ROBUST STABILITY ANALYSIS OF UNCERTAIN INTERCONNECTION IN THE BEHAVIORAL FRAMEWORK

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Abstract: This paper considers the robust stability of the interconnection of a linear time-invariant differential nominal system and passive uncertainties in the behavioral framework. A generalized version of the well-known passivity theorem is formulated by using quadratic differential forms. Based on the generalized passivity theorem, it is proved that, if the nominal system is Φ-passive, the interconnection is robustly stable against strictly $(-\Phi)$-passive uncertainty. Moreover, we show that the Φ-passivity of the nominal system is a necessary and sufficient condition for the robust stability of a "regular" interconnection.

Keywords: robust stability, interconnection, quadratic differential form, passivity, regular interconnection

1. INTRODUCTION

This paper considers the robust stability of an uncertain system from the viewpoint of the behavioral approach. The robust stability analysis is one of the most important problems in the control theory because there always exists a model uncertainty between the actual system and its mathematical model (e.g. Zhou, Doyle and Glover 1996).

The notion of passivity plays an important role in stability analysis of a feedback system in the traditional input-output framework (e.g. Vidyasagar 1993, van der Schaft 1996). That is, a feedback system is stable if it consists of a passive sub-system and a strictly passive sub-system. This result is well-known as *the passivity theorem*. Several attempts to generalize the passivity-based stability analysis have been made from various viewpoints (e.g. Iwasaki and Shibata 1999, Megretski and Rantzer 1997). It may be noted that, in the behavioral approach, the passivity or dissipativity of a linear system is characterized in terms of a quadratic difference form (QDF), and the analysis and synthesis of a passive (or dissipative) system has been extensively studied (Willems and Trentelman 1998, 2002, Belur and Trentelman 2004). Also, the modeling of an uncertain system via QDF's was considered by Petersen and Willems (2002).

An important generalization of the passivity-based robustness analysis is to derive a stability condition for an *interconnection* of a linear nominal system and a class of passive uncertainty by removing the conventional input-output assumption. The interconnection was devised to describe a more general control structure than a feedback loop in the behavioral setting (Willems 1997). Along this line, Takaba (2002) derived an LMI condition for stability of an interconnection of linear systems with the first-order representation. Pendharkar and Pillai (2004) also considered the stability analysis of an interconnection of a linear system and a class of nonlinearities.

The purpose of this paper is to derive a robust stability condition for an interconnection of a linear time-invariant nominal system and a class of linear passive uncertainties. In particular, we wish to find a necessary and sufficient condition for the robust stability under the constraint of regular interconnection. We will study the robust stability condition both for full and partial interconnections.

The organization of this paper is as follows. First, we review the basic facts on the linear differential system, the quadratic differential form and passivity, and the interconnection in Section 2. In Section 3, we consider the robust stability of a full interconnection, and derive a necessary and sufficient condition for the robust stability under the regular full interconnection. The results in Section 3 are extend to a more general case where some of manifest variables do not contribute to interconnection in Section 4. Finally, in Section 5, we give some concluding remarks and discuss the robust stabilization problem. It should be noted that the proofs of several lemmas and propositions are omitted for the limited paper length.

Notations:
$\mathbb{R}$, $\mathbb{C}$: the fields of real numbers and complex numbers
$\mathbb{C}_+ := \{\lambda \in \mathbb{C} | \ \mathrm{Re}\lambda \geq 0\}$
$\mathbb{C}^p$: the set of p-dimensional complex vectors
$\mathbb{C}^{p\times q}$: the set of $p \times q$ complex matrices
$\mathbb{C}_s^{q\times q}$: the set of $q \times q$ Hermitian matrices
$\mathbb{C}[\xi]$: the set of polynomials with complex coefficients
$\mathbb{C}^p[\xi]$: the set of p-dimensional polynomial vectors
$\mathbb{C}^{p\times q}[\xi]$: the set of $p \times q$ polynomial matrices
$\mathbb{C}^{p\times q}[\zeta,\eta]$: the set of $p \times q$ two-variable polynomial matrices in the indeterminates ζ and η
$\mathbb{C}_s^{q\times q}[\zeta,\eta]$: the set of $q \times q$ Hermitian two-variable polynomial matrices. A polynomial matrix $\Phi \in \mathbb{C}^{q\times q}[\zeta,\eta]$ is called *Hermitian* if $\Phi(\overline{\eta},\overline{\zeta})^* = \Phi(\zeta,\eta)$.
$\mathfrak{C}^\infty(\mathbb{R},\mathbb{C}^p)$: the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{C}^p$.

2. PRELIMINARIES

We will briefly review some preliminary results of the behavioral system theory (Willems 1991, Willems 1997, Willems and Trentelman 1998) which will be useful in this paper.

