AN EFFICIENT DECOMPOSITION-BASED FORMULATION FOR ROBUST CONTROL WITH CONSTRAINTS

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Abstract: This paper is concerned with the development of an efficient computational solution method for control of linear discrete-time systems subject to bounded disturbances with mixed polytopic constraints on the states and inputs. It is shown that the non-convex problem of computing an optimal affine state feedback control policy can be solved through reparameterization to an equivalent convex problem, and that if the disturbance set is the linear map of a hypercube, then this problem may be decomposed into a coupled set of finite horizon control problems. If the problem involves the minimization of a quadratic cost, then this yields a highly structured quadratic program in a tractable number of variables, which can be solved with great efficiency.

Keywords: Robust Control, Constraint Satisfaction, Optimal Control, Predictive Control

1. INTRODUCTION

This paper is concerned with the development of an efficient computational solution method for a robust optimal control problem for linear discrete-time systems subject to state and input constraints and persistent external disturbances.

The difficulty of formulating practically implementable robust solutions for constrained receding horizon control is considered an important open problem in the control literature (Mayne *et al.*, 2000). Several compromise solutions have been proposed to make the problem tractable (Bemporad, 1998; Lee and Kouvaritakis, 1999; Chisci *et al.*, 2001; Langson *et al.*, 2004), but generally at the cost of increased conservatism.

A recent proposal from the field of robust optimization (Ben-Tal *et al.*, 2002; Guslitser, 2002) suggests that the control be parameterized as an affine function of the past disturbance sequence, rather than as a state feedback law. This parameterization has been shown to have several attractive system-theoretic properties, the most important of which is that it is *equivalent* to the class of time-varying affine state feedback policies, and thus transforms the *non-convex* optimization problem of finding a constraint-admissible state feedback policy into a *convex* one, solvable using standard techniques (Goulart and Kerrigan, 2005). Furthermore, when implemented in a receding-horizon fashion, these policies enable the synthesis of stabilizing, time-invariant receding horizon control laws (Goulart *et al.*, 2005*b*).

From a computational point of view, the parameterization is attractive since it allows a robust optimal control problem to be formulated as a tractable quadratic program when the disturbance set is polytopic and the desired policy is one which minimizes a quadratic cost. The central contribution of this paper is the development of a decomposition technique that can be used to separate the problem into a set of coupled finite horizon control problems when the disturbance set is the linear map of a hypercube, and whose solution time is cubic in the horizon length at each interior-point iteration. Numerical results are presented that demonstrate that the technique is practically implementable for systems of appreciable complexity.

Notation: For matrices A and B, $A \otimes B$ is the Kronecker product of A and B, $A^\dagger$ is the one-sided or pseudo-inverse of A, $A \leq B$ denotes element-wise inequality, and $\mathrm{abs}(A)$ denotes the element-wise absolute value of A. $\mathbf{1}$ is an appropriately sized column vector of ones. e_i is an appropriately sized unit vector with a single entry in the i^{th} term. For vectors x and y, $\|x\|_Q^2 = x^T Q x$, $\mathrm{vec}(x, y) = [x^T y^T]^T$. $\mathbb{Z}_{[k,l]}$ represents the set of integers $\{k, k+1, \ldots, l\}$.

2. DEFINITIONS AND ASSUMPTIONS

Consider the discrete-time system:

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad w_k \in W \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ is the disturbance. The values of the disturbance are unknown and may change unpredictably from one time instant to the next, but are contained in a convex and compact set W, which contains the origin. It is assumed that (A, B) is stabilizable and that at each sample instant a measurement of the state is available.

Over the planning horizon $k \in \mathbb{Z}_{[0,N-1]}$, the system is subject to a mixed constraint on the states and controls of the form

$$\mathcal{Z} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\}, \tag{2}$$

where q is the number of inequality constraints that define $\mathcal{Z}$, and $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times m}$, and $b \in \mathbb{R}^q$. In addition, a further constraint is imposed on the terminal state x_N, so that at the end of the planning horizon the state must lie within a terminal constraint set X_f, defined as

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\}, \tag{3}$$

where $Y \in \mathbb{R}^{r\times n}$ and $z \in \mathbb{R}^r$.

