SIGNAL DECOUPLING WITH PREVIEW: PERFECT SOLUTION FOR NONMINIMUM-PHASE SYSTEMS IN THE GEOMETRIC APPROACH CONTEXT

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Abstract: The problem of making the output insensitive to an exogenous input signal possibly known with preview is tackled in the geometric approach context. The definition of minimal preview for decoupling is introduced. Necessary and sufficient constructive conditions for decoupling with minimal preview are proved by means of simple geometric arguments. The structural and the stabilizability conditions are considered separately. The minimal complexity of the solution is guaranteed by using the minimal self-bounded controlled invariant subspace. In the presence of unstable dynamics of that subspace, a steering along zeros technique completely devised in the state-space allows the solution with internal stability to be nonetheless achieved. Implementation is obtained by resorting to finite impulse response systems.

Keywords: Linear systems, geometric approach, decoupling with preview.

1. INTRODUCTION

Perfect decoupling, in the sense of complete rejection of exogenous input signals, and perfect tracking are difficult problems when nonminimum-phase systems are involved (Davison and Scherzinger, 1987; Qiu and Davison, 1993). Hence, a great deal of research effort has been directed towards these issues and it has been found that troubles concerned with internal stability can be overcome by accepting to face noncausal problems, where the signals to be localized or tracked are known in advance (Devasia *et al.*, 1996; Hunt *et al.*, 1996). Thus, several papers have been written on how to achieve noncausal inversion, first considering SISO systems (Gross *et al.*, 1994; Tsao, 1994; Marro and Fantoni, 1996) and, more recently, addressing MIMO systems both in the linear (Zou and Devasia, 1999; Marro *et al.*, 2002) and in the nonlinear case (Zeng and Hunt, 2000). These papers focused on so-called steering along zeros techniques, mainly devised in the transfer function approach. In this paper, we deal with the problem of decoupling for linear multivariable systems in the strict geometric context, i.e. all the arguments are set in the state space and the structural and stabilizability properties of controlled and conditioned invariant subspaces play a key role in achieving necessary and sufficient conditions for the exact solution to exist. In particular, the use of the minimal self-bounded controlled invariant subspace to treat stability issues allows us to achieve a sharper insight into connections between nonmimum-phase systems and noncausal problems. Moreover, when the minimal self-bounded controlled invariant subspace is stabilizable, it is useful to introduce the notion of minimal preview for decoupling.

According to a procedure well-settled in the geometric approach, the structural and the stabilizability conditions for decoupling are considered separately. On the assumption that the structural condition holds, two different situations must be considered depending on the stabilizability properties of the system: i) decoupling can be achieved exactly, provided that a 'short' preview of the exogenous signal is available, *minimal preview*; ii) decoupling can be achieved exactly, provided that a 'theoretically infinite' preview of the exogenous signal is given, *infinite preview*. To be more specific, while the structural condition for decoupling with preview — $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}^*$ (Willems, 1982; Imai *et al.*, 1983) — is the natural extension of the structural condition for measurable signal decoupling — $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$ (Bhattacharyya, 1974) — which extends, in turn, that for unaccessible signal decoupling — $\mathcal{H} \subseteq \mathcal{V}^*$ (Wonham and Morse, 1970; Wonham, 1985) —, as far as the stabilizability condition is concerned, we refer to that based on the use of the minimal self-bounded controlled invariant subspace satisfying the structural constraint — $\mathcal{V}_m = \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$ internally stabilizable — and we show that this is valid not only in the case of unaccessible or measurable exogenous signals (Basile and Marro, 1982; Schumacher, 1983; Basile *et al.*, 1984; Basile and Marro, 1992), but also in the case where the signals to be localized are known in advance. Hence, if both the structural and the stabilizability conditions hold, then, exact decoupling can be achieved by means of the sole minimal preview, whose length is connected to the number of steps of the algorithm for $\mathcal{S}^*$, the minimal $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B}$. Otherwise, if the structural condition holds but the stabilizability condition does not, it is herein shown that it is nonetheless possible to achieve decoupling of the exogenous signals with internal stability, provided that these are known in advance with infinite preview. Indeed, infinite preview is not necessary in practice: a preview sufficiently longer than the longest time-constant associated to the internal unassignable eigenvalues of $\mathcal{V}_m$ enables the problem to be solved with practically acceptable accuracy.

