NON-SYNCHRONIZED H_∞ ESTIMATION OF DISCRETE-TIME PIECEWISE LINEAR SYSTEMS

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Abstract: In this paper, we investigate the non-synchronized H_∞ estimation problem for a class of discrete-time piecewise linear systems. In many applications, the system is partitioned based on its state variables. Due to the estimation error, the transitions of actual state and its estimate may not be synchronized. In this paper, a Luenberger type estimator is presented to guarantee the non-synchronized H_∞ performance. Our approach employs *S-procedure* and partition-dependent slack variables to reduce design conservatism.

Keywords: H_∞ estimation; Discrete-time systems; Piecewise linear systems.

1. INTRODUCTION

Piecewise linear (PWL) systems have a wide range of applications in engineering. A large class of nonlinear components, such as relay and saturation, are piecewise linear (Johansson, 2003). Some special classes of hybrid systems and switched systems can be considered as PWL systems (Branicky, 1998). In fact, PWL systems have attracted a lot of attention recently; see e.g. (Rodrigues, 2002; Johansson, 2003; Rantzer and Johansson, 2000; Feng, 2002; Feng, 2003). Many results on stability analysis of PWL systems have appeared in recent years, especially those methods with piecewise quadratic Lyapunov functions based on linear matrix inequality (LMI) (Boyd *et al.*, 1994). Piecewise quadratic Lyapunov functions for PWL systems has been investigated in (Johansson, 2003; Rantzer and Johansson, 2000) , aiming to reduce conservativeness in analysis and design of continuous-time PWL systems. The discrete-time counterpart has been analyzed in (Feng, 2002).

Although the observer or estimator design problem for linear systems has a long history, there have been few existing works for PWL systems. In (Alessandri and Colleta, 2001*a*), a Luenberger observer for both discrete-time and continuous-time systems has been proposed, and a design based on a projection method to minimize the estimation error has been given in (Alessandri and Colleta, 2001*b*). In (Doucet *et al.*, 2000), state estimation for a finite-state Markov chain has been studied. However, in these works, the modes of the systems are known *a priori*. In (Juloski *et al.*, 2002; Juloski *et al.*, 2003), Juloski *et al.* have introduced a design procedure for the Luenberger type of observer, which does not require information on currently active dynamics of a bimodal PWL system. Also partition information is included there to alleviate the design conservatism similar to the works of (Johansson, 2003; Rantzer and Johansson, 2000).

There have also appeared several results on dynamic output feedback synthesis for PWL systems (Feng, 2003; Rodrigues *et al.*, 2000). Feng in (Feng, 2003) assumes that the partitions are de-

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fined in terms of the output of the system so that the plant and state estimator always switch to the same partition at the same time. However, many PWL systems are more likely partitioned based on the state space. Furthermore, there inevitably exists measurement noise in the output. In such situations, there is no guarantee that the system state and the estimated state always stay in the same partition at the same time. In other words, it is likely that the system state might operate in a different region as the estimated state from time to time. This type of state estimation is referred to as non-synchronized state estimation.

In this paper, we investigate the non-synchronized H_∞ estimation for discrete-time PWL systems. For the sake of simplicity of presentation, we only consider PWL systems without affine terms. However, the proposed approach can be extended to general piecewise affine systems with affine terms by a suitable transformation (Feng, 2002).

The rest of the paper is organized as follows. In Section 2 , we describe the PWL systems and the problem to be considered. Also, we present the structure of estimator under investigation. In Section 3, we give a non-synchronized H_∞ analysis result and apply it to the estimator design by an LMI approach. In Section 4, we introduce a relaxed analysis result by which the estimator design is cast into a BMI problem.

For convenience, we introduce the following notations: $A > 0$ ($A < 0$) means that A is positive definite (negative definite). $A \succeq 0$ implies that A is copositive. We also assume that all matrices mentioned in this paper are properly dimensioned.