Finally, define the stacked sequence of disturbances $\mathbf{w} \in \mathbb{R}^{nN}$ such that

$$\mathbf{w} := \mathrm{vec}(w_0, w_1, \ldots, w_{N-1}), \tag{4}$$

with $\mathbf{w} \in \mathcal{W}$ and $\mathcal{W} := W \times \cdots \times W$, and similarly define stacked versions of the state and control sequences $\mathbf{x} \in \mathbb{R}^{n(N+1)}$ and $\mathbf{u} \in \mathbb{R}^{mN}$ such that

$$\mathbf{x} := \mathrm{vec}(x_0, x_1, \ldots, x_{N-1}, x_N) \tag{5}$$

$$\mathbf{u} := \mathrm{vec}(u_0, u_1, \ldots, u_{N-1}). \tag{6}$$

3. AN AFFINE STATE FEEDBACK PARAMETERIZATION

One natural approach to controlling the system in (1), while ensuring the satisfaction of the constraints (2)–(3), is to search over the class of admissible time-varying affine state feedback control policies. We thus consider policies of the form:

$$u_i = \sum_{j=0}^{i} L_{i,j} x_j + g_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{7}$$

where each $L_{i,j} \in \mathbb{R}^{m\times n}$ and $g_i \in \mathbb{R}^m$. For notational convenience, we also define the block lower triangular matrix $\mathbf{L} \in \mathbb{R}^{mN\times n(N+1)}$ and stacked vector $\mathbf{g} \in \mathbb{R}^{mN}$ as

$$\mathbf{L} := \begin{bmatrix} L_{0,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} & 0 \end{bmatrix}, \tag{8a}$$

and

$$\mathbf{g} := \mathrm{vec}(g_0, \ldots, g_{N-1}), \tag{8b}$$

so that the control input sequence can be written as $\mathbf{u} = \mathbf{L}\mathbf{x} + \mathbf{g}$.

For a given initial state x, we define the set of admissible $(\mathbf{L}, \mathbf{g})$ satisfying (2) and (3) as

$$\Pi_N^{sf}(x) := \left\{ (\mathbf{L}, \mathbf{g}) \;\middle|\; \begin{array}{r} (\mathbf{L}, \mathbf{g}) \text{ satisfies (8)},\ x = x_0 \\ x_{i+1} = Ax_i + Bu_i + w_i \\ u_i = \sum_{j=0}^{i} L_{i,j} x_j + g_i \\ (x_i, u_i) \in \mathcal{Z},\ x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0,N-1]},\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\} \tag{9}$$

and the set of initial states x for which an admissible control policy of the form (7) exists is defined as

$$X_N^{sf} := \left\{ x \in \mathbb{R}^n \;\middle|\; \Pi_N^{sf}(x) \neq \emptyset \right\}. \tag{10}$$

In particular, we define an optimal policy pair $(\mathbf{L}^*(x), \mathbf{g}^*(x)) \in \Pi_N^{sf}(x)$ to be one which minimizes the value of a cost function that is quadratic in the disturbance-free state and input sequence. We thus define:

$$V_N(x, \mathbf{L}, \mathbf{g}, \mathbf{w}) := \sum_{i=0}^{N-1} \frac{1}{2}(\|x_i\|_Q^2 + \|u_i\|_R^2) + \frac{1}{2}\|x_N\|_P^2$$

where the matrices P, Q and R are positive definite, and u_i is given by (7), and define an optimal policy pair as

$$(\mathbf{L}^*(x), \mathbf{g}^*(x)) := \operatorname*{argmin}_{(\mathbf{L},\mathbf{g})\in\Pi_N^{sf}(x)} V_N(x, \mathbf{L}, \mathbf{g}, \mathbf{0}). \tag{11}$$