In the above-described context, it is worth pointing out some substantial technical differences between our approach and others dealing with signal decoupling. The first feature of our work is of a theoretical nature and concerns the use of $\mathcal{V}_m$ for checking stabilizability. In fact, in (Imai *et al.*, 1983; Wonham, 1985), the controlled invariant considered for stability is $\mathcal{V}_g^*$, the maximal internally stabilizable $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$. According to (Basile and Marro, 1982; Schumacher, 1983), here we consider $\mathcal{V}_m$, the minimal internally stabilizable $(A, \mathcal{B})$-controlled invariant self-bounded with respect to $\mathcal{C}$. Since an internally stabilizable $\mathcal{V}_m$ is contained in $\mathcal{V}_g^*$, assuming $\mathcal{V}_m$ in place of $\mathcal{V}_g^*$ has the advantage of yielding a control system with the minimum number of internal unassignable dynamics. Moreover, the analysis based on the use of $\mathcal{V}_m$ better clarifies the connections between steering along techniques and nonminimum-phase systems: since $\mathcal{V}_m \subseteq \mathcal{V}^*$, $\mathcal{V}_m$ may be not internally stabilizable if the system is nonminimum-phase. However, nonminimum-phase systems may not require infinite preview if the unstable internal unassignable eigenvalues of $\mathcal{V}^*$ are external to $\mathcal{V}_m$. The second relevant feature of our work is connected with implementation: the control laws steering the states of the controlled system along trajectories defined by the unstable internal unassignable eigenvalues of $\mathcal{V}_m$ are produced by precompensators including nonconventional control devices like finite impulse response (FIR) systems: the convolution profiles consist of the unstable trajectories recorded backwards in time.

2. DECOUPLING WITH MINIMAL PREVIEW

The discrete time-invariant linear system

$$x(k+1) = A\,x(k) + B\,u(k) + H\,h(k), \qquad (1)$$
$$y(k) = C\,x(k), \qquad (2)$$

is considered, with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^p$, controlled output $y \in \mathbb{R}^q$ and exogenous input $h \in \mathbb{R}^s$ (either unaccessible, or measurable, or known with preview). Matrices B, H, and C are assumed to be full-rank. The standard notation of the geometric approach is assumed (Basile and Marro, 1992). In this section, some basic results on unaccessible and measurable signal decoupling with stability are first recalled. Then, the problem of previewed signal decoupling with stability is stated and the main theorem concerning its solution is proved. Finally, the notion of minimal preview for decoupling is introduced.

Problem 1. (Unaccessible Signal Decoupling with Stability) Consider system (1),(2). Let $x(0) = 0$. Design a linear algebraic state feedback F such that $\sigma(A + BF) \subset \mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t \geq 0)$, $y(t) = 0$ for all $t \geq 0$.

Theorem 1. (Unaccessible Signal Decoupling with Stability) (Basile and Marro, 1982; Schumacher, 1983; Basile *et al.*, 1984; Basile and Marro, 1992) Consider system (1), (2). Let (A, B) be stabilizable. Problem 1 is solvable if and only if: i) $\mathcal{H} \subseteq \mathcal{V}^*$; ii) $\mathcal{V}_m = \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$ is internally stabilizable.

Problem 2. (Measurable Signal Decoupling with Stability) Consider system (1), (2). Let $x(0) = 0$. Design a linear algebraic state feedback F and a linear algebraic feedforward S of the measurable exogenous input h on the control input u

such that $\sigma(A+BF)\subset\mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t\geq 0)$, $y(t)=0$ for all $t\geq 0$.

