H_∞ STATE FEEDBACK CONTROL OF DISCRETE-TIME PIECEWISE AFFINE SYSTEMS

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Abstract: This paper addresses the H_∞ state feedback control problem for discrete-time piecewise affine systems. Our main objective is to derive design methods that take into account the partition information of the system so as to alleviate the design conservativeness embedded in conventional approaches where local Lyapunov functions are required to possess some global characteristics. To overcome the difficulty arising from non-positive definiteness of Lyapunov matrices, a transformation is introduced which converts the state feedback control problem into a bilinear matrix inequality (BMI) problem. We propose iterative linear matrix inequality (LMI) approaches to solve the BMI problem. Numerical examples demonstrate the effectiveness of the proposed design.

Keywords: Discrete-time systems; Piecewise linear systems; H_∞ control; State feedback.

1. INTRODUCTION

Piecewise affine (PWA) systems have been receiving much attention in control community because a large class of nonlinear systems, such as systems with relay, saturation and dead zone, can be modelled by PWA systems. Also, smooth nonlinear systems are approximated by PWA systems in (Johansson, 2003; Rantzer and Johansson, 2000) and fuzzy logic (neural) systems are modelled as PWA systems in (Johansson, 2003; Feng, 2003). Moreover, in (Heemels *et al.*, 2001), Heemels *et. al.* established an equivalence between some classes of hybrid systems and PWA systems. Thus PWA systems provide a powerful means for analysis and design of nonlinear systems.

There are a number of results on analysis of PWA systems, see, e.g. (Johansson, 2003; Rantzer and Johansson, 2000) for continuous-time systems and (Ferrari-Trecate *et al.*, 2001; Cuzzola and Morari, 2001; Feng, 2002) for discrete-time systems. There have been many works on stability analysis of PWA systems with piecewise quadratic Lyapunov functions. In (Johansson, 2003; Rantzer and Johansson, 2000), Johansson and Rantzer gave an inspiring idea on piecewise quadratic Lyapunov functions and ways of relaxing conservatism in analysis of continuous-time PWA systems. The discrete-time counterpart has been discussed in (Feng, 2002; Ferrari-Trecate *et al.*, 2001).

Design problems for PWA systems have also received much attention. For instance, Alessadri (Alessandri and Colleta, 2001) presented a Luenberger observer for both discrete-time and continuous-time systems, where the modes of the

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systems are known *a priori*. Trecate *et. al.* in (Ferrari-Trecate *et al.*, 2002*a*) proposed a state-smoothing algorithm for hybrid systems based on moving-horizon estimation (MHE) by exploiting the equivalence between hybrid systems modelled in the mixed logic dynamical form and PWA systems. Rodrigues in (Rodrigues, 2002) provides an observer-based output feedback controller synthesis approach for continuous-time PWA systems, where sliding mode is taken into account in detail. As for discrete-time PWA systems, the authors in (Cuzzola and Morari, 2001; Mignone *et al.*, 2000) presented a number of results on controller design. Feng in (Feng, 2003) proposed a novel method to synthesize an observer-based output feedback controller based on the so-called separation principle.

In this paper, we investigate the H_∞ state feedback control problem for PWA systems. This problem has been considered in (Cuzzola and Morari, 2001; Mignone *et al.*, 2000; Ferrari-Trecate *et al.*, 2001; Wang and Feng, 2004; Feng, 2004). In (Feng, 2004), Feng presented an H_∞ controller design method for fuzzy dynamic systems based on a continuous-time PWA model. And in (Wang and Feng, 2004), Wang and Feng extended the result for discrete-time Fuzzy systems. In (Cuzzola and Morari, 2001; Mignone *et al.*, 2000), Cuzzola *et al.* considered the H_∞ control problem, generalized H_2 problem and robust H_∞ control in (Ferrari-Trecate *et al.*, 2001). In this paper, we aim to give a less conservative design, where partition-dependent slack variables are employed with the aid of S-procedure. In particular, required properties of Lyapunov function for each partition are only to be satisfied locally. However, since the Lyapunov matrices are no longer required to be positive definite, the standard Schur complement cannot be applied to convert the design problem into an LMI or even a BMI one. In this paper, by introducing a proper transformation, the design problem is converted to a BMI problem. Iterative algorithms are then proposed to solve the BMI problem.

