TIME-VARYING DYNAMIC CONTROLLERS FOR DISCRETE-TIME LINEAR SYSTEMS WITH INPUT SATURATION

J.M. Gomes da Silva Jr. *,[1] **F. Lescher** **

* *UFRGS - Department of Electrical Engineering*
Av. Osvaldo Aranha 103, 90035-190 Porto Alegre, Brazil.
E-mail: jmgomes@eletro.ufrgs.br
** *EIGSI - ERPA*
26 Rue de Vaux le Foletier, 17000 La Rochelle, France
E-mail: fabien.lescher@eigsi.fr

Abstract: The present paper proposes a method for computing time-varying dynamic output feedback controllers for discrete-time linear systems subject to input saturation. The method is based on a locally valid polytopic representation of the saturation term. From this representation, it is shown that, at each sampling time, the matrices of the stabilizing time-varying controller can be computed from the current system output and from constant matrices obtained as a solution of some linear matrix inequalities. Optimization schemes allowing to address issues regarding the maximization of the basin attraction as well as the performance improvement of the closed-loop system, are proposed.

Keywords: dynamic output feedback, saturation, stabilization, stability domains, discrete-time systems.

1. INTRODUCTION

The physical impossibility of applying unlimited control signals makes the actuator saturation an ubiquitous problem in control systems. In particular, it is well known that the input saturation is source of performance degradation, limit cycles, different equilibrium points, and even instability. Hence, it has been great the interest in studying these negative effects and also in proposing control design procedures taking directly into account the control bounds (see for instance (Bernstein and Michel, 1995) and references therein).

Most of these works consider state feedback control laws. Although the proposition of state feedback methods allow to have a good insight into the problem, the practical applicability of these methods is limited. Considering output feedback solutions, less works are found in the literature. Most of them focus on the determination of global or semi-global stabilizing controllers (e.g see (Stoorvogel and Saberi, 1999)). The main drawback of these results is that they can only be applied to null-controllable systems. Moreover, when performance or robustness requirements must be satisfied it can be impossible to achieve global or semi-global stability. On the other hand, we can found very few works dealing with the synthesis of local stabilizing controllers via output feedback. In (Gomes da Silva Jr. *et al.*, 2001), observer-based control laws are proposed. The main problem is that the solutions consider particular quadratic Lyapunov functions (the P matrix should be block diagonal) which

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leads, in general, to very conservative solutions. In (Tyan and Bernstein, 1997), a method for designing dynamic output controllers using of the Positive Real Lemma is proposed. The main objective pursued in that paper is the minimization of an LQG criterion. A region of stability is associated to the closed-loop system. However, it should be pointed out that the size and the shape of this region are not taken into account in the design procedure, which can lead to very conservative domains of stability. Furthermore, the controller is computed from the solution of strong coupled Riccati equations which, in general, are not simple to solve. A time-varying dynamic controller is proposed in (Nguyen and Jabbari, 2000). Since the proposed approach considers only continuous-time systems, its main drawback resides in the fact that the stability properties cannot be ensured if the controller is discretized for a digital implementation. Furthermore, in that paper, no explicitly consideration is made about the region of attraction associated to the controller neither about the internal stability of the system. On the other hand, it should be pointed out that all the references above are concerned only with continuous-time systems.

The aim of this note is the proposition of a technique for the design of stabilizing dynamic output feedback controllers for discrete-time linear systems in the presence of saturating actuators. In addition to the asymptotic stability requirement, two implicit design objectives are considered: the maximization of the region of attraction of the closed-loop system and the guarantee of a certain degree of time-domain performance for the system operation in a neighborhood of the origin (equilibrium point). The theoretical conditions for solving the synthesis problem are based on a polytopic representation of the closed-loop system subject to saturation (Gomes da Silva Jr and Tarbouriech, 2001). Using then the classical variables transformations as proposed in (Scherer *et al.*, 1997) and (de Oliveira *et al.*, 2000), it is possible to formulate conditions that allow to compute a linear time-varying dynamic controller that stabilizes the closed-loop system. The matrices of the controller are computed, at each sampling time, from the current system output and from constant matrices obtained as solution of some linear matrix inequalities (LMIs) constraints. Optimization problems to the determination of the controller in order to enlarge the basin of attraction of the closed-loop as well as enhance the time-domain performance of the closed-loop system are therefore proposed. A numerical example is provided to illustrate the application of the proposed method.

