2)$, where we obtain a second order approximation of exactly linear error dynamics (4).

We want to express the observer gain (26) in $\boldsymbol{x}$-coordinates. From (12) and (13) we conclude

$$\frac{\partial}{\partial z_{\nu_i}} S(\boldsymbol{z}) = ad_{-\boldsymbol{f}}^{\kappa_i - 1} \boldsymbol{v}_i(\boldsymbol{x}) \ . \tag{27}$$

Eq. (20) can be rewritten as

$$S'(\boldsymbol{z}) \frac{\partial}{\partial z_{\nu_i}} \boldsymbol{\alpha}(C\boldsymbol{z}) = ad_{-\boldsymbol{f}}^{\kappa_i} \boldsymbol{v}_i(\boldsymbol{x}) \tag{28}$$

because $\boldsymbol{y} = C\boldsymbol{z} = (z_{\nu_1}, \ldots, z_{\nu_p})^T$. Differentiating (28) w.r.t. z_{ν_j} yields

$$\begin{aligned}&\frac{\partial}{\partial z_{\nu_j}} \left(S'(\boldsymbol{z}) \frac{\partial}{\partial z_{\nu_i}} \boldsymbol{\alpha}(C\boldsymbol{z}) \right) \\ &= S'(\boldsymbol{z}) \frac{\partial^2}{\partial z_{\nu_i} \partial z_{\nu_j}} \boldsymbol{\alpha}(C\boldsymbol{z}) \\ &\quad + \left(\frac{\partial}{\partial z_{\nu_j}} S'(\boldsymbol{z}) \right) \frac{\partial}{\partial z_{\nu_i}} \boldsymbol{\alpha}(C\boldsymbol{z}) \ .\end{aligned} \tag{29}$$

according to the product rule. The left hand side of (29) results in

$$\begin{aligned}
&\frac{\partial}{\partial z_{\nu_j}}\left(S'(\boldsymbol{z})\frac{\partial}{\partial z_{\nu_i}}\boldsymbol{\alpha}(C\boldsymbol{z})\right)\\
&= \frac{\partial}{\partial z_{\nu_j}}\, ad^{\kappa_i}_{-\boldsymbol{f}}\boldsymbol{v}_i(\boldsymbol{x})\Big|_{\boldsymbol{x}=S(\boldsymbol{z})}\\
&= \left(\frac{\partial}{\partial \boldsymbol{x}} ad^{\kappa_i}_{-\boldsymbol{f}}\boldsymbol{v}_i(\boldsymbol{x})\right)\cdot\frac{\partial}{\partial z_{\nu_j}}S(\boldsymbol{z})\\
&= \left(\frac{\partial}{\partial \boldsymbol{x}} ad^{\kappa_i}_{-\boldsymbol{f}}\boldsymbol{v}_i(\boldsymbol{x})\right)\cdot ad^{\kappa_j-1}_{-\boldsymbol{f}}\boldsymbol{v}_j(\boldsymbol{x}) \ .
\end{aligned} \tag{30}$$

From (27) we conclude

$$\begin{aligned}
\frac{\partial}{\partial z_{\nu_j}}S'(\boldsymbol{z}) &= \frac{\partial}{\partial z_{\nu_j}}\frac{\partial}{\partial \boldsymbol{z}}S(\boldsymbol{z})\\
&= \frac{\partial}{\partial \boldsymbol{z}}\frac{\partial}{\partial z_{\nu_j}}S(\boldsymbol{z})\\
&= \frac{\partial}{\partial \boldsymbol{z}}\, ad^{\kappa_j-1}_{-\boldsymbol{f}}\boldsymbol{v}_j(\boldsymbol{x})\Big|_{\boldsymbol{x}=S(\boldsymbol{z})}\\
&= \frac{\partial}{\partial \boldsymbol{x}} ad^{\kappa_j-1}_{-\boldsymbol{f}}\boldsymbol{v}_j(\boldsymbol{x})\cdot S'(\boldsymbol{z}) \ .
\end{aligned}$$