Theorem 2. (Measurable Signal Decoupling with Stability) (Basile and Marro, 1992) Consider system (1), (2). Let (A,B) be stabilizable. Problem 2 is solvable if and only if: i) $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B}$; ii) $\mathcal{V}_m=\mathcal{V}^*\cap\min\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{H})$ is internally stabilizable.

Problem 3. (Previewed Signal Decoupling with Stability) Consider system (1), (2). Let $\sigma(A)\subset\mathbb{C}^{\odot}$. Let $x(0)=0$. Let $h(k)$ be known with preview of k_p steps, $\rho_M\leq k_p<\infty$. Design a linear dynamic feedforward compensator (A_c,B_c,C_c,D_c), having $h_p(k)=h(k+k_p)$ as input, such that $\sigma(A_c)\subset\mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t\geq 0)$, $y(t)=0$ for all $t\geq 0$.

Lemma 1. For any $\mathcal{Q}\subseteq\mathbb{R}^n$,

$$\min\,\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{Q})=\min\,\mathcal{S}(A,\mathcal{C},\mathcal{S}^*+\mathcal{Q}).$$

Proof: By construction, the subspaces generated by the standard algorithms for the minimal $(A,\mathcal{C})$-conditioned invariant subspaces respectively containing $\mathcal{B}+\mathcal{Q}$ and $\mathcal{B}$ satisfy the inclusions $\mathcal{S}_1'=\mathcal{B}+\mathcal{Q}\supseteq\mathcal{S}_1=\mathcal{B}$ and

$$\mathcal{S}_i'=A(\mathcal{S}_{i-1}'\cap\mathcal{C})+\mathcal{B}+\mathcal{Q}\supseteq \mathcal{S}_i=A(\mathcal{S}_{i-1}\cap\mathcal{C})+\mathcal{B},$$

for $i=2,3,\ldots,\rho_M$, where ρ_M is the number of steps for evaluating $\mathcal{S}^*$. These algorithms do not necessarily converge within the same number of steps, but the last inclusion implies $\min\,\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{Q})\supseteq\mathcal{S}^*$. Hence, it implies $\min\,\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{Q})\supseteq\mathcal{S}^*+\mathcal{B}+\mathcal{Q}\supseteq\mathcal{S}^*+\mathcal{Q}$. The latter inclusion means that $\min\,\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{Q})$ is an $(A,\mathcal{C})$-conditioned invariant containing $\mathcal{S}^*+\mathcal{Q}$, therefore

$$\min\,\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{Q})\supseteq\min\,\mathcal{S}(A,\mathcal{C},\mathcal{S}^*+\mathcal{Q}).$$

On the other hand, $\mathcal{B}+\mathcal{Q}\subseteq\mathcal{S}^*+\mathcal{Q}$ implies

$$\min\,\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{Q})\subseteq\min\,\mathcal{S}(A,\mathcal{C},\mathcal{S}^*+\mathcal{Q}).\quad\blacksquare$$

Theorem 3. (Previewed Signal Decoupling with Stability) Problem 3 is solvable if and only if: i) $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{S}^*$; ii) $\mathcal{V}_m=\mathcal{V}^*\cap\min\,\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{H})$ is internally stabilizable.

Proof: Since condition i) is well settled in the literature (Willems, 1982; Imai *et al.*, 1983), this proof will focus on condition ii).