Notations. $A_{(i)}$ denotes the ith row of matrix A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A. $\star$ stands for symmetric blocks; $\bullet$ stands for an element that has no influence on the development

2. PROBLEM STATEMENT

Consider the discrete-time linear system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) \qquad\; = Cx(k) \end{cases} \tag{1}$$

where $x(k) \in \Re^n$, $u(k) \in \Re^m$, $y(k) \in \Re^p$ are the state, the input and the measured output vectors, respectively, and $k \in \mathcal{N}$. Matrices A, B and C are real constant matrices of appropriate dimensions. Pairs (A, B) and (C, A) are assumed to be controllable and observable respectively. The input vector u is subject to amplitude limitations defined as follows:

$$|u_{(i)}| \leq u_{0(i)}, \quad i = 1, ..., m \tag{2}$$

where $u_{0(i)} > 0, i = 1, ..., m$ denote the control amplitude bounds.

We suppose that only the output $y(k)$ is available for measurement. Hence our aim is to compute an n-order stabilizing dynamic compensator in the following form

$$\begin{aligned} x_c(k+1) &= A_c(k)x_c(k) + B_c(k)y(k) \\ v_c(k) &= C_c x_c(k) + D_c y(k) \end{aligned} \tag{3}$$

where $x_c(k) \in \Re^n$ is the controller state, $v_c(k) \in \Re^m$ is the controller output, matrices $A_c(k)$, $B_c(k)$, C_c have appropriate dimensions. Note that $A_c(k)$ and $B_c(k)$ are possibly time-varying matrices. As a consequence of the control bounds, the effective control signal applied to system (1) is a saturated one:

$$u(k) = sat(v_c(k)) = sat(C_c x_c(k) + D_c C x(k)) \tag{4}$$

where each component, $i = 1, \ldots, m$, is defined by

$$sat(v_c(k))_{(i)} = sign(v_c(k)_{(i)}) \max\{u_{0(i)}, |v_c(k)_{(i)}|\}$$

The resulting closed-loop system is nonlinear and can be written as

$$\begin{aligned} x(k+1) &= Ax(k) + Bsat(C_c x_c(k) + D_c C x(k)) \\ x_c(k+1) &= A_c(k)x_c(k) + B_c(k)Cx(k) \end{aligned} \tag{5}$$

3. MAIN RESULTS

Defining an augmented state vector $z = [x' \;\; x_c']' \in \Re^{2n}$ and the matrices $\mathcal{A}(k) = \begin{pmatrix} A & 0 \\ B_c(k)C & A_c(k) \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$ and $K = \begin{pmatrix} D_c C & C_c \end{pmatrix}$, the closed-loop system (5) can be re-written as

$$z(k+1) = \mathcal{A}(k)z(k) + \mathcal{B}sat(Kz(k)) \tag{6}$$

Note now that each control entry, $i = 1, \ldots, m$, can be re-written as follows:

$$u(k)_{(i)} = sat(K_{(i)}z(k)) = \alpha(k)_{(i)}K_{(i)}z(k) \quad (7)$$

with $\alpha(k)_{(i)} = sign(K_{(i)}z(k)) \min\{\frac{u_{0(i)}}{|K_{(i)}z(k)|}, 1\}$. Note that $\alpha(k)_{(i)}$ depends on the value of $v_{c(i)}(k) = C_{c(i)}x_c(k) + D_{c(i)}y(k) = K_{(i)}z(k)$ at each instant. Considering now the vector $\alpha(k)$ and a diagonal matrix $\Gamma(\alpha(k)) = diag(\alpha(k))$, it follows that

$$u(k) = \Gamma(\alpha(k))Kz(k) \quad (8)$$

and the closed-loop system reads:

$$z(k+1) = (\mathcal{A}(k) + \mathcal{B}\Gamma(\alpha(k))K)z(k) = \tilde{\mathcal{A}}_k z(k)$$