2.1 *linear time-invariant differential system*

In the behavioral approach, a dynamical system is characterized by *its behavior*. The behavior is the set of all possible trajectories which meet the dynamic laws of the system. Throughout this paper, we will identify a dynamical system with its behavior for ease of notation. We are mainly interested in a linear time-invariant differential system described by a differential-algebraic equation with constant coefficients

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_L \frac{d^L}{dt^L} w = 0,$$

or equivalently

$$R\left(\frac{d}{dt}\right) w = 0,$$

where $R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L \in \mathbb{C}^{p\times q}[\xi]$. This representation is called *a kernel representation*. The variable $w : \mathbb{R} \to \mathbb{C}^q$ is called *a manifest variable*. Then, the behavior is defined by

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{C}^q) \,\middle|\, R\left(\frac{d}{dt}\right) w = 0 \right\}.$$

In short, we denote this behavior as $\mathfrak{B} = \ker R(\frac{d}{dt})$. We define $\mathfrak{L}^q$ as the set of such linear time-invariant differential behaviors with q variables. Note that we can define the behavior in the class of $\mathfrak{C}^\infty$-functions without loss of generality, because we are interested in the robust stability of interconnections of linear time-invariant systems.

Recall that there are more than one polynomial matrices which induce kernel representations of $\mathfrak{B}$. A polynomial matrix $R(\xi)$ satisfying $\mathfrak{B} = \ker R(d/dt)$ is said to be *minimal* if the number of rows of $R(\xi)$ is less than or equal to that of any other polynomial matrix which induces a kernel representation of $\mathfrak{B}$.

A system $\mathfrak{B}$ is called *controllable* if, for any $w_1, w_2 \in \mathfrak{B}$, there exist a $w \in \mathfrak{B}$ and a positive constant T such that $w(t) = w_1(t)$ $(t \leq 0)$ and $w(t) = w_2(t-T)$ $(t \geq T)$. The family of controllable linear time-invariant differential systems is denoted by $\mathfrak{L}^q_{\mathrm{cont}}$. When a kernel representation of $\mathfrak{B}$ is induce by $R(\xi)$, $\mathfrak{B}$ is controllable iff $\mathrm{rank}(R(\lambda))$ is constant for all $\lambda \in \mathbb{C}$. If $R(\xi)$ induces a minimal kernel representation of a controllable system, then $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$.

If $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, there exists a polynomial matrix $M \in \mathbb{C}^{q\times m}[\xi]$ such that $R(\xi)M(\xi) = 0$. In this case, $\mathfrak{B}$ can be rewritten as

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{C}^q) \,\middle|\, \exists \ell \text{ s.t. } w = M\left(\frac{d}{dt}\right)\ell \right\},$$

or in short $\mathfrak{B} = \mathrm{im} M(\frac{d}{dt})$, where $\ell : \mathbb{R} \to \mathbb{C}^m$ is an auxiliary variable called *a latent variable*. The representation $w = M(\frac{d}{dt})\ell$ is called *an image representation*. The image representation is said to be *observable* if $M(\frac{d}{dt})\ell = 0$ implies $\ell = 0$. The behavior $\mathfrak{B} = \mathrm{im} M(\frac{d}{dt})$ is observable if and only if $M(\lambda)$ has full column rank for any $\lambda \in \mathbb{C}$.

Suppose that $R \in \mathbb{C}^{p\times q}[\xi]$ induces a minimal kernel representation of $\mathfrak{B} \subset \mathfrak{L}^q$. Then, there exists a nonsingular permutation matrix Π such that

$$R(\xi)\Pi^{-1} = \begin{pmatrix} Q(\xi) & -P(\xi) \end{pmatrix}, \ \det P \neq 0,$$
$$\Pi w = \begin{pmatrix} u \\ y \end{pmatrix}, \ u : \mathbb{R} \to \mathbb{C}^m, \ y : \mathbb{R} \to \mathbb{C}^p, \ p + m = q.$$

Then, u and y serve as the input and output of $\mathfrak{B}$, respectively, and the transfer function from u to y is defined by

$$G(\xi) = P(\xi)^{-1}Q(\xi)$$

For the obvious reason, the above partition is called *the input/output (I/O) partition of* $\mathfrak{B}$. It should be noted that the choice of inputs and ouputs is not unique, and is *not* given *a priori*. The dimensions of u and y (namey, m and p) are invariant for any choice of inputs and outputs and for any representation of $\mathfrak{B}$. We refer to these dimensions as *input and output* *cardinalities of* $\mathfrak{B}$, and denote them by $\mathtt{m}(\mathfrak{B})$ and $\mathtt{p}(\mathfrak{B})$, respectively. It should also be noted that, the system $\mathfrak{B} \in \mathfrak{L}^q$ is autonomous if and only if $\mathtt{m}(\mathfrak{B}) = 0$ and $\mathtt{p}(\mathfrak{B}) = q$.