If. First note that, owing to condition i), subspaces $\mathcal{H}_{\mathcal{S}^*}\subseteq\mathcal{S}^*$ and $\mathcal{H}_{\mathcal{V}^*}\subseteq\mathcal{V}^*$ exist such that $\mathcal{H}=\mathcal{H}_{\mathcal{S}^*}+\mathcal{H}_{\mathcal{V}^*}$. By superposition, assuming $h(k)=e_i\,\delta(k-\rho_M)$, with $k=0,1,\ldots$ and e_i $(i=0,1,\ldots,s)$ denoting the generic i-th vector of the main basis of $\mathbb{R}^s$, does not cause any loss of generality. The input $h(k)$ is assumed to be previewed of ρ_M time instants. Let τ be defined as $\tau=H\,e_i\,\delta(k-\rho_M)$ with $k=\rho_M$. Then, τ can be expressed as $\tau=\tau_{\mathcal{S}^*}+\tau_{\mathcal{V}^*}$ with $\tau_{\mathcal{S}^*}\in\mathcal{H}_{\mathcal{S}^*}$ and $\tau_{\mathcal{V}^*}\in\mathcal{H}_{\mathcal{V}^*}$. The decomposition of τ as $\tau_{\mathcal{S}^*}$ and $\tau_{\mathcal{V}^*}$ is not unique if $\mathcal{H}_{\mathcal{S}^*}\cap\mathcal{H}_{\mathcal{V}^*}\neq\{0\}$, which may occur if the system is not left-invertible, but the arguments herein presented hold for any decomposition considered. By definition of $\mathcal{S}^*$, any state belonging to $\mathcal{H}_{\mathcal{S}^*}$ can be reached from the origin in ρ_M steps at most, along a trajectory belonging to $\mathcal{C}$, therefore invisible at the output, until the last step but one. Hence, the component $\tau_{\mathcal{S}^*}$ can be nulled by applying the control input sequence driving the state from the origin to its opposite, $-\tau_{\mathcal{S}^*}$. On the other hand, the component $\tau_{\mathcal{V}^*}$ can be localized on $\mathcal{V}^*$, since both the conditions of Theorem 1 are satisfied. In fact, $\mathcal{H}_{\mathcal{V}^*}\subseteq\mathcal{V}^*$ by construction, and $\mathcal{V}^*\cap\min\,\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{H}_{\mathcal{V}^*})$ is internally stabilizable since $\mathcal{V}_m$ is internally stabilizable by assumption and, by Lemma 1,

$$\begin{aligned}&\mathcal{V}^*\cap\min\,\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{H}_{\mathcal{V}^*})\\&\quad=\mathcal{V}^*\cap\min\,\mathcal{S}(A,\mathcal{C},\mathcal{S}^*+\mathcal{H}_{\mathcal{V}^*})\\&\quad=\mathcal{V}^*\cap\min\,\mathcal{S}(A,\mathcal{C},\mathcal{S}^*+\mathcal{H})\\&\quad=\mathcal{V}^*\cap\min\,\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{H})\\&\quad=\mathcal{V}_m.\end{aligned}$$

Only if. If $\mathcal{H}\not\subseteq\mathcal{V}^*+\mathcal{S}^*$, then the effect of the input $h(k)$ cannot be made invisible at the output because of the maximality of the respective subspaces $\mathcal{V}^*$ and $\mathcal{S}^*$. In fact, $\mathcal{V}^*$ is the maximal set of initial states in $\mathcal{C}$ corresponding to trajectories indefinitely controllable on $\mathcal{C}$, while $\mathcal{S}^*$ is the maximal set of states that can be reached from the origin in a finite number of steps with all the intermediate states in $\mathcal{C}$ except the last. On the other hand, if the structural condition holds, but $\mathcal{V}_m$ is not internally stabilizable, since $\mathcal{V}_m$ is the minimal $(A,\mathcal{B}+\mathcal{H})$-controlled invariant self-bounded with respect to $\mathcal{C}$, no internally stabilizable $(A,\mathcal{B})$-controlled invariant $\mathcal{V}$ exists satisfying both $\mathcal{V}\subseteq\mathcal{C}$ and $\mathcal{H}\subseteq\mathcal{V}+\mathcal{S}^*$. $\blacksquare$