This paper is organized as follows. In Section 2, we give the PWA system model under investigation and some preliminaries. In Sections 3 and 4, we provide new design methods for the H_∞ state feedback controller design. Finally, we draw some conclusions in the last section.

For convenience, we introduce the following notations. $A > 0$ means that A is positive definite. $A \geq 0$ means A is nonnegative definite. $A \succeq 0$ implies that A is copositive. Due to space limitation, we omit some proofs.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following discrete-time PWA systems:

$$\begin{aligned} x_{t+1} &= A_i x_t + B_i u_t + E_i w + a_i \\ z_t &= C_i x_t + D_i u_t + G_i w \\ &\text{for } x_t \in S_i, i \in \mathcal{I} \end{aligned} \tag{1}$$

where $\{S_i = \{x_t | F_i x_t + f_i \geq 0\}\}_{i \in \mathcal{I}} \subseteq \mathcal{R}^n$ denotes a set of polyhedral partitions/subspaces of the state space, $\mathcal{I}$ is the index set of these partitions/subspaces. $x \in \mathcal{R}^n$ is the system state. $u \in \mathcal{R}^m$ is the controlled input vector, $w \in \mathcal{R}^l$ is the noise and $z \in \mathcal{R}^r$ is the output vector. We assume $m \leq n$. All the matrices mentioned in this paper are appropriately dimensioned. Let Ω represent possible index pairs of transitions: $\Omega = \{i, j | x_t \in S_i, x_{t+1} \in S_j,\ i, j \in \mathcal{I}\}$.

For simplicity of presentation, we do not consider PWA systems with affine term, i.e. we assume that $a_i \equiv 0$ and $f_i \equiv 0$. However, the proposed approach can be extended to the general PWA systems with affine term by a suitable transformation (Feng, 2002).

Consider the following state feedback controller:

$$u_t = K_i x_t,\ x_t \in S_i,\ i \in \mathcal{I} \tag{2}$$

where K_i is the state feedback gain of the i-th partition which is to be determined.

Thus the closed-loop system (1) and the controller (2) is given by

$$\begin{aligned} x_{t+1} &= \bar{A}_i x_t + E_i w \\ z_t &= \bar{C}_i x_t + G_i w \end{aligned} \qquad \text{for } x_t \in S_i, i \in \mathcal{I} \tag{3}$$

where $\bar{A}_i = A_i + B_i K_i$ and $\bar{C}_i = C_i + D_i K_i$.

The H_∞ state feedback control problem is stated as follows. Given a prescribed level of disturbance attenuation $\gamma > 0$, design a state feedback controller (2), such that the closed-loop system is exponentially stable and satisfies

$$\| z_t \|^2_{l_2[0,N]} < \gamma^2 \| w_t \|^2_{l_2[0,N]} + v(x_0),\ \forall N \geq 0 \tag{4}$$

where $v(x_0) \geq 0$.

Observe that most of the existing controller design methods for PWA systems are based on the stability results from Trecate and Cuzzola *et. al.* in (Ferrari-Trecate *et al.*, 2001; Cuzzola and Morari, 2001; Cuzzola and Morari, 2002), where piecewise Lyapunov functions are used to analyze the stability of switched systems. These results employ different Lyapunov functions for different partitions and are better than results based on a common Lyapunov function with global properties. A less conservative result was suggested by

Feng (Feng, 2002) and Trecate (Ferrari-Trecate *et al.*, 2002*b*), where the partition information of the systems are further employed in the analysis.

Lemma 1. (Feng, 2002; Ferrari-Trecate *et al.*, 2002*b*) The system (3) is exponentially stable, if there exist some ($P_i = P_i^T, V_i \succeq 0, U_{ij} \succeq 0$) such that

$$P_i - F_i^T V_i F_i > 0, i \in \mathcal{I} \tag{5}$$

$$\bar{A}_i^T P_j \bar{A}_i - P_i + F_i^T U_{ij} F_i < 0, i, j \in \Omega \tag{6}$$

Remark 2. A special case of Lemma 1 is to set $V_i \equiv 0$. Thus $P_i > 0$. Usually, this brings convenience for the controller design as an LMI approach may be applied by using the Schur complement. Note that when this condition is not met, the inequality (6) cannot lead to an LMI for the control design problem since the Schur complement is not applicable for the linearization of (6). In this case, the term $\bar{A}_i^T P_j \bar{A}_i$ in (6) is neither linear nor bilinear. To convert the design problem to a BMI one, we introduce the following lemma.