2. DISCRETE-TIME PWL SYSTEMS

Consider the general discrete-time PWL system:

$$\begin{aligned} x_{t+1} &= A_i x_t + B_i w_t \\ y_t &= C_i x_t + D_i w_t \quad\quad x_t \in S_i, i \in \mathcal{I} \\ z_t &= E_i x_t \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$ is the system state vector, $y \in \mathcal{R}^m$ is the measurement, $w \in \mathcal{R}^r$ is the noise input and $z \in \mathcal{R}^p$ is the signal to be estimated. $\{S_i = \{x_t | F_i x_t \geq 0\}\}_{i \in \mathcal{I}} \subseteq \mathcal{R}^n$ denotes partitions of the state space into a set of convex polyhedral subspaces. We assume that when the state of the system transits from region S_i to S_j at time t, the dynamics of the system is governed by the dynamics of the local model of S_i at that time. We also define a set Ω that represents all possible transitions of the state of the system, that is, $\Omega = \{i, j | x_t \in S_i, x_{t+1} \in S_j, i, j \in \mathcal{I}\}$.

2.1 Stability Analysis

Several stability analysis results on discrete-time PWL systems have been reported in recent years. Trecate and Cuzzola *et. al.* in (Cuzzola and Morari, 2001; Cuzzola and Morari, 2002) showed that

$$A_i^T P_j A_i - P_i < 0, \ \forall i, j \in \Omega \tag{2}$$

where $P_i > 0$, is a sufficient condition for the exponential stability of the unforced system of (1). This result is based on piecewise quadratic Lyapunov functions and is less conservative than methods based on a common quadratic Lyapunov function, where $P_i = P, \forall i \in \mathcal{I}$. To further relax the conservatism, Feng (Feng, 2002) and Trecate *et.al.* (Ferrari-Trecate *et al.*, 2002) has given the following result:

Lemma 1. (Feng, 2002; Ferrari-Trecate *et al.*, 2002) The unforced system of (1) is exponentially stable, if there exists some ($P_i = P_i^T$,$V_i \succeq 0$, $U_{ij} \succeq 0$) such that

$$P_i - F_i^T V_i F_i > 0, \forall i \in \mathcal{I} \tag{3}$$

$$A_i^T P_j A_i - P_i + F_i^T U_{ij} F_i < 0, \forall i, j \in \Omega \tag{4}$$

2.2 Estimator Structure

We consider the following Luenberger-type of estimator

$$\begin{aligned} \hat{x}_{t+1} &= A_j \hat{x}_t + L_j (y_t - C_j \hat{x}_t) \\ \hat{z}_t &= E_j \hat{x}_t \end{aligned} \quad \hat{x}_t \in S_j, \ j \in \mathcal{I} \tag{5}$$

where L_j is the estimator gain to be designed.

In view of the system dynamics $x_t \in S_i$ and the estimator dynamics $\hat{x}_t \in S_j$, the state estimation error dynamic can be obtained by combining the system (1) and the state estimator (5):

$$(\Sigma_d) : \begin{aligned} \xi_{t+1} &= \tilde{A}_{ij} \xi_t + \tilde{B}_{ij} w_t \\ \epsilon_t &= z_t - \hat{z}_t = \tilde{E}_{ij} \xi_t \end{aligned} \tag{6}$$

where $e_t \triangleq x_t - \hat{x}_t$, $\xi_t \triangleq [x_t^T \ \ e_t^T]^T$, and

$$\begin{aligned} \tilde{A}_{ij} &= \begin{bmatrix} A_i & 0 \\ A_i - A_j - L_j C_i + L_j C_j & A_j - L_j C_j \end{bmatrix}; \\ \tilde{B}_{ij} &= \begin{bmatrix} B_i \\ B_i - L_j D_i \end{bmatrix}; \ \tilde{E}_{ij} = [E_i - E_j \ \ E_j] \end{aligned} \tag{7}$$

Note that the transition of the estimator state $\hat{x}_t$ is based on its estimated value. If x_t and $\hat{x}_t$ are close enough such that e_t can be ignored, i.e., x_t and $\hat{x}_t$ are synchronized in transition from one region to another, then the method in (Feng, 2003; Rodrigues *et al.*, 2000) may be applied here. However, in practice, x_t and $\hat{x}_t$ may not always be close to each other, especially in the initial period. Thus they are non-synchronized.

Remark 2. For the case $x_t \in S_i$ and $\hat{x}_t \in S_j$, we have

$$\tilde{F}_{ij}\xi_t \triangleq \begin{bmatrix} F_i & 0 \\ F_j & -F_j \end{bmatrix} \xi_t \geq 0 \tag{8}$$

We define the new region $\tilde{S}_{ij} = \{\xi_t | \tilde{F}_{ij}\xi_t \geq 0\}$. It is easy to check that $\tilde{S}_{ij}$ is still a convex polyhedron.