Remark 1. In Theorem 3, the assumption of stability of A is not restrictive with respect to those of stabilizability of (A,B) and detectability of (A,C) usually considered. On these hypotheses, a stable system can be obtained by dynamic output feedback according to the block diagram shown in Fig. 1. In (Marro and Zattoni, 2004), it was proved

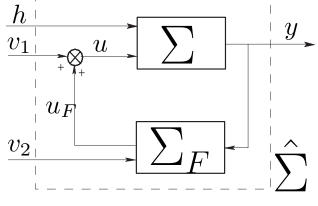

Fig. 1. Block diagram for prestabilization.

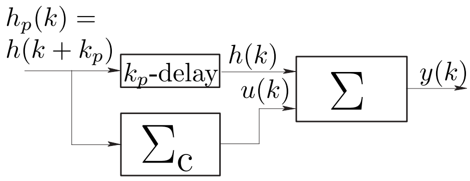

Fig. 2. Block diagram for previewed signal decoupling.

that in the extended state space of the stabilized system, the internal unassignable eigenvalues of the minimal self-bounded controlled invariant are the same as those in the state space of the original system.

Definition 1. (Minimal Preview for Decoupling). Consider system (1),(2). Let conditions i) and ii) of Theorem 3 hold. Let $\mathcal{S}_i$ be the generic i-th subspace of the sequence for the computation of $\mathcal{S}^*$, i.e. $\mathcal{S}_1 = \mathcal{B}$, $\mathcal{S}_i = A\,(\mathcal{S}_{i-1} \cap \mathcal{C}) + \mathcal{B}$, with $i = 2, \ldots, \rho_M$, and ρ_M being the minimal integer such that $\mathcal{S}_{\rho_M+1} = \mathcal{S}_{\rho_M}$. The minimal preview for decoupling is defined as $\rho - 1$, where ρ is the least integer such that $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}_\rho$ holds.

The block diagram for previewed signal decoupling with stability if the conditions of Theorem 3 are satisfied is shown in Fig. 2. The block Σ stands for the system (1),(2), Σ_c for the feedforward compensator (A_c, B_c, C_c, D_c), and the block 'k_p-delay' for a cascade of k_p unit delays inserted on the input h signal flow to model its preview of k_p steps. According to Definition 1, the preview $k_p = \rho$ is sufficient to solve the problem.

3. DECOUPLING WITH INFINITE PREVIEW: AN ALGORITHMIC SETTING

In this section, an algorithmic solution to a relaxed version of Problem 3 is devised, where the preview available is not necessarily finite and the precompensator Σ_c is not necessarily a standard dynamic system defined by a quadruple (A_c, B_c, C_c, D_c). By releasing these constraints, theoretically perfect decoupling with stability can be achieved also in the presence of unstable internal unassignable eigenvalues of $\mathcal{V}_m$. In this case, the precompensator should include not only a dynamic unit reproducing the stable dynamics associated to the motion on $\mathcal{V}_m$ and the dead-beat associated to the motion on $\mathcal{S}^*$, but it should also include a convolution unit reproducing the unstable dynamics associated to the motion on $\mathcal{V}_m$. In practice, the convolution is truncated and the additional unit working in connection with the standard dynamic unit is an FIR system. In order to guarantee a negligible truncation error, the preview k_p, which is equal to the length of the FIR window, should be sufficiently longer than the longest time constant associated to the unstable internal unassignable eigenvalues of $\mathcal{V}_m$. The algorithmic procedure herein presented encompasses also the case of decoupling with minimal preview as a special case. The structural condition of Theorem 3, namely $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}^*$, is assumed to be satisfied. Two different strategies are outlined according to whether the stabilizability condition is satisfied or not: in the former case, the minimal preview is required to obtain exact decoupling, in the latter an infinite preview is theoretically demanded.