A system $\mathfrak{B}$ is said to be *asymptotically stable* if $w(t) \to 0$ $(t \to \infty)$ holds for all $w \in \mathfrak{B}$. Clearly, $\mathfrak{B}$ must be autonomous in order to be asymptotically stable. The behavior $\mathfrak{B} = \ker R(\frac{d}{dt})$ is asymptotically stable if and only if $R(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}_+$. In the case where $R(\xi)$ is square, $\mathfrak{B}$ is asymptotically stable iff $R(\xi)$ is Hurwitz, namely $\det R(\xi) = 0$ has all roots in $\mathrm{Re}\,\xi < 0$.

2.2 *Quadratic differential form and passivity*

A quadratic differential form (QDF) $Q_\Phi(w)$ is defined as a quadratic form of $w : \mathbb{R} \to \mathbb{C}^q$ and its derivatives. Namely,

$$Q_\Phi(w) = \sum_{i=0}^{k}\sum_{j=0}^{k} \left(\frac{d^i w}{dt^i}\right)^* \Phi_{ij} \left(\frac{d^j w}{dt^j}\right)$$

where $\Phi_{ij} \in \mathbb{C}^{q\times q}$ and $\Phi_{ji}^* = \Phi_{ij}$. $(i = 0,1,\ldots,k)$. We can associate Q_Φ with a Hermitian two-variable polynomial matrix

$$\Phi(\zeta,\eta) = \sum_{i=0}^{k}\sum_{j=1}^{k} \zeta^i \eta^j \Phi_{ij} \in \mathbb{C}_s^{q\times q}[\zeta,\eta].$$

Notice that the indeterminates ζ and η correspond to the differentiations on w^* and w, respectively. The detailed discussion on the fundamental theory of QDFs can be found in Willems and Trentelman (1998).

A QDF $Q_\Phi(w)$ is said to be *nonnegative* if $Q_\Phi(w)(t) \geq 0$ $\forall t \in \mathbb{R}$ holds for all $w : \mathbb{R} \to \mathbb{C}^q$. Furthermore, $Q_\Phi(w)$ is called *positive* if it is nonnegative and $Q_\Phi(w)(t) = 0\ \forall t$ implies $w(t) = 0\ \forall t$. In the same way, we can define the nonnegativity and positivity along the behavior $\mathfrak{B}$.

We are now at the position to define the passivity in the behaviral framework.

Definition 1. The system $\mathfrak{B}$ is said to be *passive with respect to* Q_Φ or simply Φ*-passive* if

$$\int_{-\infty}^{0} Q_\Phi(w)(t)dt \geq 0 \ \ \forall w \in \mathfrak{B} \cap \mathfrak{D} \qquad (1)$$

where $\mathfrak{D}$ denotes the family of infinitely often differentiable functions with compact support.

Moreover, $\mathfrak{B}$ is said to be *strictly passive with respect to* Q_Φ or simply *strictly* Φ*-passive* if there exists a positive constant ε such that

$$\int_{-\infty}^{0} Q_\Phi(w)(t)dt \geq \varepsilon \int_{-\infty}^{0} \|w(t)\|^2 dt \ \ \forall w \in \mathfrak{B}\cap\mathfrak{D} \quad (2)$$

The passivity is closely related to the dissipativity. The dissipativity of a dynamical system is characterized in terms of a dissipation inequality and a storage function. Namely, $\mathfrak{B}$ is dissipative with respect to Q_Φ if there exists a QDF Q_Ψ satisfying

$$\frac{d}{dt} Q_\Psi(w)(t) \leq Q_\Phi(w)(t) \ \ \forall t \in \mathbb{R}, \forall w \in \mathfrak{B}. \qquad (3)$$

This inequality is called *a dissipation inequality*, and $Q_\Psi(w)$ is called *a storage function.* A Φ-passive system is also called Φ*-dissipative on* $\mathbb{R}_-$ in the literature. The next lemma establishes the relation between the passivity and the dissipation inequality (Willems and Trentelman 1998, Willems and Trentelman 2002).

Lemma 1. Let $\mathfrak{B} \in \mathfrak{L}^q_{\mathrm{cont}}$ and $\Phi \in \mathbb{C}_s^{q\times q}[\zeta,\eta]$ be given. The following statements are equivalent.

(i) The behavior $\mathfrak{B}$ is Φ-passive.
(ii) There exists a nonnegative storage function Q_Ψ for $\mathfrak{B}$ and Q_Φ.
(iii) $M(\lambda)^*\Phi(\bar{\lambda},\lambda)M(\lambda) \geq 0$ holds for any $\lambda \in \mathbb{C}_+$, where $M(\xi)$ is the polynomial matrix that induces an image representation of $\mathfrak{B}$.