As mentioned before, the two consecutive system states x_t and x_{t+1} may belong to different regions. Thus we define a set $\breve{\Omega}$ that represents all transitions from one region to another which happen in the system state and estimated state, that is $\breve{\Omega} \triangleq \{i, j | x_t \in S_i, \hat{x}_t \in S_j, i \neq j, \ i, j \in \mathcal{I}\}$. As a consequence, there are four cases about the dynamics transitions of the combined system (6) from ξ_t to ξ_{t+1}, where $x_t \in S_i$, $x_{t+1} \in S_k$, $\hat{x}_t \in S_j$ and $\hat{x}_{t+1} \in S_l$:

$$\Psi \triangleq \left\{ i, j, k, l \middle| \left\{ \begin{array}{l} \text{Case 1}: \ i = j, \ k = l \in \mathcal{I} \\ \text{Case 2}: \ i = j \in \mathcal{I}, \ k, l \in \breve{\Omega} \\ \text{Case 3}: \ i, j \in \breve{\Omega}, \ k = l \in \mathcal{I} \\ \text{Case 4}: \ i, j \in \breve{\Omega}, \ k, l \in \breve{\Omega} \end{array} \right. \right\} \tag{9}$$

Note that if x_t and $\hat{x}_t$ always synchronize in transition from one region to another, there only exists Case 1. We also define the set $\check{\Omega} \triangleq \{i, j | x_t \in S_i, \hat{x}_t \in S_j, \ i, j \in \mathcal{I}\}$.

Next, consider an index pair $i, j, k, l \in \Psi$. We define a piecewise quadratic Lyapunov function candidate as follows:

$$V(\xi_t) = \sum_{i,j \in \check{\Omega}} \rho_{ij} V_{ij}(\xi_t), \ where \ V_{ij}(\xi) = \xi_t^T P_{ij} \xi_t, \ \xi_t \in \tilde{S}_{ij}$$

$$\rho_{ij} = \begin{cases} 1, & \xi_t \in \tilde{S}_{ij} \\ 0, & otherwise \end{cases} \tag{10}$$

The problem under consideration is stated as follows:

H_∞ ESTIMATION PROBLEM: Consider the PWL system (6). Our objective is to design a H_∞ estimator of form (5) for a given scalar $\gamma > 0$ such that for $\forall N \geq 0$

$$\| \epsilon_t \|^2_{\ell_2[0,N]} < \gamma^2 \| w_t \|^2_{\ell_2[0,N]} + v(x_0, \hat{x}_0), \tag{11}$$

where $v(x_0, \hat{x}_0)$ is a non-negative function.

3. LINEAR MATRIX INEQUALITY APPROACH

In this section, we consider the H_∞ estimation problem for the system (1). We shall develop an H_∞ estimator design method via an LMI approach. As usual, we set the Lyapunov matrices $P_{ij} > 0$ for $i, j \in \mathcal{I}$, though it is not a necessary assumption as noted in Lemma 1. The following theorem gives an analytical result.

Theorem 3. Consider the system defined by (6). A given estimator satisfies the H_∞ performance γ, if there exits a solution $(P_{ij} > 0, U_{ijkl} \succeq 0)$ to the following inequalities for $\forall (i, j, k, l) \in \Psi$:

$$\begin{bmatrix} \varpi_{ijkl} & \tilde{A}_{ij}^T P_{kl} \tilde{B}_{ij} \\ \tilde{B}_{ij}^T P_{kl} \tilde{A}_{ij} & \tilde{B}_{ij}^T P_{kl} \tilde{B}_{ij} - \gamma^2 I \end{bmatrix} < 0 \tag{12}$$

where $\varpi_{ijkl} = \tilde{A}_{ij}^T P_{kl} \tilde{A}_{ij} - P_{ij} + \tilde{F}_{ij}^T U_{ijkl} \tilde{F}_{ij} + \tilde{E}_{ij}^T \tilde{E}_{ij}$.