The algorithmic setting presented in this section is built on the following basic concepts of the geometric approach. Recall that $\mathcal{V}_m$ is a locus of initial states in $\mathcal{C}$ corresponding to trajectories indefinitely controllable in $\mathcal{C}$ and that $\mathcal{S}^*$ is the maximal set of states that can be reached from the origin in ρ_M steps along trajectories with all the intermediate states in $\mathcal{C}$. Then, suppose that an impulse is applied to the input h at the time ρ_M, thus producing a component of the state $x_h \in \mathcal{H}$, which is decomposable as $x_h = x_{h,S} + x_{h,V}$, with $x_{h,S} \in \mathcal{S}^*$ and $x_{h,V} \in \mathcal{V}_m$ — note that $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{S}^*$ is implied by the structural condition (Basile and Marro, 1982; Schumacher, 1983; Basile and Marro, 1992). The component $x_{h,S}$ can be nulled by applying the control sequence that drives the state from the origin to $-x_{h,S}$ along a trajectory in $\mathcal{S}^*$. The component $x_{h,V}$ can be maintained on $\mathcal{V}_m$ by a suitable control action in the time interval $\rho_M \le k < \infty$ while avoiding state divergence, if all the internal unassignable modes of $\mathcal{V}_m$ are stable (or stabilizable). Otherwise, $x_{h,V}$ must be further decomposed as $x_{h,V} = x_{h,V_S} + x_{h,V_U}$, with x_{h,V_S} belonging to the subspace of the stable (or stabilizable) internal modes of $\mathcal{V}_m$ and x_{h,V_U} belonging to that of the unstable modes. The former component can be maintained on $\mathcal{V}_m$, avoiding state divergence, by a suitable control action in the time interval $\rho_M \le k < \infty$, while the latter can be nulled by reaching $-x_{h,V_U}$ with a control action, applied in the time interval $-\infty < k \le \rho_M - 1$, corresponding to a trajectory in $\mathcal{V}_m$ from the origin. The hypothesis that $\mathcal{V}_m$ does not have internal unassignable eigenvalues on the unit circle is required in order to discriminate between stable and unstable modes when $\mathcal{V}_m$ is not internally stabilizable. Moreover, Algorithms 1 and 2 require that system (1),(2) is left-invertible with respect to the control input. Algorithm 3 is a means to deal with non left-invertible systems. Algorithms 1 and 2 provide the control and state sequences for motions on $\mathcal{S}^*$ and $\mathcal{V}_m$, respectively, assuming $h(k) = I\,\delta(k - \rho_M)$. This particular choice of the input h directly yields the FIR system convolution profiles and the matrices of the dynamic unit. Matrix H must be decomposed as $H = VH'_1 + SH'_2$, where V and S denote basis matrices of $\mathcal{V}_m$ and $\mathcal{S}^*$, respectively. Let F be such that $(A+BF)\,\mathcal{V}_m \subseteq \mathcal{V}_m$ and let $T = [V\ S\ T_1]$ be a state space basis transformation. The system matrices in the new basis have the structures

$$A' = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ 0 & A'_{22} & A'_{23} \\ 0 & A'_{32} & A'_{33} \end{bmatrix}, \tag{3}$$

$$B' = \begin{bmatrix} 0 \\ B'_2 \\ 0 \end{bmatrix}, \qquad H' = \begin{bmatrix} H'_1 \\ H'_2 \\ 0 \end{bmatrix}, \tag{4}$$

$$C' = \begin{bmatrix} 0 & C'_2 & C'_3 \end{bmatrix}, \ F' = \begin{bmatrix} F'_1 & F'_2 & F'_3 \end{bmatrix}. \tag{5}$$

Algorithm 1. (Motion on $\mathcal{S}^*$). The controls $U_1(k)$, $k = 0, \ldots, \rho_M - 1$, and the corresponding states $X_1(k)$, $k = 1, \ldots, \rho_M$, are derived through the following steps.