Suppose now that $z(k)$ belongs to the set

$$S(\underline{\alpha}) = \{z \in \Re^{(2n)}; |K_{(i)}z| \leq \frac{u_{0(i)}}{\underline{\alpha}_{(i)}}, i = 1, ..., m\} \quad (9)$$

where $0 < \underline{\alpha}_{(i)} \leq 1$. In this case, it follows that $\underline{\alpha}_{(i)} \leq \alpha(k)_{(i)} \leq 1$, $i = 1, ..., m$. Hence, by convexity there exists $0 \leq \lambda_{j,k} \leq 1$, $j = 1, \ldots, 2^m$, such that $\sum_{j=1}^{2^m} \lambda_{j,k} = 1$ and

$$z(k+1) = (\mathcal{A}(k) + \mathcal{B}\sum_{j=1}^{2^m} \lambda_{j,k}\Gamma_j K)z(k) = \tilde{\mathcal{A}}_k z(k)$$

where Γ_j, $j = 1, \ldots, 2^m$, are the vertices of a polytope of diagonal matrices whose diagonal elements $\Gamma_{j(i,i)}$ take the value $\underline{\alpha}_{(i)}$ or 1 (Gomes da Silva Jr and Tarbouriech, 2001).

Theorem 1. If there exists symmetric positive definite matrices $Y \in \Re^{(n*n)}$, $X \in \Re^{(n*n)}$, matrices $\hat{A}$,$\hat{B}$,$\hat{C}$,$\hat{D}$ of appropriate dimensions, and scalars $0 < \underline{\alpha}_{(i)} \leq 1$, $i = 1, \ldots, m$, satisfying the following matrix inequalities

$$\begin{pmatrix} X & I_n & AX + B\Gamma_j\hat{C} & A + B\Gamma_j\hat{D}C \\ I_n & Y & \hat{A} & YA + \hat{B}C \\ \star & \star & X & I_n \\ \star & \star & I_n & Y \end{pmatrix} > 0 \quad (10)$$
$$j = 1, ..., 2^m.$$

$$\begin{pmatrix} X & I_n & \underline{\alpha}_{(i)}\hat{C}'_{(i)} \\ I_n & Y & \underline{\alpha}_{(i)}(\hat{D}C)'_{(i)} \\ \star & \star & u_{0(i)}^2 \end{pmatrix} > 0 \quad i = 1, ..., m. \quad (11)$$

then the dynamic controller (3) with

$$D_c = \hat{D}, \quad C_c = (\hat{C} - D_cCX)(M')^{-1},$$
$$B_c(k) = N^{-1}(\hat{B} - YB\Gamma(\alpha(k))D_c) \text{ and}$$
$$A_c(k) = N^{-1}(\hat{A} - NB_cCX - YB\Gamma(\alpha(k))C_cM' - Y(A + B\Gamma(\alpha(k))D_cC)X)(M')^{-1},$$

where matrices M and $N \in \Re^{2n*2n}$ verify the relation $MN' = I - XY$ guarantees that the region $\mathcal{E}(P, 1) = \{z(k) \in \Re^{2n} \; ; \; z(k)'Pz(k) \leq 1\}$ with $P = \begin{pmatrix} Y & N \\ N' & \bullet \end{pmatrix}$ is a domain of asymptotic stability for the closed-loop system (5).