The next lemma plays an important role in the analysis of the case where Φ is a constant matrix.

Lemma 2. Let a nonsingular matrix $\Phi \in \mathbb{C}_s^{q\times q}$ be given. If the controllable behavior $\mathfrak{B} \in \mathfrak{L}^q_{\mathrm{cont}}$ is Φ-passive, then we have

$$\sigma_+(\Phi) \geq \mathtt{m}(\mathfrak{B}).$$

where $\sigma_+(\ \cdot\ )$ and $\sigma_-(\ \cdot\ )$ denote the numbers of positive and negative eigenvalues of an Hermitian matrix, respectively.

Proof: See Willems and Trentelman (2002). ■

2.3 *Interconnection*

We introduce the notion of an interconnection of two linear time-invariant differential systems $\mathfrak{B}, \mathfrak{B}'$.

2.3.1. *Full interconnection* We first consider the simplest interconnection where both $\mathfrak{B}$ and $\mathfrak{B}'$ belong to $\mathfrak{L}^q$ and all the manifest variables contribute to the interconnection. Such an interconnection is defined by $\mathfrak{B} \cap \mathfrak{B}'$, and is referred to as *a full interconnection*. Obviously, $w \in \mathfrak{B} \cap \mathfrak{B}'$ implies that the manifest variable w must satisfy the laws of both systems.

Let $M(\xi)$ and $L(\xi)$ induce the image representations of $\mathfrak{B}$ and $\mathfrak{B}'$, respectively. Then, $w \in \mathfrak{B} \cap \mathfrak{B}'$ is expressed as

$$w = M\left(\frac{d}{dt}\right)\ell, \ \ \begin{pmatrix} M(\frac{d}{dt}) & -L(\frac{d}{dt}) \end{pmatrix} \begin{pmatrix} \ell \\ \ell' \end{pmatrix} = 0.$$

Also, if $R(\xi)$ and $K(\xi)$ induce the kernel representations of $\mathfrak{B}$ and $\mathfrak{B}'$, the kernel representation of $\mathfrak{B} \cap \mathfrak{B}'$ is given by

$$\begin{pmatrix} R(\frac{d}{dt}) \\ K(\frac{d}{dt}) \end{pmatrix} w = 0. \qquad (4)$$

The interconnection $\mathfrak{B} \cap \mathfrak{B}'$ is said to be *regular* if

$$\mathtt{p}(\mathfrak{B} \cap \mathfrak{B}') = \mathtt{p}(\mathfrak{B}) + \mathtt{p}(\mathfrak{B}'), \tag{5}$$

or equivalently

$$\mathtt{m}(\mathfrak{B} \cap \mathfrak{B}') = \mathtt{m}(\mathfrak{B}) + \mathtt{m}(\mathfrak{B}') - q. \tag{6}$$

The notion of regularity is very important because a regular interconnection admits a feedback structure with an appropriate choice of I/O partition under the above condition (Willems 1997). It should be noted that, if $\mathfrak{B} \cap \mathfrak{B}'$ is autonomous and regular, then $\mathtt{p}(\mathfrak{B}) + \mathtt{p}(\mathfrak{B}') = q$ and $\mathtt{m}(\mathfrak{B}) + \mathtt{m}(\mathfrak{B}') = q$.

Let $R(\xi)$ and $K(\xi)$ induce minimal kernel representations of $\mathfrak{B}$ and $\mathfrak{B}'$, respectively. It is obvious from the previous discussions that $\mathfrak{B} \cap \mathfrak{B}'$ is regular and stable iff $\begin{pmatrix} R(\xi) \\ K(\xi) \end{pmatrix}$ is square and Hurwitz.

Lemma 3. Consider two controllable systems $\mathfrak{B}, \mathfrak{B}' \in \mathfrak{L}^q_{\mathrm{cont}}$. Let $R(\xi)$ and $K(\xi)$ induce minimal kernel representations of $\mathfrak{B}$ and $\mathfrak{B}'$, respectively. Similarly, let $M(\xi)$ and $L(\xi)$ induce observable image representations of $\mathfrak{B}$ and $\mathfrak{B}'$, respectively. The following are equivalent.

(i) $\mathfrak{B} \cap \mathfrak{B}'$ is regular and asymptotically stable.
(ii) $\begin{pmatrix} R(\xi) \\ K(\xi) \end{pmatrix}$ is square and Hurwitz.
(iii) $\begin{pmatrix} M(\xi) & -L(\xi) \end{pmatrix}$ is square and Hurwitz.
(iv) $K(\xi)M(\xi)$ is square and Hurwitz.
(v) $R(\xi)L(\xi)$ is square and Hurwitz.