Together with (28) we get

$$\begin{aligned}
&\left(\frac{\partial}{\partial z_{\nu_j}}S'(\boldsymbol{z})\right)\frac{\partial}{\partial z_{\nu_i}}\boldsymbol{\alpha}(C\boldsymbol{z})\\
&= \left(\frac{\partial}{\partial \boldsymbol{x}} ad^{\kappa_j-1}_{-\boldsymbol{f}}\boldsymbol{v}_j(\boldsymbol{x})\right)\cdot S'(\boldsymbol{z})\frac{\partial}{\partial z_{\nu_i}}\boldsymbol{\alpha}(C\boldsymbol{z})\\
&= \left(\frac{\partial}{\partial \boldsymbol{x}} ad^{\kappa_j-1}_{-\boldsymbol{f}}\boldsymbol{v}_j(\boldsymbol{x})\right)\cdot S'(\boldsymbol{z})(S'(\boldsymbol{z}))^{-1} ad^{\kappa_i}_{-\boldsymbol{f}}\boldsymbol{v}_i(\boldsymbol{x})\\
&= \left(\frac{\partial}{\partial \boldsymbol{x}} ad^{\kappa_j-1}_{-\boldsymbol{f}}\boldsymbol{v}_j(\boldsymbol{x})\right)\cdot ad^{\kappa_i}_{-\boldsymbol{f}}\boldsymbol{v}_i(\boldsymbol{x}) \ .
\end{aligned} \tag{31}$$

Putting (29), (30) and (31) together, we finally obtain

$$\begin{aligned}
&S'(\boldsymbol{z})\frac{\partial^2}{\partial z_{\nu_i}\partial z_{\nu_j}}\boldsymbol{\alpha}(C\boldsymbol{z})\\
&= \frac{\partial}{\partial z_{\nu_j}}\left(S'(\boldsymbol{z})\frac{\partial}{\partial z_{\nu_i}}\boldsymbol{\alpha}(C\boldsymbol{z})\right)\\
&\quad - \left(\frac{\partial}{\partial z_{\nu_j}}S'(\boldsymbol{z})\right)\frac{\partial}{\partial z_{\nu_i}}\boldsymbol{\alpha}(C\boldsymbol{z})\\
&= \left(\frac{\partial}{\partial \boldsymbol{x}} ad^{\kappa_i}_{-\boldsymbol{f}}\boldsymbol{v}_i(\boldsymbol{x})\right)\cdot ad^{\kappa_j-1}_{-\boldsymbol{f}}\boldsymbol{v}_j(\boldsymbol{x})\\
&\quad - \left(\frac{\partial}{\partial \boldsymbol{x}} ad^{\kappa_j-1}_{-\boldsymbol{f}}\boldsymbol{v}_j(\boldsymbol{x})\right)\cdot ad^{\kappa_i}_{-\boldsymbol{f}}\boldsymbol{v}_i(\boldsymbol{x})\\
&= [ad^{\kappa_j-1}_{-\boldsymbol{f}}\boldsymbol{v}_j, ad^{\kappa_i}_{-\boldsymbol{f}}\boldsymbol{v}_i](\boldsymbol{x}) \ .
\end{aligned}$$

Therefore, the observer gain (26) can be written as

$$\boldsymbol{k}_{2ij}(\hat{\boldsymbol{x}}) = \frac{1}{2}[ad^{\kappa_j-1}_{-\boldsymbol{f}}\boldsymbol{v}_j, ad^{\kappa_i}_{-\boldsymbol{f}}\boldsymbol{v}_i](\hat{\boldsymbol{x}}) \ . \tag{32}$$

If we arrange the p^2 vector fields $\boldsymbol{k}_{211},\ldots,\boldsymbol{k}_{2pp}$ into a $n\times p^2$ matrix $K_2(\hat{\boldsymbol{x}}) = (\boldsymbol{k}_{211}(\hat{\boldsymbol{x}}),\ldots,\boldsymbol{k}_{21p}(\hat{\boldsymbol{x}}),\ldots,\boldsymbol{k}_{2p1}(\hat{\boldsymbol{x}}),\ldots,\boldsymbol{k}_{2pp}(\hat{\boldsymbol{x}}))$, the observer (23) can be written as

$$\begin{aligned}
\dot{\hat{\boldsymbol{x}}} &= \boldsymbol{f}(\hat{\boldsymbol{x}}) + K_1(\hat{\boldsymbol{x}})(\boldsymbol{y}-\boldsymbol{h}(\hat{\boldsymbol{x}}))\\
&\quad + K_2(\hat{\boldsymbol{x}})\left((\boldsymbol{y}-\boldsymbol{h}(\hat{\boldsymbol{x}}))\otimes(\boldsymbol{y}-\boldsymbol{h}(\hat{\boldsymbol{x}}))\right) ,
\end{aligned} \tag{33}$$

where $\otimes$ denotes the Kronecker tensor product.