1. Compute basis matrices M_i of the subspace $\mathcal{S}_i \cap \mathcal{C}$ for $i = 1, \ldots, \rho_M - 1$.
2. Compute the sequences $\beta(i)$ and $U_1(i)$, $i = 1, \ldots, \rho_M - 1$, as
$$\begin{bmatrix} \beta(\rho_M - j) \\ U_1(\rho_M - j) \end{bmatrix} = \begin{bmatrix} A\, M_{\rho_M - j} & B \end{bmatrix}^{\#} M_{\rho_M - j+1} \beta(\rho_M - j + 1),$$
for $j = 1, \ldots, \rho_M - 1$, with $M_{\rho_M} = S$ and $\beta(\rho_M) = -\,H'_2$.
3. Compute $U_1(0)$ driving the states from the origin to $M_1 \beta(1)$ as
$$U_1(0) = B^{\#} M_1\, \beta(1).$$
4. Compute the states $X_1(i)$, $i = 1, \ldots, \rho_M$, as
$$X_1(i) = M_i\, \beta(i), \quad i = 1, \ldots, \rho_M.$$

Algorithm 2. (Motion on $\mathcal{V}_m$). Two strategies are outlined depending on whether $\mathcal{V}_m$ is internally stabilizable or not.

1. If $\mathcal{V}_m$ is internally stabilizable, the motion on $\mathcal{V}_m$ is provided by the pair (A'_{11}, H'_1) in (3),(4), i.e. the states restricted to $\mathbb{R}^{n_\mathcal{V}}$, $n_\mathcal{V} = \dim(\mathcal{V}_m)$, are
$$X_2(\rho_M + i) = (A'_{11})^i\, H'_1, \quad i = 0, 1, \ldots,$$
and the controls are
$$U_2(\rho_M + i) = F'_1\, (A'_{11})^i\, H'_1, \quad i = 0, 1, \ldots.$$
2. If $\mathcal{V}_m$ is not internally stabilizable, a second state space basis transformation T', whose aim is to separate the stable and unstable modes of $\mathcal{V}_m$, is required. The matrices A''_{11}, H''_1 and F''_1, respectively corresponding to A'_{11}, H'_1 and F'_1 in the new basis, have the structures
$$A''_{11} = \begin{bmatrix} A_S & 0 \\ 0 & A_U \end{bmatrix}, \quad H''_1 = \begin{bmatrix} H_S \\ H_U \end{bmatrix},$$
$$F''_1 = \begin{bmatrix} F_S & F_U \end{bmatrix}.$$
A preaction, nulling the unstable component of the state H_U at the time instant ρ_M must be computed backwards through the matrix A_U. The states restricted to $\mathbb{R}^{n_u}$, $n_u = \dim(\mathcal{V}_m^U)$, are
$$X_3(\rho_M - j) = -\,A_U^{-j} H_U, \quad j = 0, 1, \ldots,$$
and the controls are
$$U_3(\rho_M - j) = -\,F_U A_U^{-j} H_U, \quad j = 1, \ldots.$$
The stable component of the state H_S is managed as in the case of $\mathcal{V}_m$ stabilizable.

Algorithms 1 and 2 directly yield the compensator. If all the internal modes of $\mathcal{V}_m$ are stable, decoupling is achieved by means of the minimal preaction (dead-beat, motion on $\mathcal{S}^*$) and postaction (motion on $\mathcal{V}_m$ along the stable zeros). The first can be obtained as the output of a ρ_M-step FIR system with suitable convolution profiles, the latter can be realized as the output of a stable dynamic unit. Hence, the compensator turns out to be the parallel of a ρ_M-step FIR system and a dynamic unit. The FIR system is

$$u_\mathrm{F}(k) = \sum_{\ell=0}^{\rho_M - 1} \varPhi(\ell)\, h(k-\ell), \quad k = 0, 1, \ldots, \tag{6}$$