Lemma 3. The inequality (6) in Lemma 1 is satisfied if and only if for some matrices ($P_i = P_i^T, V_i \succeq 0, U_{ij} \succeq 0, \Upsilon_{ij}, \Psi_j$), the following inequality holds for $\forall\ (i, j) \in \Omega$:

$$\begin{bmatrix} -P_i + F_i^T U_{ij} F_i + \bar{A}_i^T \Upsilon_{ij} + \Upsilon_{ij}^T \bar{A}_i & -\Upsilon_{ij}^T + \bar{A}_i^T \Psi_j \\ -\Upsilon_{ij} + \Psi_j^T \bar{A}_i & P_j - \left(\Psi_j + \Psi_j^T\right) \end{bmatrix} < 0 \tag{7}$$

Proof: (7) $\Longrightarrow$ (6): Multiplying (7) from the left and the right by $\Theta_i^T = \begin{bmatrix} I & \bar{A}_i^T \\ 0 & I \end{bmatrix}$ and Θ_i, respectively, we obtain

$$\begin{bmatrix} \bar{A}_i^T P_j \bar{A}_i - P_i + F_i^T U_{ij} F_i & -\Upsilon_{ij}^T + \bar{A}_i^T P_j - \bar{A}_i^T \Psi_j \\ * & P_j - \left(\Psi_j + \Psi_j^T\right) \end{bmatrix} < 0 \tag{8}$$

which implies (6).

(6) $\Longrightarrow$ (7): Given P_j that satisfies (6), let $\Psi_j = \frac{1}{2}(P_j + I)$, $\Upsilon_{ij}^T = \bar{A}_i^T P_j - \bar{A}_i^T \Psi_j$. By substituting them into (8), we have:

$$\begin{bmatrix} A_i^T P_j A_i - P_i + F_i^T U_{ij} F_i & 0 \\ 0 & -I \end{bmatrix} < 0 \tag{9}$$

Note that Θ_i is invertible, so (9) $\Longrightarrow$ (7). Thus we get the result. ■

Remark 4. Observe that (7) is a BMI. Further, the coupling between Lyapunov matrices and system matrices is now removed, which provides a possible way to a less conservative controller design.

3. LINEAR MATRIX INEQUALITY APPROACH AND SEQUENTIAL LINEAR PROGRAMMING MATRIX METHOD

In this section, we consider the H_∞ state controller design problem based on an LMI technique. First, we assume that the Lyapunov matrix P_i is positive definite. The following lemma is well-known:

Lemma 5. (Ferrari-Trecate *et al.*, 2001) Given a prescribed $\gamma > 0$, the PWA system (3) is stable and has an H_∞ disturbance attenuation γ if there exists a solution $P_i > 0$ such that for $\forall\ i, j \in \Omega$:

$$\begin{bmatrix} \bar{A}_i^T P_j \bar{A}_i - P_i + \bar{C}_i^T \bar{C}_i & \bar{C}_i^T G_i + \bar{A}_i^T P_j E_i \\ * & \boldsymbol{\vartheta}_{ij} \end{bmatrix} < 0 \tag{10}$$

where

$$\boldsymbol{\vartheta}_{ij} = E_i^T P_j E_i + G_i^T G_i - \gamma^2 I \tag{11}$$

It is easy to see that (10) is equivalent to the following inequality:

$$\begin{bmatrix} -Q_i & Q_i \bar{A}_i^T & Q_i \bar{C}_i^T & 0 \\ \bar{A}_i Q_i & -Q_j & 0 & E_i \\ \bar{C}_i Q_i & 0 & -I & G_i \\ 0 & E_i^T & G_i^T & -\gamma^2 I \end{bmatrix} < 0 \tag{12}$$

where $Q_i = P_i^{-1}$.