Proof: Suppose that (10) is verified. Then, for all convex sum $\sum_{j=1}^{2^m} \lambda_{j,k} = 1$, it follows that

$$\sum_{j=1}^{2^m} \lambda_{j,k} \begin{pmatrix} X & I_n & AX + B\Gamma_j\hat{C} & A + B\Gamma_j\hat{D}C \\ I_n & Y & \hat{A} & YA + \hat{B}C \\ \star & \star & X & I_n \\ \star & \star & I_n & Y \end{pmatrix} > 0 \quad (12)$$

The terms of the secondary diagonal of (12) can therefore be re-written as:

$$\begin{pmatrix} AX + B\sum_{j=1}^{2^m}\lambda_{j,k}\Gamma_j\hat{C} & A + B\sum_{j=1}^{2^m}\lambda_{j,k}\Gamma_j\hat{D}C \\ \hat{A} & YA + \hat{B}C \end{pmatrix} \quad (13)$$

Considering $P^{-1} = \begin{pmatrix} X & M \\ M' & \bullet \end{pmatrix}$ and defining $\Pi_1 = \begin{pmatrix} X & I_n \\ M' & 0 \end{pmatrix}$ and $\Pi_2 = \begin{pmatrix} I_n & Y \\ 0 & N' \end{pmatrix}$, it follows that $P\Pi_1 = \Pi_2$ and $\Pi_1'P\Pi_1 = \Pi_1'\Pi_2 = \begin{pmatrix} X & I_n \\ I_n & Y \end{pmatrix}$

Consider now the following changing of variables (Scherer *et al.*, 1997)(de Oliveira *et al.*, 2000):
$\hat{A} = NA_c(k)M' + NB_c(k)CX +$
$YB\Gamma(\alpha(k))C_cM' + Y(A + B\Gamma(\alpha(k))D_cC)X$,
$\hat{B} = YB\Gamma(\alpha(k))D_c + NB_c(k)$,
$\hat{C} = D_cCX + C_cM'$, $\hat{D} = D_c$.
From the definition of $\tilde{\mathcal{A}}_k$, one obtains $\Pi_1'P\tilde{\mathcal{A}}_k\Pi_1 = \begin{pmatrix} AX + B\Gamma(\alpha(k))\hat{C} & A + B\Gamma(\alpha(k))\hat{D}C \\ \hat{A} & YA + \hat{B}C \end{pmatrix}$.

Note that if (10) is verified, it follows that $X - Y^{-1} > 0$, which implies that $I - XY$ is nonsingular and it is always possible to compute square and nonsingular matrices N and M verifying the equation $NM' = I - XY$. This fact ensures that Π_1 is nonsingular. Then, right and left multiplying (12) respectively by $\begin{pmatrix} \Pi_1^{-1} & 0 \\ 0 & \Pi_1^{-1} \end{pmatrix}$ and its transpose, with $\lambda_{j,k}$ such that $\sum_{j=1}^{2^m} \lambda_{j,k}\Gamma_j = \Gamma(\alpha(k))$, one obtains

$$\begin{pmatrix} P & P\tilde{\mathcal{A}}_k \\ \tilde{\mathcal{A}}'_k P & P \end{pmatrix} > 0 \Leftrightarrow P - \tilde{\mathcal{A}}_k' P \tilde{\mathcal{A}}_k > 0 \quad (14)$$

Right and left multiplying (11) respectively by $\begin{pmatrix} (\Pi_1')^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ and its transpose it follows that $\begin{pmatrix} P & \underline{\alpha}_{(i)}K'_{(i)} \\ \underline{\alpha}_{(i)}K_{(i)} & u_{0(i)}^2 \end{pmatrix} > 0$, $i = 1, ..., m$, which ensures that $\mathcal{E}(P, 1) \subset S(\underline{\alpha})$.

Suppose now that $z(k) \in \mathcal{E}(P, 1)$. If (11) is satisfied, it follows that $z(k) \in S(\underline{\alpha})$ and there exist

$\lambda_{j,k}$ such that $\sum_{j=1}^{2^m} \lambda_{j,k}\Gamma_j = \Gamma(\alpha(k))$. Hence, if (10) is verified, from (14) it follows that

$$z(k)'Pz(k) - z(k)'\tilde{\mathcal{A}}_k' P\tilde{\mathcal{A}}_k z(k) > 0$$

Hence, considering $V(z(k)) = z(k)'Pz(k)$, it follows that $V(z(k+1)) < V(z(k)$. Since this reasoning can be applied $\forall z(k) \in \mathcal{E}(P,1)$, we can conclude that $\mathcal{E}(P,1)$ is a contractive set and $V(z(k))$ is a strictly decreasing Lyapunov function for system (5) in $\mathcal{E}(P,1)$. $\square$