For the receding-horizon control case, a *time-invariant* control law $\mu_N : X_N^{sf} \to \mathbb{R}^m$ can be implemented by using the first part of this control policy at each time instant, i.e. by implementing the time-invariant receding horizon control law

$$\mu_N(x) := L^*_{0,0}(x)x + g^*_0(x). \tag{12}$$

This control law is, in general, a *nonlinear* function with respect to the current state, and has been shown to have many attractive geometric and system-theoretic properties. In particular, its implementation renders the set $\Pi^{sf}_N(x)$ robust positively invariant subject to certain conditions on the set X_f, and, when $\mathcal{W}$ is a polytope, is a continuous function of x. Furthermore, the closed loop system is guaranteed to be input-to-state stable (ISS) under suitable assumptions on Q, P, R, and X_f. See (Goulart *et al.*, 2005*b*) for a proof of these and other results. Unfortunately, such a control policy is seemingly very difficult to implement, since the set $\Pi^{sf}_N(x)$ and cost function $V_N(x,\cdot,\cdot,0)$ are non-convex; however, it is possible to convert this non-convex optimization problem to an equivalent *convex* problem through an appropriate reparameterization. This parameterization is introduced in the following section.

4. AN AFFINE DISTURBANCE FEEDBACK PARAMETERIZATION

A recent result in the literature on robust optimization suggests a control policy that is affine in the sequence of past disturbances, so that

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}. \tag{13}$$

A scheme of this type was suggested in (Ben-Tal *et al.*, 2002; Guslitser, 2002), and later independently proposed in (Löfberg, 2003). Note that since full state feedback is assumed, the past disturbance sequence is easily calculated as:

$$w_i = x_{i+1} - Ax_i - Bu_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}. \tag{14}$$

Define the variable $\mathbf{v} \in \mathbb{R}^{mN}$ and the block lower triangular matrix $\mathbf{M} \in \mathbb{R}^{mN\times nN}$ such that

$$\mathbf{v} := \operatorname{vec}(v_0, \dots, v_{N-1}) \tag{15a}$$

and

$$\mathbf{M} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \tag{15b}$$

so that the control input sequence can be written as $\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}$. Define the set of admissible $(\mathbf{M}, \mathbf{v})$, for which (2) and (3) are satisfied, as:

$$\Pi^{df}_N(x) := \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M},\mathbf{v}) \text{ satisfies (15)} \\ x = x_0 \\ x_{i+1} = Ax_i + Bu_i + w_i \\ u_i = \sum_{j=0}^{i-1} M_{i,j}w_j + v_i \\ (x_i, u_i) \in \mathcal{Z},\; x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0,N-1]},\; \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}, \tag{16}$$

and define the set of initial states x for which an admissible control policy of the form (13) exists as

$$X^{df}_N := \{x \in \mathbb{R}^n \mid \Pi^{df}_N(x) \neq \emptyset\}. \tag{17}$$

We are interested in this control policy parameterization primarily due to the following property, proof of which may be found in (Goulart and Kerrigan, 2005):

Theorem 1. For a given state $x \in X^{df}_N$, the set of admissible policies $\Pi^{df}_N(x)$ is convex, and the set of admissible states $X^{df}_N = X^{sf}_N$. For any admissible $(\mathbf{L}, \mathbf{g})$ an admissible $(\mathbf{M}, \mathbf{v})$ can be found that yields the same input and state sequence for all $\mathbf{w} \in \mathcal{W}$, and vice-versa.

This result enables implementation of the control law $u = \mu_N(x)$ by replacing the non-convex optimization problem (11) with an equivalent convex one. If we define the nominal states $\hat{x}_i \in \mathbb{R}^n$ to be the states when no disturbances occur, i.e. $\hat{x}_{i+1} = A\hat{x}_i + Bv_i$. and define $\hat{\mathbf{x}} \in \mathbb{R}^{nN}$ as

$$\hat{\mathbf{x}} := \operatorname{vec}(x, \hat{x}_1, \dots, \hat{x}_N) = \mathbf{B}\mathbf{v} + \mathbf{A}x. \tag{18}$$

where $\mathbf{A} \in \mathbb{R}^{n(N+1)\times n}$ and $\mathbf{B} \in \mathbb{R}^{n(N+1)\times mN}$ are suitably defined, then a quadratic cost function similar to that in (11) can be written as

$$V^{df}_N(x,\mathbf{v}) := \frac{1}{2}(\|\mathbf{B}\mathbf{v}\|^2_{\mathcal{Q}} + \|\mathbf{v}\|^2_{\mathcal{R}}) + (\mathbf{A}x)^T \mathcal{Q}\mathbf{B}\mathbf{v}, \tag{19}$$

where $\mathcal{Q} := \left[\begin{smallmatrix} I\otimes Q & \\ & P \end{smallmatrix}\right]$ and $\mathcal{R} := I \otimes R$. As a direct result of the equivalence of the two parameterizations, the minimum value of (19) evaluated over the admissible policies $\Pi^{df}_N(x)$ is equal to the minimum value of V_N in (11), i.e.