Thus we may simply let $W_i = K_i Q_i$ in (12), and obtain an LMI approach to the H_∞ state feedback controller design problem, see (Ferrari-Trecate *et al.*, 2001). However, as noted, the design does not take into account the partition information and is likely to be conservative.

In order to obtain a less conservative design, we may consider Lemma 1. For the case that $V_i = 0$, i.e. the Lyapunov matrix is required to be positive definite, we give a design procedure below by the projection lemma.

Lemma 6. (Projection Lemma). (Boyd *et al.*, 1994) Given a symmetric matrix Σ and two matrices $\mathcal{M}$ and $\mathcal{N}$, let $\mathcal{M}^\perp$ and $\mathcal{N}^\perp$ be the null spaces of $\mathcal{M}^T$ and $\mathcal{N}^T$, respectively. There exists some matrix Θ such that

$$\Sigma + \mathcal{M}\Theta^T\mathcal{N}^T + \mathcal{N}\Theta\mathcal{M}^T < 0 \tag{13}$$

if and only if

$$\mathcal{M}^{\perp^T}\Sigma\mathcal{M}^\perp < 0; \quad \mathcal{N}^{\perp^T}\Sigma\mathcal{N}^\perp < 0 \tag{14}$$

Firstly, we give a relaxed result compared with Lemma 5.

Lemma 7. Given a prescribed $\gamma > 0$, the PWA system (3) is exponentially stable and has the H_∞

disturbance attenuation γ if there exists a solution $(P_i > 0, K_i, U_{ij} \succeq 0)$ such that for $\forall\ i, j \in \Omega$

$$\begin{bmatrix} \bar{A}_i^T P_j \bar{A}_i - P_i + F_i^T U_{ij} F_i + \bar{C}_i^T \bar{C}_i & * \\ G_i^T \bar{C}_i + E_i^T P_j \bar{A}_i & \boldsymbol{\vartheta}_{ij} \end{bmatrix} < 0 \quad (15)$$

where $\boldsymbol{\vartheta}_{ij}$ is as in (11).

Applying the Schur complement to (15), we obtain

$$\begin{bmatrix} -P_i + F_i^T U_{ij} F_i & \bar{A}_i^T P_j & \bar{C}_i^T & 0 \\ P_j \bar{A}_i & -P_j & 0 & P_j E_i \\ \bar{C}_i & 0 & -I & G_i \\ 0 & E_i^T P_j & G_i^T & -\gamma^2 I \end{bmatrix} < 0 \quad (16)$$

(16) can be rewritten as

$$\Sigma_{ij} + \mathcal{N} K_i^T \mathcal{M}_{ij}^T + \mathcal{M}_{ij} K_i \mathcal{N}^T < 0 \quad (17)$$

where $\Sigma_{ij} = \begin{bmatrix} -P_i + F_i^T U_{ij} F_i & A_i^T P_j & C_i^T & 0 \\ P_j A_i & -P_j & 0 & P_j E_i \\ C_i & 0 & -I & G_i \\ 0 & E_i^T P_j & G_i^T & -\gamma^2 I \end{bmatrix}$, $\mathcal{N} = \begin{bmatrix} I_{n\times n} \\ 0_{n\times n} \\ 0_{r\times n} \\ 0_{l\times n} \end{bmatrix}$, $\mathcal{M}_{ij} = \begin{bmatrix} 0_{n\times m} \\ P_j B_i \\ D_i \\ 0_{l\times m} \end{bmatrix}$.