with $\varPhi(\ell) = U_1(\ell)$, $\ell = 0, \ldots, \rho_M - 1$. The dynamic unit is

$$w(k+1) = N\, w(k) + L\, h(k - \rho_M), \ \ k = 0, 1, .., \tag{7}$$
$$u_\mathrm{D}(k) = M\, w(k), \tag{8}$$

where $N = A'_{11}$, $L = H'_1$, $M = F'_1$. Hence, the control input is $u(k) = u_\mathrm{F}(k) + u_\mathrm{D}(k)$, $k = 0, 1, \ldots$. As aforementioned, in this case, the precompensator achieving perfect decoupling with stability can also be implemented as a unique standard dynamic unit (A_c, B_c, C_c, D_c) also including the FIR system. Otherwise, if unstable modes are also present in $\mathcal{V}_m$, infinite preview is required. The evolution of the state along the unstable modes of $\mathcal{V}_m$ can only be computed backwards in time and reproduced through a convolutor with an infinitely large window. In practice, an FIR system is considered, whose window should be large enough to make the truncation error negligible. In conclusion, to achieve perfect decoupling with stability in the presence of unstable unassignable internal eigenvalues of $\mathcal{V}_m$ a convolutor with an infinitely large window would be required and this cannot be reduced to a standard dynamic unit

(A_c, B_c, C_c, D_c). However, practical implementation requirements introduce truncation, which implies that: i) only an approximate solution is achievable in practice; ii) the convolution unit with truncated profile can be implemented as an FIR system, which, in the discrete-time case, can be reduced to a quadruple (A_c, B_c, C_c, D_c) with a peculiar structure. In this case, (6) is modified into

$$u_{\mathrm{F}}(k) = \sum_{\ell=-k_a}^{\rho_M-1} \Phi(\ell)\, h(k-\ell), \quad k=0,1,\ldots, \quad (9)$$

with $\Phi(\ell) = U_1(\ell) + U_3(\ell)$, $\ell = -k_a, \ldots, \rho_M - 1$, (with a slight abuse of notation the control sequences are assumed to be zero wherever they are not explicitly defined). The dynamic unit is described by (7),(8) with $N = A_S$, $L = H_S$, $M = F_S$. If the triple (A, B, C) is not left-invertible, the previous procedure can be applied anyhow, provided that a preliminary manipulation is performed to obtain a left-invertible triple and the results thus obtained are adapted to fit the original system.

Algorithm 3. (Extension to Non Left-Invertible Systems) (Marro and Zattoni, 2004; Marro and Zattoni, 2005). If the triple (A, B, C) is not left-invertible, the previous procedure should be applied to (A^*, B^*, C), with

1. $A^* = A + BF^*$, where F^* is a state feedback matrix such that $(A + BF^*)\,\mathcal{V}^* \subseteq \mathcal{V}^*$ and all the elements of $\sigma(A + BF^*)|_{\mathcal{R}_{\mathcal{V}^*}}$ are stable;
2. $B^* = B\,U^*$, where U^* is a basis matrix of the subspace $\mathcal{U}^* = (B^{-1}\,\mathcal{V}^*)^{\perp}$, the orthogonal complement of the inverse image of $\mathcal{V}^*$ with respect to B.

Let $\bar{U}_i(k)$ and $\bar{X}_i(k)$, with $i = 1, 2, 3$ and k consistently defined, be the sequences of controls and states provided by Algorithms 1 and 2 applied to (A^*, B^*, C). The corresponding control sequences for (A, B, C) must be computed as $U_i(k) = U^* \bar{U}_i(k) + F^* \bar{X}_i(k)$, $i = 1, 2, 3$.

4. CONCLUSIONS

The problem of making the output totally insensitive to an exogenous input signal has been solved in the geometric context. Necessity of previewing the signal to be decoupled has been precisely related to structural and stabilizability properties of the system.

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