2.3.2. *Partial interconnection* We consider a more general situation where some of the manifest variables do not contribute to the interconnection. Such an interconnection is called *a partial interconnection*, and denoted by $\mathfrak{B} \wedge \mathfrak{B}'$. Suppose that $\mathfrak{B}$ and $\mathfrak{B}'$ belong to $\mathfrak{L}^{q+n}$ and $\mathfrak{L}^q$, respectively. Then, the partial interconnection $\mathfrak{B} \wedge \mathfrak{B}'$ is defined by

$$\mathfrak{B} \wedge \mathfrak{B}' = \left\{ (w,x) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{C}^{q+n}) \,|\, (w,x) \in \mathfrak{B},\, w \in \mathfrak{B}' \right\}.$$

We also define the projection $\pi_w : \mathfrak{L}^{q+n} \to \mathfrak{L}^q$ as

$$\pi_w(\mathfrak{B}) = \{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{C}^q) \,|\, \exists x \text{ s.t. } (w,x) \in \mathfrak{B} \}.$$

Suppose that $\mathfrak{B}$ and $\mathfrak{B}'$ are respectively described by

$$R\left(\frac{d}{dt}\right) w + X\left(\frac{d}{dt}\right) x = 0, \tag{7}$$

$$K\left(\frac{d}{dt}\right) w = 0, \tag{8}$$

where $w : \mathbb{R} \to \mathbb{C}^q$ represents the interconnection variable through which $\mathfrak{B}$ is connected to $\mathfrak{B}$. The variable $x : \mathbb{R} \to \mathbb{C}^n$ is said to be *detectable from* w in $\mathfrak{B}$, if $w = 0$ implies $x(t) \to 0$ $(t \to \infty)$. Putting the above equations together yields a kernel representation of $\mathfrak{B} \wedge \mathfrak{B}'$ is given by

$$\begin{pmatrix} R(\frac{d}{dt}) & X(\frac{d}{dt}) \\ K(\frac{d}{dt}) & 0 \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} = 0. \tag{9}$$

In the same way as the full interconnection case, we can define the regularity of $\mathfrak{B} \wedge \mathfrak{B}'$. That is, the partial interconnection is said to be regular if

$$\mathtt{p}(\mathfrak{B} \wedge \mathfrak{B}') = \mathtt{p}(\mathfrak{B}) + \mathtt{p}(\mathfrak{B}').$$

Also, $\mathfrak{B} \wedge \mathfrak{B}'$ is said to be asymptotically stable if $(w(t), x(t)) \to (0,0)$ $(t \to \infty)$ holds for all $(w,x) \in \mathfrak{B} \wedge \mathfrak{B}'$. Clearly, $\mathfrak{B} \wedge \mathfrak{B}'$ is asymptotically stable iff $\begin{pmatrix} R(\lambda) & X(\lambda) \\ K(\lambda) & 0 \end{pmatrix}$ has full column rank for all $\lambda \in \mathbb{C}_+$.

The following lemmas explain important relationships between the full and partial interconnections.

Lemma 4. The partial interconnection $\mathfrak{B} \wedge \mathfrak{B}'$ is asymptotically stable if and only if the following conditions are satisfied simultaneously.

(i) The variable x is detectable from w in $\mathfrak{B}$.
(ii) $\pi_w(\mathfrak{B}) \cap \mathfrak{B}'$ is asymptotically stable.

Lemma 5. The partial interconnection $\mathfrak{B} \wedge \mathfrak{B}'$ is regular if and only if the full interconnection $\pi_w(\mathfrak{B}) \cap \mathfrak{B}'$ is regular.

It may be noted that similar issues to the above lemmas are discussed from the viewpoint of regular implementability by Belur and Trentelman (2002).

3. ROBUST STABILITY ANALYSIS OF FULL INTERCONNECTION

We first present a fundamental result for the stability of a full interconnection $\mathfrak{B} \cap \mathfrak{B}'$. This can be considered as a generalized version of the well-known passivity theorem.

Proposition 1. Let $\Phi(\zeta, \eta) \in \mathbb{C}_s^{q \times q}[\zeta, \eta]$ induce a QDF Q_Φ. Assume that $\mathfrak{B} \in \mathfrak{L}^q_{\mathrm{cont}}$ is Φ-passive and $\mathfrak{B}' \in \mathfrak{L}^q_{\mathrm{cont}}$ is strictly $(-\Phi)$-passive. Then, the interconnection $\mathfrak{B} \cap \mathfrak{B}'$ is asymptotically stable.