We assume that $rank([B_i' \ \ D_i']) = m_i$. Thus we have

$$\mathcal{N}^{\perp} = \begin{bmatrix} 0_{n\times n} & 0_{n\times r} & 0_{n\times l} \\ I_{n\times n} & 0_{n\times r} & 0_{n\times l} \\ 0_{r\times n} & I_{r\times r} & 0_{r\times l} \\ 0_{l\times n} & 0_{l\times r} & I_{l\times l} \end{bmatrix},$$

$$\mathcal{M}_{ij}^{\perp} = \begin{bmatrix} I_{n\times n} & 0_{n\times(n+r-m_i)} & 0_{n\times l} \\ 0_{(n+r)\times n} & \begin{bmatrix} P_j^{-1} & 0 \\ 0 & I_{r\times r} \end{bmatrix} [B_i' \ \ D_i']^{\perp} & 0_{(n+r)\times l} \\ 0_{l\times n} & 0_{l\times(n+r-m_i)} & I_{l\times l} \end{bmatrix} \quad (18)$$

$$\mathcal{N}^{\perp T} \Sigma_{ij} \mathcal{N}^{\perp} = \begin{bmatrix} -P_j & 0 & P_j E_i \\ 0 & -I & G_i \\ E_i^T P_j & G_i^T & -\gamma^2 I \end{bmatrix} < 0 \quad (19)$$

$$\mathcal{M}_{ij}^{\perp T} \Sigma_{ij} \mathcal{M}_{ij}^{\perp} = \begin{bmatrix} -P_i + F_i^T U_{ij} F_i & * & * \\ [B_i' \ \ D_i']^{\perp T} \begin{bmatrix} A_i \\ C_i \end{bmatrix} & -[B_i' \ \ D_i']^{\perp T} \begin{bmatrix} Q_j & 0 \\ 0 & I \end{bmatrix} [B_i' \ \ D_i']^{\perp} & * \\ 0 & [E_i^T \ \ G_i^T][B_i' \ \ D_i']^{\perp} & -\gamma^2 I \end{bmatrix} \quad (20)$$

$$P_j Q_j = I \quad (21)$$

Remark 8. Consider the system (3). The following two statements are equivalent:

(1) There exists a solution $P_i > 0, K_i$ satisfying (15);
(2) There exists a solution $P_i > 0, Q_i > 0$ satisfying (19), (20) and (21).

Thus the state feedback control problem can be solved by first finding a solution for (19), (20) and (21). If they are solvable, we can obtain K_i by substituting them into (15).

Now the problem is how to address the equality constraint (21). Note that (21) can be weakened to the following well-known semi-definite programming relaxation:

$$\begin{bmatrix} -P_j & I \\ I & -Q_j \end{bmatrix} \leq 0 \quad (22)$$

Observe that the condition (21) is equivalent to $trace(P_j Q_j) = n$, thus we can solve the constraint by solving the following optimization problem

$$min \sum_{j \in \mathcal{I}} trace(P_j Q_j), \ \ subject \ to \ (22) \quad (23)$$

The above problem is not convex since the cost function in (23) is bilinear. This bilinear problem has been investigated by many researchers in static output feedback control for continuous-time systems. In fact, some efficient computational algorithms, such as the cone complementarity linearization method (Ghaoui *et al.*, 2001) and sequential linear programming matrix method (SLPMM) (Leibfritz, 2001), were proposed. In this paper, we borrow the main idea of SLPMM because SLPMM always generates a strictly decreasing sequence of the objective function value which is bounded below by some integer, and thus it is convergent.

Now we extend the SLPMM to solve the state feedback control problem and have the following result.

Algorithm 1. SLPMM Based on Projection Lemma

Step 1 Obtain an initial set (P_i^0, Q_i^0) by solving (19), (20) and(22) for $\forall (i, j) \in \Omega$.
Step 2 Given P_i^k and Q_i^k, solve the following optimization problem for some $P_i > 0, Q_i > 0$:

$$\begin{gathered} min \sum_{j \in \mathcal{I}} trace(P_j Q_j^k + P_j^k Q_j), \\ s.t. \ (19), \ (20) \ and \ (22) \ for \ \forall (i, j) \in \Omega \end{gathered} \quad (24)$$

Step 3 If $\sum_{j \in \mathcal{I}} trace(P_j Q_j^k + P_j^k Q_j - 2 P_j^k Q_j^k) \leq \epsilon$, stop, where ϵ is a pre-defined sufficient small positive scalar.
Step 4 Compute $\alpha \in [0 \ 1]$ by solving

$$min \sum_{j \in \mathcal{I}} trace((1-\alpha) P_j + \alpha P_j^k)((1-\alpha) Q_j + \alpha Q_j^k))$$

Set $P_j^{k+1} = (1-\alpha) P_j + \alpha P_j^k$, $Q_j^{k+1} = (1-\alpha) Q_j + \alpha Q_j^k$. $k = k+1$. Go to Step 2.