$$\min_{(\mathbf{M},\mathbf{v})\in\Pi^{df}_N(x)} V^{df}_N(x,\mathbf{v}) = \min_{(\mathbf{L},\mathbf{g})\in\Pi^{sf}_N(x)} V_N(x,\mathbf{L},\mathbf{g},\mathbf{0})$$

The control law $\mu_N(\cdot)$ can then be implemented using the first part of the optimal $\mathbf{v}^*(\cdot)$ at each step, i.e. $\mu_N(x) = v^*_0(x) = L^*_{0,0}(x)x + g^*_0(x)$, where

$$(\mathbf{M}^*(x), \mathbf{v}^*(x)) := \operatorname*{argmin}_{(\mathbf{M},\mathbf{v})\in\Pi^{df}_N(x)} V^{df}_N(x,\mathbf{v}) \tag{20}$$

which requires the minimization of a *convex* function over a *convex* set.

This minimization is particularly easy when W is polyhedral or norm-bounded. In the remainder of this paper, we will consider the particular case where W is generated as the linear map of a hypercube. Define

$$W = \{w \in \mathbb{R}^n \mid w = Ed,\; \|d\|_\infty \leq 1\}, \tag{21}$$

where $E \in \mathbb{R}^{n\times l}$ is full column rank, so that the stacked generating disturbance sequence $\mathbf{d} \in \mathbb{R}^{lN}$ is

$$\mathbf{d} = \operatorname{vec}(d_0, \dots, d_{N-1}), \tag{22}$$

and define the matrix $J := I \otimes E$, so that $\mathbf{w} = J\mathbf{d}$. As shown in (Kerrigan and Maciejowski, 2004), it is then possible to eliminate the universal quantifier in (16) and construct matrices $F \in \mathbb{R}^{(qN+r)\times mN}$, $G \in \mathbb{R}^{(qN+r)\times nN}$ and $T \in \mathbb{R}^{(qN+r)\times n}$, and vector $c \in \mathbb{R}^{qN+r}$ such that the set of feasible pairs $(\mathbf{M}, \mathbf{v})$ can be written in terms of purely affine constraints:

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \left| \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (15), } \exists \mathbf{\Lambda} \text{ s.t.} \\ F\mathbf{v} + \mathbf{\Lambda}\mathbf{1} \le c + Tx \\ -\mathbf{\Lambda} \le (F\mathbf{M}J + GJ) \le \mathbf{\Lambda} \end{array} \right. \right\}. \tag{23}$$

The minimization in (20) then reduces to a tractable and convex QP in $\mathbf{M}$, $\mathbf{\Lambda}$, and $\mathbf{v}$. The total number of variables in (23) is mN in $\mathbf{v}$, mnN^2 in $\mathbf{M}$, and $(qlN^2 + rlN)$ in $\mathbf{\Lambda}$, with a number of constraints equal to $(qN + r) + (qlN^2 + rlN))$, or $\mathcal{O}(N^2)$ overall. For a naive interior point computational approach using a dense factorization method, the resulting quadratic program would thus require computation time of $\mathcal{O}(N^6)$ at each iteration.

Next, define the following variable transformation:

$$\mathbf{U} := \mathbf{M}J \tag{24}$$

such that $\mathbf{U} \in \mathbb{R}^{mN\times lN}$ has a block lower triangular structure similar to that defined in (15) for $\mathbf{M}$.