Note that for computing the controller matrices at the sampling time k it is necessary to obtain $\Gamma(\alpha(k))$. Since the matrices C_c and D_c are time invariant, the output $y(k)$ and the controller state $x_c(k)$ are available, it follows that $\alpha(k)_i = sign(v_{c(i)}(k))\min\{1, u_{0(i)}/|v_{c(i)}(k)|\}$, where $v_c(k) = Kz(k) = C_c x_c(k) + D_c y(k)$. Hence, at each sampling time the following algorithm should be executed.

Algorithm 1.

(1) compute $v_c(k) = C_c x_c(k) + D_c y(k)$
(2) apply $u(k) = sat(v_c(k))$ to the process
(3) $\alpha(k)_{(i)} = sign(v_{c(i)}(k))\min\{1, u_{0(i)}/|v_{c(i)}(k)|\}$
(4) compute matrices $A_c(k)$ and $B_c(k)$.
(5) $x_c(k+1) = A_c(k)x_c(k) + B_c(k)y(k)$
(6) $x_c(k) \leftarrow x_c(k+1)$

Note that in Theorem 1, the same matrices $\hat{A}$ and $\hat{B}$ are considered for all vertices $j = 1,\ldots,2^m$. In order to reduce the conservatism of the condition, the result can be adapted in order to consider different matrices in each vertex. This result can be summarized as follows.

Theorem 2. If there exists symmetric positive definite matrices $Y \in \Re^{(n*n)}$, $X \in \Re^{(n*n)}$, matrices $\hat{A}_j$,$\hat{B}_j$, $\hat{C}$,$\hat{D}$ of appropriate dimensions, $j = 1,\ldots,2^m$, and scalars $0 < \underline{\alpha}_{(i)} \le 1$, $i = 1,\ldots,m$, satisfying

$$\begin{pmatrix} X & I_n & AX + B\Gamma_j\hat{C} & A + B\Gamma_j\hat{D}C \\ I_n & Y & \hat{A}_j & YA + \hat{B}_jC \\ \star & \star & X & I_n \\ \star & \star & I_n & Y \end{pmatrix} > 0 \qquad (15)$$
$$j = 1,...,2^m$$

and relation (11), then, considering the $\lambda_{j,k}$ such that $\Gamma(\alpha(k)) = \sum_{j=1}^{2^m} \lambda_{j,k}\Gamma_j$, the dynamic controller (3) with
$D_c = \hat{D}$, $C_c = (\hat{C} - D_cCX)(M')^{-1}$,
$B_c(k) = N^{-1}(\sum_{j=1}^{2^m} \lambda_{j,k}\hat{B}_j - YB\Gamma(\alpha(k))D_c)$,
$A_c(k) = N^{-1}(\sum_{j=1}^{2^m} \lambda_{j,k}\hat{A}_j - NB_cCX - YB\Gamma(\alpha(k))C_cM' - Y(A+B\Gamma(\alpha(k))D_cC)X)(M')^{-1}$,
where matrices M and $N \in \Re^{2n*2n}$ verify the relation $MN' = I - XY$, guarantees that the region $\mathcal{E}(P,1) = \{z(k) \in \Re^{2n}\ ;\ z(k)'Pz(k) \le 1\}$ is a domain of asymptotic stability for the closed-loop system (5).