6. EXAMPLE

Consider the hyperchaotic Rössler system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} -x_2 - x_3 \\ x_1 + 0.25x_2 + x_4 \\ 3 + x_1x_3 \\ -0.5x_3 + 0.05x_4 \end{pmatrix} \tag{34}$$

with the output

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) = \begin{pmatrix} x_2 \\ \ln x_3 \end{pmatrix} ,$$

see (Rössler, 1979; Peng *et al.*, 1996; Morgül, 1999). This system has the observability indices $\kappa_1 = \kappa_2 = 2$. The selection matrix (8) given by

$$Q(\boldsymbol{x}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & x_3^{-1} & 0 \\ 1 & 0 & -3x_3^{-1} & 0 \end{pmatrix}$$

is regular provided $x_3 > 0$. From (9) and (13) we get the matrix

$$\Pi(\boldsymbol{x}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_3 \\ 1 & 0.05 & -1 & -0.05 \end{pmatrix} . \tag{35}$$

For the extended Luenberger observer (15) we obtain the gain vectors

$$\boldsymbol{k}_{11}(\boldsymbol{x}) = \begin{pmatrix} -1 \\ p_{11} + 0.3 \\ 0 \\ p_{10} + 0.05p_{11} + 0.0025 \end{pmatrix}$$

and

$$\boldsymbol{k}_{12}(\boldsymbol{x}) = \begin{pmatrix} p_{20} - x_3 \\ -0.05 \\ p_{21}x_3x_3 - 3 \\ -p_{20} - 0.05p_{21} - 0.5x_3 - 0.0025 \end{pmatrix} .$$

For the second order approximation discussed in Sect. 5, Eq. (32) yields

$$\boldsymbol{k}_{222}(\boldsymbol{x}) = \begin{pmatrix} -0.5x_3 \\ 0 \\ 1.5 \\ -0.25x_3 \end{pmatrix}$$

and $\boldsymbol{k}_{2ij}(\boldsymbol{x}) = (0,0,0,0)^T$ otherwise.

For this example we can also solve (12) with (35) symbolically. The transformation T has the following form:

$$T(\boldsymbol{x}) = \begin{pmatrix} x_1 - 0.05x_2 + x_4 + 0.05\ln x_3 \\ x_2 \\ x_1 \\ \ln x_3 \end{pmatrix} .$$

Based on this change of coordinates one can compute (6) with the output injection

$$\boldsymbol{\alpha}(\boldsymbol{y}) = \begin{pmatrix} -1.0125y_1 - 1.5\,\mathrm{e}^{y_2} + 0.15\,\mathrm{e}^{-y_2} \\ 0.3y_1 - 0.05y_2 \\ -y_1 - \mathrm{e}^{y_2} \\ 3\,\mathrm{e}^{-y_2} \end{pmatrix} .$$

The numerical simulation was carried out with the CACSD package Scilab (Gomez, 1999). All observer eigenvalues were placed at -3. We used the initial values $\boldsymbol{x}(0) = (-20,0,1,15)^T$ and $\hat{\boldsymbol{x}}(0) = (0,0,1,0)^T$. Fig. 1 shows the Euclidean norm of the observation error $\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$. It can be seen that the new observer (23) based on a second order approximation converges faster than the extended Luenberger observer (15).

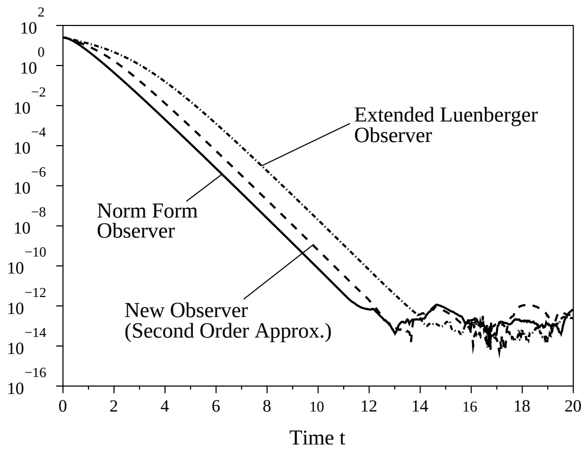

Fig. 1. Norm of the observation error

7. SUMMARY

We discussed an approximation of the normal form observer. Our approach can be considered as a generalization of the extended Luenberger observer. The computation of the observer gain involves derivatives and matrix inversion. The method proposed here can easily be extended to systems with inputs.

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