Optimization of the cost function (19) over the set of feasible disturbance feedback policies thus yields a QP in $\mathbf{U}$, $\mathbf{\Lambda}$ and $\mathbf{v}$:

$$\min_{\mathbf{U},\mathbf{\Lambda},\mathbf{v}} \frac{1}{2}(\|\mathbf{B}\mathbf{v}\|^2_{\mathcal{Q}} + \|\mathbf{v}\|^2_{\mathcal{R}}) + (\mathbf{A}x)^T\mathbf{B}\mathbf{v} \tag{25a}$$

subject to $\mathbf{U}$ strictly block lower triangular, and:

$$F\mathbf{v} + \mathbf{\Lambda}\mathbf{1} \le c + Tx \tag{25b}$$
$$-\mathbf{\Lambda} \le (F\mathbf{U} + GJ) \le \mathbf{\Lambda}. \tag{25c}$$

Note that, since E is assumed full column rank, $J^\dagger J = I$, and an admissible $\mathbf{M}$ may always be recovered by selecting $\mathbf{M} = \mathbf{U}J^\dagger$.

Remark 2. The critical feature of the quadratic program (25) is that the columns of the variables $\mathbf{U}$ and $\mathbf{\Lambda}$ are decoupled in the constraint (25c). This allows columnwise separation of the constraint into a number of subproblems, subject to the coupling constraint (25b).

5. DIAGONALIZING THE ROBUST OPTIMAL CONTROL PROBLEM

Following the type of strategy proposed in (Rao *et al.*, 1998), the QP defined in (25) can be rewritten in a more computationally attractive form by reintroducing the eliminated state variables to achieve greater structure. This is done by separating the original problem into subproblems; a nominal problem, consisting of the part of the state resulting from the nominal control vector $\mathbf{v}$, and a set of perturbation problems, each representing those components of the state resulting from each of the columns of (25c) in turn.

Nominal States and Inputs We first define a constraint contraction vector $\delta\mathbf{c} \in \mathbb{R}^{qN+r}$ such that

$$\delta\mathbf{c} := \text{vec}(\delta c_0, \ldots, \delta c_N) = \mathbf{\Lambda}\mathbf{1}, \tag{26}$$

so that the constraint (25b) becomes

$$F\mathbf{s} + \delta\mathbf{c} \le c + Tx. \tag{27}$$

Recalling that the nominal states $\hat{x}_i$ are defined in (18) as the expected states given no disturbances, it is easy to show that the constraint (27) can be written explicitly in terms of the nominal controls v_i and states $\hat{x}_i$ as

$$\hat{x}_0 = x, \tag{28a}$$
$$\hat{x}_{i+1} - A\hat{x}_i - Bv_i = 0, \quad \forall i \in \mathbb{Z}_{[0,N-1]} \tag{28b}$$
$$C\hat{x}_i + Dv_i + \delta c_i \le b, \quad \forall i \in \mathbb{Z}_{[0,N-1]} \tag{28c}$$
$$Y\hat{x}_N + \delta c_N \le z, \tag{28d}$$

which is in a form that is *exactly the same* as that in a conventional receding horizon control problem with no disturbances, but with the state and input constraints at each stage i modified by δc_i.

Perturbed States and Inputs We next consider the effects of each of the columns of $(F\mathbf{U} + GJ)$ in turn, and seek to construct a set of problems similar to that in (28). We treat each column as the output of a system subject to a unit impulse, and construct a sub-problem that calculates the contribution of that disturbance to the total constraint contraction vector $\delta\mathbf{c}$.

From the original QP constraint (25c), the constraint contraction vector $\delta\mathbf{c}$ can be written as

$$\text{abs}(F\mathbf{U} + GJ)\mathbf{1} \le \mathbf{\Lambda}\mathbf{1} = \delta\mathbf{c}, \tag{29}$$

the left hand side of which can be rewritten as

$$\text{abs}(F\mathbf{U} + GJ)\mathbf{1} = \sum_{p=1}^{lN} \text{abs}((F\mathbf{U} + GJ)e_p). \tag{30}$$

Define $\mathbf{y}^p \in \mathbb{R}^{qN+r}$ and $\delta\mathbf{c}^p \in \mathbb{R}^{qN+r}$ as

$$\mathbf{y}^p := (F\mathbf{U} + GJ)e_p \tag{31}$$
$$\delta\mathbf{c}^p := \text{abs}(\mathbf{y}^p). \tag{32}$$