Proof: Let now $0 \le \lambda_{j,k} \le 1$ such that, at instant k, $\sum_{j=1}^{2^m}\lambda_{j,k}\Gamma_j = \Gamma(\alpha(k))$ with $\sum_{j=1}^{2^m}\lambda_{j,k} = 1$. Considering now that (15) is verified $\forall j = 1,\ldots,2^m$, it follows that:

$$\begin{pmatrix} X & I_n & AX + B\Gamma(\alpha(k))\hat{C} & A + B\Gamma(\alpha(k))\hat{D}C \\ I_n & Y & \sum_{j=1}^{2^m}\lambda_{j,k}\hat{A}_j & YA + \sum_{j=1}^{2^m}\lambda_{j,k}\hat{B}_jC \\ \star & \star & X & I_n \\ \star & \star & I_n & Y \end{pmatrix} > 0$$

From this point the proof mimics the one of Theorem 1. $\square$

Differently from the previous result, now, in order to compute the controller matrices at each sampling time, it is necessary to determine explicitly the coefficients $\lambda_{j,k}$ such that $\Gamma(\underline{\alpha}(k)) = \sum_{j=1}^{2^m} \lambda_{j,k}\Gamma_j$. This can be easily accomplished by obtaining a feasible solution for the following linear program:

$$\begin{gathered} \min \sum_{j=1}^{2^m}\lambda_{j,k} \\ \text{subject to} \\ \sum_{j=1}^{2^m}\lambda_{j,k}\Gamma_j = \Gamma(\alpha(k)),\quad \sum_{j=1}^{2^m}\lambda_{j,k} = 1, \\ 0 \le \lambda_{j,k} \le 1 \end{gathered} \qquad (16)$$

The algorithm to be performed at each sampling time can now be slightly changed as follows.

Algorithm 2.

(1) compute $v_c(k) = C_c x_c(k) + D_c y(k)$
(2) apply $u(k) = sat(v_c(k))$ to the process
(3) $\alpha(k)_{(i)} = sign(v_{c(i)}(k))\min\{1, u_{0(i)}/|v_{c(i)}(k)|\}$
(4) compute $\lambda_{k,j}$ from (16)
(5) compute matrices $A_c(k)$ and $B_c(k)$
(6) $x_c(k+1) = A_c(k)x_c(k) + B_c(k)y(k)$
(7) $x_c(k) \leftarrow x_c(k+1)$

Although in this case it is necessary to find a feasible solution to a linear program at each sampling time, it is worth to notice that this linear program is very simple and in general the computational burden involved in the solution is not prohibitive even for relatively fast dynamics systems.

4. OPTIMIZATION PROBLEMS

4.1 Enlargement of the basin of attraction

An implicit objective in the synthesis of the stabilizing controller (3) is the maximization of estimates of the basin of attraction associated to the closed-loop system. In other words, we want to compute (3) such that the associated region of asymptotic stability is as large as possible considering some size criterion. This can be addressed if we consider a set Ξ_0 with a given shape and a scaling factor β. This shape set can be easily defined as a polyhedral described by the convex hull of its vertices:

$$\Xi_0 \triangleq Co\{v_1, v_2, \ldots, v_{n_r}\},\ v_l \in \Re^{2n},\ l = 1, \ldots, n_r$$

Hence, recalling Theorems 1 and 2, we aim at searching for matrices $X, Y, \hat{A}, \hat{B}, \hat{C}, \hat{D}$ and a vector $\underline{\alpha}$ in order to obtain $\beta\ \Xi_0 \subset \mathcal{E}(P,1)$ with β as large as possible

Note that in this case, the vectors v_l can be viewed as directions in which we want to maximize the region of attraction. In particular it is interesting to maximize the region of stability in directions associated to the states of the plant. In this case the vectors v_l assume the form $\left(v_{l1}'\ 0\right)'$.

Noticing that $\beta\left(v_{l1}'\ 0\right)' \in \mathcal{E}(P,1)$ is equivalent to $\beta\left(v_{l1}'\ 0\right) P \left(v_{l1}'\ 0\right)' \beta \leq 1$ and considering $\mu = 1/\beta^2$, it follows that $\beta\ \Xi_0 \subset \mathcal{E}(P,1)$ is equivalent to:

$$v_{l1}' Y v_{l1} \leq \mu, \quad l = 1, \ldots, r \tag{17}$$

Hence, maximize the ellipsoid $\mathcal{E}(P,1)$ along the directions v_{l1} is equivalent to minimize μ.