4. BILINEAR MATRIX INEQUALITY APPROACH

In existing controller, observer and estimator design methods for discrete-time PWA systems, $P_i > 0$, for $i \in \mathcal{I}$ seems to be a default setting. But based on the stability theory stated as in Lemma 1, it is not a necessary condition. A sufficient condition for a proper piecewise quadratic Lyapunov function $V(x_t)$ only requires that $V(x_t)$ be positive in each partition. In this section we shall do away with the assumption that $P_i > 0$ with the aid of Lemmas 1 and 3.

Lemma 9. Given a prescribed $\gamma > 0$, the PWA system (3) is exponentially stable and has the H_∞ disturbance attenuation γ if there exists a solution $(P_i = P_i^T, K_i, V_i \succeq 0, U_{ij} \succeq 0)$ such that

$$P_i - F_i^T V_i F_i > 0, \ \forall \ i \in \mathcal{I} \tag{25}$$

and (15) is satisfied for all $i, j \in \Omega$.

As mentioned earlier, (15) is neither an LMI nor a BMI. By applying Lemma 3, we obtain an analysis result which will lead to an H_∞ state feedback design via BMI techniques.

Theorem 10. Given a prescribed $\gamma > 0$, the PWA system (3) is exponentially stable and has the H_∞ disturbance attenuation γ, if there exists a set of solution $(P_i = P_i^T, \Upsilon_{ij}, \Psi_j, V_i \succeq 0, U_{ij} \succeq 0)$, such that (25) and following inequality are satisfied for $\forall \ i, j \in \Omega$:

$$\begin{bmatrix} -I & * & * & * \\ \bar{C}^T & -P_i + F_i^T U_{ij} F_i + \bar{A}_i^T \Upsilon_{ij} + \Upsilon_{ij}^T \bar{A}_i & * & * \\ 0 & -\Upsilon_{ij} + \Psi_j^T \bar{A}_i & \boldsymbol{\theta}_j & * \\ 0 & G_i^T \bar{C}_i + E_i^T P_j \bar{A}_i & 0 & \boldsymbol{\vartheta}_{ij} \end{bmatrix} < 0 \tag{26}$$

where $\boldsymbol{\theta}_j = P_j - \left(\Psi_j + \Psi_j^T\right)$ and $\boldsymbol{\vartheta}_{ij} = E_i^T P_j E_i + G_i^T G_i - \gamma^2 I$.

Note that by defining $\tilde{A}_i = \begin{bmatrix} \bar{A}_i & E_i \\ \bar{C}_i & G_i \end{bmatrix}$, $\tilde{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & I \end{bmatrix}$ and $\tilde{U}_{ij} = \begin{bmatrix} F_i^T U_{ij} F_i & 0 \\ 0 & (1-\gamma^2) I \end{bmatrix}$, (15) can be rewritten as

$$\tilde{A}_i^T \tilde{P}_j \tilde{A}_i - \tilde{P}_i + \tilde{U}_{ij} < 0 \tag{27}$$

Thus we can conclude that (27) is equivalent to the following inequality for some $(\tilde{P}_i = P_i^T, \Upsilon_i, \Psi_{ij})$ for $\forall \ (i, j) \in \Omega$:

$$\begin{bmatrix} -\tilde{P}_i + \tilde{U}_{ij} + \tilde{A}_i^T \Upsilon_{ij} + \Upsilon_{ij}^T \tilde{A}_i & -\Upsilon_{ij}^T + \tilde{A}_i^T \Psi_j \\ -\Upsilon_{ij} + \Psi_j^T \tilde{A}_i & \tilde{P}_j - \left(\Psi_j + \Psi_j^T\right) \end{bmatrix} < 0 \tag{28}$$

There are several existing iterative (local) algorithms to BMI problems, such as V-K iterative algorithm (Goh *et al.*, 1994), path-following algorithm (Hassibi *et al.*, 1999), and method-of-centers-like algorithm (Kanev *et al.*, 2004) for local region, branch and bound algorithm (Beran *et al.*, 1997) and trust region strategy (J. Thevenet, 2004) for global optimization. We can also apply the commercial software: PENBMI to solve this problem (Stingl, 2004). We omit the detail steps here.