The unit vector e_p represents a unit disturbance in some element j of the generating disturbance d_k at some time step k, with no disturbances at any other step [1]. If we denote the j^{th} column of E as $E_{(j)}$, then it is easy to recognize $\mathbf{y}^p$ as the stacked output vector of the system

$$(u_i^p, x_i^p, y_i^p) = 0, \quad \forall i \in \mathbb{Z}_{[0,k]} \tag{33a}$$
$$x_{k+1}^p = E_{(j)}, \tag{33b}$$
$$x_{i+1}^p - Ax_i^p - Bu_i^p = 0, \quad \forall i \in \mathbb{Z}_{[k+1,N-1]} \tag{33c}$$
$$y_i^p - Cx_i^p - Du_i^p = 0, \quad \forall i \in \mathbb{Z}_{[k+1,N-1]} \tag{33d}$$
$$y_N^p - Yx_N^p = 0, \tag{33e}$$

[1] Note that this implies $p = lk + j$, $k = (p - j)/l$ and $j = 1 + (p - 1) \mod l$.

where $\mathbf{y}^p = \mathrm{vec}(y_0^p, \ldots, y_N^p)$. The inputs u_i^p of this system come directly from the p^{th} column of the matrix $\mathbf{U}$, and thus represent columns of the submatrices $U_{i,k}$. If the constraint terms $\delta\mathbf{c}^p$ for each subproblem are similarly written as $\delta\mathbf{c}^p = \mathrm{vec}(\delta c_0^p, \ldots, \delta c_N^p)$, then each component must satisfy $\delta c_i^p = \mathrm{abs}(y_i^p)$, or in linear inequality constraint form, $-\delta c_i^p \le y_i^p \le \delta c_i^p$.

Note that for the p^{th} subproblem, representing a disturbance at stage k, the constraint contraction terms are zero prior to stage $(k+1)$. By defining

$$\bar{C} := \begin{bmatrix} +C \\ -C \end{bmatrix} \; \bar{D} := \begin{bmatrix} +D \\ -D \end{bmatrix} \; \bar{Y} := \begin{bmatrix} +Y \\ -Y \end{bmatrix} \tag{34}$$

$$H := \begin{bmatrix} -I_q \\ -I_q \end{bmatrix} \; H_f := \begin{bmatrix} -I_r \\ -I_r \end{bmatrix}, \tag{35}$$

equations (33d) and (33e) can be combined to give

$$\bar{C}x_i^p + \bar{D}u_i^p + H\delta c_i^p \le 0, \quad \forall i \in \mathbb{Z}_{[k+1,N-1]} \tag{36a}$$

$$\bar{Y}x_N^p + H_f\delta c_N^p \le 0. \tag{36b}$$

The elements of the total constraint contraction vector in (29) can be found by summing the contributions from each of the subproblems:

$$\delta c_i = \sum\nolimits_{p=1}^{lN} \delta c_i^p, \quad \forall i \in \mathbb{Z}_{[0,N]}, \tag{37}$$

Complete Robust Control Problem We can now restate the robust optimization problem (25) as:

$$\min \sum_{i=0}^{N-1} \left(\frac{1}{2}\|\hat{x}_i\|_Q^2 + \frac{1}{2}\|v_i\|_R^2 \right) + \frac{1}{2}\|\hat{x}_N\|_P^2 \tag{38a}$$

subject to:

$$(28),\ (37) \tag{38b}$$

$$\text{(33a)–(33c) and (36)} \quad \forall p \in \mathbb{Z}_{[1,lN]} \tag{38c}$$

The decision variables in this problem are the nominal states and controls $\hat{x}_i$ and v_i at each stage (the initial state $\hat{x}_0$ is known), plus the perturbed states, controls, and constraint contractions terms x_i^p, u_i^p, and δc_i^p for each subproblem at each stage. It is critical to note that this convex, tractable QP is equivalent to the non-convex problem (11).

Finally, note that the constraints in (38) can be rewritten in diagonalized form by interleaving the variables by time index. The complete set of constraints is then easily written in singly bordered block diagonal form with considerable structure and sparsity.

6. RESULTS

Two sparse QP solvers were used to evaluate the proposed formulation. The first, OOQP (Gertz and Wright, 2003), uses a primal-dual interior approach configured with the sparse factorization code MA27 from the HSL library (HSL, 2002) and the OOQP version of the multiple-corrector interior point method of (Gondzio, 1996). The second sparse solver used was the QP interface to the PATH (Dirske and Ferris, 1995) solver, a code which solves the more general mixed complementarity problem using an active-set method, of which the quadratic programming problem is a special case. All results reported in this section were generated on a 3Ghz Pentium 4 single processor machine with 1GB of RAM.