Proof: We see from Lemma 1 that, under the assumptions, there exist nonnegative QDF's Q_Ψ, Q_Θ and a positive constant ε such that

$$\frac{d}{dt} Q_\Psi(w)(t) \le Q_\Phi(w)(t) \quad \forall t \in \mathbb{R},\ \forall w \in \mathfrak{B}. \tag{10}$$

$$\frac{d}{dt} Q_\Theta(w)(t) + \varepsilon \|w(t)\|^2 \le -Q_\Phi(w)(t) \qquad \forall t \in \mathbb{R},\ \forall w \in \mathfrak{B}', \tag{11}$$

By putting these inequalities together, we obtain

$$Q_{\Psi+\Theta}(w)(t) \ge 0 \ \ \forall w \in \mathfrak{B} \cap \mathfrak{B}', \tag{12}$$

$$\frac{d}{dt} Q_{\Psi+\Theta}(w)(t) \le -\varepsilon \|w(t)\|^2 \ \ \forall w \in \mathfrak{B} \cap \mathfrak{B}' \tag{13}$$

Since $Q_{\Phi+\Theta}(w)(t)$ is monotone non-increasing and bounded below from the above inequalities, $Q_{\Psi+\Theta}(w)(t)$ converges as t goes to infinity. This implies that $\frac{d}{dt} Q_{\Psi+\Theta}(w)(t) \to 0$ and hence $w(t) \to 0$ from (13). This completes the proof. ■

Remark 1. The QDF $Q_{\Psi+\Theta}(w)$ serves as a Lyapunov function for $\mathfrak{B}\cap\mathfrak{B}'$. For the detail of the Lyapunov theory in the behavioral framework, the readers should refer to Willems and Trentelman (1998) and Peeters and Rapisarda (2001).

Consider the interconnection of the nominal system $\mathfrak{B}\in\mathfrak{L}^q_{\text{cont}}$ and the uncertainty $\mathfrak{B}_\Delta\in\Delta_\Phi$, where the uncertainty set Δ_Φ is defined as

$$\Delta_\Phi := \left\{\mathfrak{B}_\Delta\in\mathfrak{L}^q_{\text{cont}} \mid \mathfrak{B}_\Delta:\ \text{strictly } (-\Phi)\text{-passive}\right\}.$$

The interconnection $\mathfrak{B}\cap\mathfrak{B}_\Delta$ is said to be *robustly stable against* Δ_Φ if it is asymptotically stable for all $\mathfrak{B}_\Delta\in\Delta_\Phi$. A sufficient condition for the robust stability of $\mathfrak{B}\cap\mathfrak{B}_\Delta$ immediately follows from Proposition 1.

Theorem 1. Let $\Phi\in\mathbb{C}_s^{q\times q}[\zeta,\eta]$ be given. Assume that $\mathfrak{B}\in\mathfrak{L}^q_{\text{cont}}$ is Φ-passive. Then, the full interconnection $\mathfrak{B}\cap\mathfrak{B}_\Delta$ is robustly stable against Δ_Φ.

This theorem gives a sufficient condition for robust stability of a "general" full interconnection. The theorem guarantees the robust stability even for an irregular interconnection. For example, consider $\mathfrak{B}\in\mathfrak{L}^3_{\text{cont}}$ and Φ defined by

$$\mathfrak{B}=\ker R\left(\frac{d}{dt}\right),\ R(\xi)=\begin{pmatrix}\sqrt{3}/3 & 0 & -1\\ 0 & -\xi & 1\end{pmatrix}$$
$$\Phi=\operatorname{diag}(-1,-1,3).$$

As a member of Δ_Φ, we choose

$$\mathfrak{B}_\Delta=\ker K\left(\frac{d}{dt}\right),\ K(\xi)=\begin{pmatrix}\xi+1 & 0 & -1\\ 0 & 1 & -1\end{pmatrix}$$

By Theorem 1, $\mathfrak{B}\cap\mathfrak{B}_\Delta$ is asymptotically stable. However, it is straightforward to verify that $\begin{pmatrix}R(\lambda)\\K(\lambda)\end{pmatrix}$ has full column rank for all $\lambda\in\mathbb{C}$, implying $\mathfrak{B}\cap\mathfrak{B}_\Delta=\{0\}$. Clearly, this is an impractical situation because it means that $\mathfrak{B}_\Delta$ forces the trajectory $w\in\mathfrak{B}$ to be identically zero.

In order to consider the robust stability in more practical situations, we need to impose the regularity on $\mathfrak{B}\cap\mathfrak{B}_\Delta$. Therefore, we introduce a subset of Δ_Φ as

$$\Delta_\Phi^p := \{\mathfrak{B}_\Delta\in\Delta_\Phi : \mathtt{m}(\mathfrak{B}_\Delta)=p\},$$

where p is the output cardinality of $\mathfrak{B}$, i.e. $p=\mathtt{p}(\mathfrak{B})$. Noting $q=\mathtt{p}(\mathfrak{B})+\mathtt{m}(\mathfrak{B})$, we obtain $\mathtt{m}(\mathfrak{B})+\mathtt{m}(\mathfrak{B}_\Delta)=q$ for $\mathfrak{B}_\Delta\in\Delta_\Phi^p$. This is a necessary condition for $\mathfrak{B}\cap\mathfrak{B}_\Delta$ to be an autonomous regular interconnection.