Note that matrix N, that appears in matrix P, is not optimized in the above problem. On the other hand, once one obtains a matrix Y, by maximizing $\mathcal{E}(P,1)$ in the space of the system state, it is possible to explore the degrees of freedom in the choice of matrix N. For instance, $\mathcal{E}(P,1)$ can be maximized along the directions given by generic vectors $v_l = \left(v_{l1}'\ v_{l2}'\right)'$ where $v_{l1} \in \Re^n$ and $v_{l2} \in \Re^n$, by solving the following problem:

$$\begin{array}{c} \min\limits_{N,\mu} \mu \\ \text{subject to} \\ \begin{pmatrix} \mu & v_{l1}' & v_{l1}'Y + v_{l2}'N' \\ v_{l1} & X & I_n \\ Y v_{l1} + N v_{l2} & I_n & Y \end{pmatrix} > 0 \\ l = 1, \ldots, r \end{array} \tag{18}$$

where X et Y are given matrices verifying the conditions of Theorem 1 (or 2).

Note that (18) is equivalent to

$$\beta\left(v_{l1}'\ v_{l2}'\right) P \left(v_{l1}'\ v_{l2}'\right)' \beta < 1$$

In order to proof this, it suffices to apply Schur's complement, pre and post multiply the obtained matrix inequality respectively by $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & Y & N \end{pmatrix}$ and F'.

4.2 Performance Issues

In addition to the guarantee of stability for a region as large as possible, the controller should be designed in order to ensure some degree of time-domain performance in a neighborhood of the origin (Gomes da Silva Jr and Tarbouriech, 2001). Here we consider this neighborhood as the region of linear behavior of the closed-loop system, i.e., the region where the control inputs do not saturate:

$$R_L = \{z \in \Re^{2n}\ ;\ |K_{(i)} z| \leq u_{0(i)}, \quad i = 1, \ldots, m\}$$

When the system operates inside R_L it follows that $\Gamma(\alpha(k)) = I$ and $A_c(k)$ and $B_c(k)$ are constant matrices which we denote as A_c and B_c. In this case, the time-domain performance can be achieved if we consider the pole placement of the matrix $\mathcal{A} = \begin{pmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{pmatrix}$ in a suitable region inside the unit circle. Considering an LMI framework, the results stated in (Scherer *et al.*, 1997) can be used to place the poles in a called LMI region in the complex plane. For example, if we verify the following LMI,

$$\begin{pmatrix} rX & rI_n & AX + B\hat{C} & A + B\hat{D}C \\ rI_n & rY & \hat{A} & YA + \hat{B}C \\ \star & \star & rX & rI_n \\ \star & \star & rI_n & rY \end{pmatrix} > 0 \tag{19}$$

it is easy to show that the poles of $\mathcal{A}$ will be placed in a circle centered in zero and with ray $r < 1$. In this case, smaller is r faster will be the decay rate of the time-response inside the linearity region.

4.3 Optimization Problem

Based on the considerations stated in sections 4.1 and 4.2, the following optimization problem can be formulated in order to compute a controller with the aim of maximizing the basin of attraction of the closed-loop system while ensuring some time-domain performance near the equilibrium point.

$$\begin{array}{c} \min\limits_{X,Y,\hat{A},\hat{B},\hat{C},\hat{D},\underline{\alpha}} \mu \\ \text{subject to} \\ v_{l1}' Y v_{l1} < \mu \quad l = 1, \ldots, r \\ (10)\ (\text{or } (15)),\ (11) \text{ and } (19) \end{array} \tag{20}$$

For a fixed $\underline{\alpha}$, all the constraints in (20) are LMIs. Hence, for systems with one or two the inputs, the optimal solution can be easily found by an iterative search on a grid defined by the components of $\underline{\alpha}$. On the other hand, when the system presents more inputs two steps can be performed iteratively in order to obtain a suboptimal solution (Gomes da Silva Jr and Tarbouriech, 2001):