Table 1. Average Solution Times (sec)

	$(\mathbf{M}, \mathbf{v})$		Decomposition	
Problem Size	OOQP	PATH	OOQP	PATH
4 states, 4 stages	0.026	0.047	0.021	0.033
4 states, 8 stages	0.201	0.213	0.089	0.117
4 states, 12 stages	0.977	2.199	0.287	2.926
4 states, 16 stages	3.871	39.83	0.128	10.93
4 states, 20 stages	12.99	76.46	1.128	31.03
8 states, 4 stages	0.886	4.869	0.181	1.130
8 states, 8 stages	7.844	49.15	0.842	19.59
8 states, 12 stages	49.20	303.7	2.949	131.6
8 states, 16 stages	210.5	x	7.219	x
8 states, 20 stages	501.7	x	13.14	x
12 states, 4 stages	4.866	24.66	0.428	6.007
12 states, 8 stages	95.84	697.1†	3.458	230.5†
12 states, 12 stages	672.2	x	11.86	x
12 states, 16 stages	x	x	33.04	x
12 states, 20 stages	x	x	79.06	x

x – Solver failed all test cases

† – Based on limited data set due to failures

A set of test cases was generated to compare the performance of the two sparse solvers using the $(\mathbf{M}, \mathbf{v})$ formulation of Section 4 with the decomposition based method of Section 5. Each test case is defined by its number of states n and horizon length N. The remaining parameters were chosen using the following rules:

- There are twice as many states as inputs.
- All constraints represent randomly selected symmetric bounds subjected to a random similarity transformation.
- The matrices A and B are randomly generated, with (A, B) controllable, A stable.
- The dimension l of the generating disturbance is chosen as half the number of states, with randomly generated E of full rank.
- All test cases have feasible solutions. The current state x is selected such that at least some of the inequality constraints in (38b) are active at the optimal solution.

The average computational times required by each of the two solvers for the two problem formulations for a range of problem sizes are shown in Table 1. Each entry represents the average of ten test cases, unless otherwise noted.

It is clear from these results that, as expected, the decomposition-based formulation can be solved much more efficiently than the original $(\mathbf{M}, \mathbf{v})$ formulation in every case, and that the difference in solution times increases dramatically with increased problem size. Figure 1 shows that the

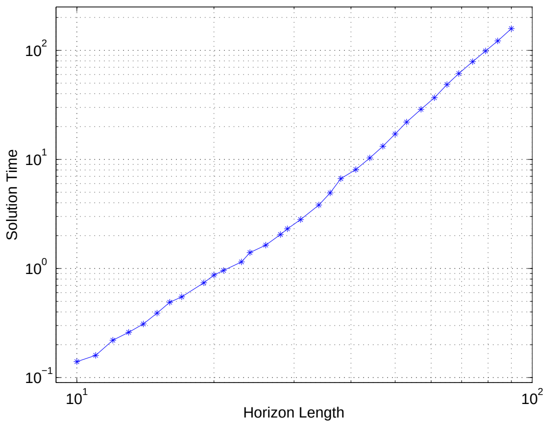

Fig. 1. Computation time vs. horizon length for a 4 state system, using decomposition method

interior point solution time increases cubicly with horizon length for a randomly generated problem with 4 states. It can be shown that, when using an interior-point algorithm, the problem (38) can *always* be solved in $\mathcal{O}(N^3)$ at each iteration, given a suitable factorization procedure (Goulart *et al.*, 2005*a*).

7. CONCLUSIONS AND FUTURE WORK

The results presented rely on the solvers employed to exploit the underlying structure of the quadratic program. It is also possible to exploit this structure directly, by developing specialized factorization algorithms for the factorization of each interior point step. It may also be possible to achieve considerably better performance by placing further constraints on the structure of the disturbance feedback matrix $\mathbf{M}$, though this appears difficult to do if the attractive system-theoretic properties of the present formulation are to be preserved. These are topics of ongoing research.

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