Furthermore, we make the following assumption.

Assumption 1: Φ is a Hermitian nonsingular matrix in $\mathbb{C}_s^{q\times q}$, and $\sigma_+(\Phi)\le\mathtt{m}(\mathfrak{B})$.

It follows from Lemma 2 that the second condition in Assumption 1 is a necessary condition for $\Delta_\Phi^p\neq\emptyset$.

A necessary and sufficient condition for robust stability under the constraint of regular interconnection is given by the next theorem.

Theorem 2. Let $\mathfrak{B}\in\mathfrak{L}^q_{\text{cont}}$ be given. Under Assumption 1, the interconnection $\mathfrak{B}\cap\mathfrak{B}_\Delta$ is regular and robustly stable against Δ_Φ^p if and only if $\sigma_+(\Phi)=\mathtt{m}(\mathfrak{B})$ and $\mathfrak{B}$ is Φ-passive.

We refer to the condition $\sigma_+(\Phi)=\mathtt{m}(\mathfrak{B})$ as the *liveness* condition.

Proof: (Sufficiency) The robust stability of $\mathfrak{B}\cap\mathfrak{B}_\Delta$ immediately follows from Theorem 1 because $\Delta_\Phi^p\subset\Delta_\Phi$. Since stability implies autonomy, we get $\mathtt{p}(\mathfrak{B}\cap\mathfrak{B}_\Delta)=q$. By the definition of Δ_Φ^p, it follows that $\mathtt{p}(\mathfrak{B}_\Delta)=q-p$ and hence $\mathtt{p}(\mathfrak{B})+\mathtt{p}(\mathfrak{B}_\Delta)=q$. Thus, $\mathfrak{B}\cap\mathfrak{B}_\Delta$ is a regular interconnection.

(Necessity) Recall from Lemma 2 that $\mathfrak{B}$ cannot be Φ-passive if $\sigma_+(\Phi)<\mathtt{m}(\mathfrak{B})$. Hence, to prove the necessity, we have only to deduce a contradiction under the assumption that $\mathfrak{B}$ is not Φ-passive. Suppose on the contrary that $\mathfrak{B}$ is not Φ-passive. Then, there exist a complex number $\mu\in\mathbb{C}_+$ and a nonzero vector $v\in\mathbb{C}^m$ such that

$$v^*M(\lambda)^*\Phi M(\lambda)v<0. \tag{14}$$

By the inertia theorem, there exists a nonsingular matrix $D\in\mathbb{C}^{q\times q}$ such that

$$\Phi=D^*JD,\quad J=\begin{pmatrix}I_{\sigma_+(\Phi)} & 0\\ 0 & -I_{\sigma_-(\Phi)}\end{pmatrix}. \tag{15}$$

We partition $DM(\xi)$ as

$$DM(\xi)=\begin{pmatrix}W(\xi)\\Z(\xi)\end{pmatrix},$$
$$W\in\mathbb{C}^{\sigma_+(\Phi)\times m}[\xi],\ \ Z\in\mathbb{C}^{\sigma_-(\Phi)\times m}[\xi]. \tag{16}$$

Then, (14) is equivalent to

$$\|W(\mu)v\|^2-\|Z(\mu)v\|^2<0. \tag{17}$$

This implies that $Z(\mu)v\neq0$. For simplicity, we choose v so that $\|Z(\mu)v\|=1$. Then, there exists a unitary matrix $V\in\mathbb{C}^{p\times(p-1)}$ such that

$$\left(Z(\mu)v\ \ V\right)^*\left(Z(\mu)v\ \ V\right)=I_m. \tag{18}$$

We form a constant matrix

$$L=D^{-1}\begin{pmatrix}W(\mu)v & 0\\ Z(\mu)v & V\end{pmatrix}\in\mathbb{C}^{q\times p}.$$

It follows from (17) and (18) that L has full column rank, and there exists an $\varepsilon>0$ satisfying $L^*\Phi L\le-\varepsilon L^*L$. This implies that $L\mathfrak{C}^\infty(\mathbb{R},\mathbb{C}^q)$ is strictly $(-\Phi)$-passive, namely $L\mathfrak{C}^\infty(\mathbb{R},\mathbb{C}^q)\subset\Delta_\Phi^p$. Furthermore, it is clear that the rank of $\left(M(\xi)\ -L\right)$ degenerates at $\xi=\mu$. In fact,

$$\det\left(M(\mu)\ \ -L\right)=\det\left[D^{-1}\left(\begin{array}{c:cc}W(\mu) & -W(\mu)v & 0\\ Z(\mu) & -Z(\mu)v & -V\end{array}\right)\right]=0.$$