5. EXAMPLES

Example 1. Consider the system with the following parameters:

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1 \end{bmatrix}, \ A_2 = A_4 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & 1 \end{bmatrix},$$

$$B_1 = B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ B_2 = B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$E_1 = E_3 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, \ E_2 = E_4 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix},$$

$$G_1 = G_2 = G_3 = G_4 = 0, \ C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$D_1 = D_2 = D_3 = D_4 = 0.1,$$

$$F_1 = -F_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \ F_2 = -F_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

This example is borrowed from (Feng, 2004) where an optimal $\gamma = 0.2$ was reported. Using the SLPMM introduced in Section 3, we obtain a much better optimal γ of 0.0309 than that in (Feng, 2004). Further, if we use the obtained result and apply the result in Theorem 10 using the path-following algorithm (Hassibi *et al.*, 1999), we can get a better result $\gamma = 0.0300$. One set of possible controller gains are

$$K_1 = K_3 = \begin{bmatrix} -0.514161 & -1.10141 \end{bmatrix},$$

$$K_2 = K_4 = \begin{bmatrix} -1.01279 & -0.372147 \end{bmatrix}$$

We input a white noise with power 0.01. Figure 1 shows the trajectories of z from initial state $[0\ 0]^T$ and $[0.1\ 0.1]^T$, respectively.

Example 2. Consider the system with the following parameters

$$A_1 = A_3 = \begin{bmatrix} 0.28 & 0.315 \\ 0.63 & -0.84 \end{bmatrix}, \ A_2 = A_4 = \begin{bmatrix} 0.525 & 0.77 \\ -0.7 & -0.07 \end{bmatrix},$$

$$B_1 = B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ B_2 = B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$E_1 = E_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ E_2 = E_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$G_1 = G_2 = G_3 = G_4 = 0, \ C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$D_1 = D_2 = D_3 = D_4 = 0,$$

$$F_1 = -F_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ F_2 = -F_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

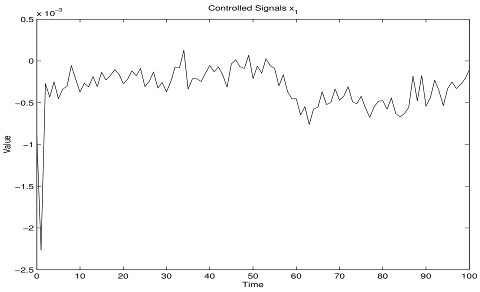
This example is borrowed from (Ferrari-Trecate *et al.*, 2001). If the SLPMM algorithm in Section 3 is applied to this example, the performance is significantly improved to $\gamma = 1.37$ compared with the optimal $\gamma = 1.73$ given in (Ferrari-Trecate *et al.*, 2001). The corresponding controller gains are

$$K_1 = K_3 = \begin{bmatrix} -0.241011 & -0.366977 \end{bmatrix}$$

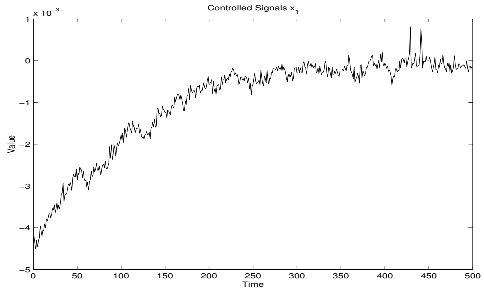
$$K_2 = K_4 = \begin{bmatrix} 0.736137 & 0.123002 \end{bmatrix}$$

6. CONCLUSION

In this paper, we provided several improved H_∞ state feedback control design methods for discrete-time PWA systems. Our methods take into consideration of the partition information of the system.



initial state $[0\ 0]^T$



initial state $[0.1\ 0.1]^T$

Fig. 1. Output z via the H_∞ controller.

The less conservative designs are achieved by applying S-procedure and partition dependent slack variables. The examples illustrate the advantages of our methods.

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