	$r = 0.95$		$r = 0.85$	
$\underline{\alpha}$	β Th. 1	β Th. 2	β Th. 1	β Th. 2
1	15.801	15.801	8.700	8.700
0.8	18.506	19.202	10.689	10.873
0.7	18.534	19.055	10.700	12.098

Table 1. Trade-offs

(1) Fix $\underline{\alpha}$ and solve (20) with X, Y, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ as variables.
(2) Fix $\hat{C}$ and $\hat{D}$ and solve (20) with X, Y, $\hat{A}$, $\hat{B}$ and $\underline{\alpha}$ as variables.

Note that smaller are $\underline{\alpha}$ larger tends to be the region $S(\underline{\alpha})$, which allows to include larger regions of stability $\mathcal{E}(P, 1)$ (see detailed comments in (Gomes da Silva Jr and Tarbouriech, 2001)).

5. NUMERICAL EXAMPLE AND CONCLUDING REMARKS

Consider the discrete-times system (1) with the following matrices:

$$A = \begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{pmatrix}; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \; C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

The control bounds is given by $u_0 = 7$.

Note that matrix A is unstable (its eigenvalues are $1+0.4j$ et $1-0.4j$). Our objective is to design a dynamic controller, in the form (3), that ensures the stability of the system in an ellipsoidal region as large as possible in the following directions $\begin{pmatrix} 1 & 1 \end{pmatrix}'$ and $\begin{pmatrix} 1 & -1 \end{pmatrix}'$. In addition to the maximization of the stability region, a performance requirement in terms of a pole placement inside a disk with $r = 0.8$ should be satisfied.

In order to determine the dynamic controller satisfying the requirements above, the optimization problem (20) is considered. Using a grid procedure in $\underline{\alpha}$, the optimal value for μ is obtained with $\underline{\alpha} = 0.625$. In this case, the following is obtained:

$$P = \begin{pmatrix} 0.0042 & 0.0017 & -0.0002 & -0.0004 \\ 0.0017 & 0.0064 & -0.0004 & -0.0014 \\ -0.0002 & -0.0004 & 2.057 * 10^4 & 5.681 * 10^3 \\ -0.0004 & -0.0014 & 5.681 * 10^3 & 2.489 * 10^3 \end{pmatrix}$$

$$A_c(k) = \begin{pmatrix} 0.7818 & 2.0689 \\ -0.2802 & -0.7212 \end{pmatrix} + \alpha(k) \begin{pmatrix} -0.0115 & 1.5535 \\ 0.0287 & -3.8809 \end{pmatrix}$$

$$B_c(k) = \begin{pmatrix} -2.9213 \\ 4.3621 \end{pmatrix} + \alpha(k) \begin{pmatrix} 0.0013 \\ 0.0049 \end{pmatrix}$$

$$C_c = \begin{pmatrix} 0.0074 & 0.0183 \end{pmatrix} \quad D_c = \begin{pmatrix} -0.7728 \end{pmatrix}$$

Considering the optimization problem (20), Table 5 depicts the values obtained for $\beta = 1/\sqrt{\mu}$, for different values of r and $\underline{\alpha}$, and using conditions of Theorems 1 and 2.

Similar to (Gomes da Silva Jr. *et al.*, 2003), where saturating state feedback control laws were studied, here we can also notice a trade-off between the time-domain performance (measured by the pole placement of the unsaturated system), the size of the region of stability and the effective saturation of the control law. Note that for smaller values of r (more stringent performance requirement), smaller are the values of β (i.e. smaller is the region where the asymptotic stability is ensured) are obtained. Moreover, in the presence of the performance constraints, larger regions of stability are obtained with effective saturating control laws. Observe that the value obtained with $\underline{\alpha} = 1$ corresponds to the linear solution, i.e., the saturation is avoided. Hence, for a given r, allowing saturation, i.e. with values of $\underline{\alpha}$ smaller than 1, larger regions of stability are obtained. On the other hand, as expected, the condition of Theorem 2 gives less conservative regions of stability.

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