Therefore, $\left(M(\xi)\ -L\right)$ is not a Hurwitz matrix, and hence $\mathfrak{B}\cap\left(L\mathfrak{C}^\infty(\mathbb{R},\mathbb{C}^q)\right)$ is unstable by Lemma 3. Since this contradicts the robust stability against Δ_Φ^p, the proof of the necessity has been completed. ■

Remark 2. Since $\Delta_\Phi^p\subset\Delta_\Phi$, Theorem 2 asserts that the Φ-passivity of $\mathfrak{B}$ is also a necessary condition for the robust stability against Δ_Φ under Assumption 1.

4. ROBUST STABILITY ANALYSIS OF PARTIAL INTERCONNECTION

We consider the robust stability of a partial interconnection. Let the nominal behavior be given by $\mathfrak{B} \in \mathfrak{L}^{q+n}$. The uncertainty set Δ_Φ and Δ_Φ^p are defined in the same manner as in the previous section, while p denotes the output cardinality of $\pi_w(\mathfrak{B})$, i.e. $p = \mathtt{p}(\pi_w(\mathfrak{B}))$. The partial interconnection $\mathfrak{B} \wedge \mathfrak{B}_\Delta, \mathfrak{B}_\Delta \in \Delta_\Phi$ is given by

$$\mathfrak{B} \wedge \mathfrak{B}_\Delta = \{(w,x) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{C}^{q+n}) |\ (w,x) \in \mathfrak{B},\ w \in \mathfrak{B}_\Delta\}.$$

The partial interconnection $\mathfrak{B} \wedge \mathfrak{B}_\Delta$ is said to be robustly stable against Δ_Φ if $\mathfrak{B} \wedge \mathfrak{B}_\Delta$ is asymptotically stable for all $\mathfrak{B}_\Delta \in \Delta_\Phi$.

Assumption 2:
(i) x is detectable from w in $\mathfrak{B}$.
(ii) $\pi_w(\mathfrak{B})$ is controllable.

Note that $\mathtt{m}(\mathfrak{B}) = \mathtt{m}(\pi_w(\mathfrak{B}))$ holds under Assumption 2 (i).

It is easily seen from Lemma 4 that, under Assumption 2, $\mathfrak{B} \wedge \mathfrak{B}_\Delta$ is robustly stable against $\Delta_\Phi$if and only if the full interconnection $\pi_w(\mathfrak{B}) \cap \mathfrak{B}_\Delta$ is robustly stable. Thus, we obtain the following results from Lemmas 4,5 and Theorems 1, 2.

Theorem 3. Let $\mathfrak{B} \in \mathfrak{L}^{q+n}_{\text{cont}}$ and $\Phi \in \mathbb{C}_{\text{s}}^{q\times q}[\zeta,\eta]$ be given. Under Assumption 2, the partial interconnection $\mathfrak{B} \wedge \mathfrak{B}_\Delta$ is robustly stable against Δ_Φ if $\pi_w(\mathfrak{B})$ is Φ-passive.

Theorem 4. Let p denote the output cardinality of $\pi_w(\mathfrak{B})$. Under Assumptions 1 and 2, $\mathfrak{B} \wedge \mathfrak{B}_\Delta$ is regular and robustly stable against Δ_Φ^p if and only if $\pi_w(\mathfrak{B})$ is Φ-passive and the liveness condition $\sigma_+(\Phi) = \mathtt{m}(\mathfrak{B})$ holds.

5. CONCLUDING REMARKS

In this paper, we have studied the robust stability of an uncertain interconnection with strictly $(-\Phi)$-passive uncertainty. In the behavavioral framework, we have given a self-contained proof of the robust stability condition that the Φ-passivity of the nominal system together with the liveness condition must be satisfied in order that the interconnection is regular and robustly stable against the uncertainty set Δ_Φ^p.

It may be noted that we can easily adapt the present results to the case of real-valued behaviors, though we have studied the robust stability of complex-valued behaviors in this paper.

The analysis result derived in this paper provides an important insight into the robust stabilization problem in the behavioral setting. When we solve the robust stabilization problem under the constraint of regular interconnection with all possible uncertainty, we need to find a controller that satisfies the liveness condition as well as the Φ-passivity. Fortunately, the synthesis of a Φ-passive system with the liveness condition was resolved by Willems and Trentelman (2002) and Belur and Trentelman (2004). Their results are applicable to the robust stabilization with the regularity constraint.

As a future research topic, it is interesting to examine the robust stability againt nonstrictly $(-\Phi)$-passive uncertainty.

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