To enable an estimator design, we can apply the Schur complement to (12) to obtain

$$\begin{bmatrix} -P_{kl} & P_{kl}\tilde{A}_{ij} & P_{kl}\tilde{B}_{ij} \\ * & \vartheta_{ijkl} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{13}$$

where $\vartheta_{ijkl} = -P_{ij} + \tilde{F}_{ij}^T U_{ijkl} \tilde{F}_{ij} + \tilde{E}_{ij}^T \tilde{E}_{ij}$ and '*' denotes an entry that can be deduced from the symmetry of the matrix.

Remark 4. Based on the inequality (13), a direct design can be obtained by letting $P_{ij} = \begin{bmatrix} P_{ij}^{(1)} & 0 \\ 0 & P^{(3)} \end{bmatrix}$, $i, j, k, l \in \Psi$. However, this approach will be very conservative in general.

In the following, we shall focus on how to alleviate the conservatism. Some technical lemmas will be presented first.

Lemma 5. The inequality (12) of Theorem 3, can be implied by

$$\begin{bmatrix} P_{kl} - 2P_{ij} & P_{ij}\tilde{A}_{ij} & P_{ij}\tilde{B}_{ij} \\ * & \vartheta_{ijkl} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \ \forall i, j, k, l \in \Psi \tag{14}$$

Remark 6. Note that $(P_{ij} - \sigma P_{kl}) P_{kl}^{-1} (P_{ij} - \sigma P_{kl}) \geq 0$, where σ is a real scalar, thus we have $P_{ij} P_{kl}^{-1} P_{ij} \geq 2\sigma P_{ij} - \sigma^2 P_{kl}$. So the (1,1)-block of inequality (14), in fact, can be replaced by $\sigma^2 P_{kl} - 2\sigma P_{ij}$. The additional σ may bring some flexibilities.

Further, to remove the structural constraint on P_{ij}, we resort to the following lemma.

Lemma 7. There exists a solution $(P_{ij} > 0, U_{ijkl} \succeq 0)$ to inequality (14) of Lemma 5, if and only if there exists a solution $(P_{ij} > 0, U_{ijkl} \succeq 0, G_{ij})$ to the following inequality for $\forall i, j, k, l \in \Psi$:

$$\begin{bmatrix} P_{kl} - 2P_{ij} & G_{ij}^T \tilde{A}_{ij} & G_{ij}^T \tilde{B}_{ij} & G_{ij}^T - P_{ij} \\ * & \vartheta_{ijkl} & 0 & -\varepsilon \tilde{A}_{ij}^T G_{ij} \\ * & * & -\gamma^2 I & -\varepsilon \tilde{B}_{ij}^T G_{ij} \\ * & * & * & -\varepsilon (G_{ij} + G_{ij}^T) \end{bmatrix} < 0 \tag{15}$$

where ε is a positive scalar.

Remark 8. The key idea of the above two lemmas is to eliminate the coupling between the Lyapunov matrices P_{kl} and the system matrices.

Now we can let $P_{ij} = \begin{bmatrix} P_{ij}^{(1)} & P_{ij}^{(2)} \\ P_{ij}^{(2)T} & P_{ij}^{(3)} \end{bmatrix}$,$i, j \in \Omega$. In order to obtain a design method based on LMIs, we let $G_{ij} = \begin{bmatrix} G_{ij}^{(1)} & G_{ij}^{(2)} \\ 0 & G_{j}^{(3)} \end{bmatrix}$ for $i, j \in \Omega$ in (15).We note that G_{ij} is invertible, so is $G_j^{(3)}$. Based on the above lemma, we obtain th following theorem.

Theorem 9. Consider the system defined by (1). Given a scalar $\gamma > 0$, there exists an estimator (5) that solves the H_∞ estimation problem if for some $\varepsilon > 0$, there exists a solution $(P_{ij} > 0, G_{ij}, U_{ijkl} \succeq 0, W_j)$ to the following LMIs for $\forall i, j, k, l \in \Psi$:

$$\begin{bmatrix} P_{kl}^{(1)} - 2P_{ij}^{(1)} & P_{kl}^{(2)} - 2P_{ij}^{(2)} & {G_{ij}^{(1)}}^T A_i & 0 & {G_{ij}^{(1)}}^T B_i & {G_{ij}^{(1)}}^T - P_{ij}^{(1)} & -P_{ij}^{(2)} \\ & P_{kl}^{(3)} - 2P_{ij}^{(3)} & \boldsymbol{\chi}_{ij}^{(1)} & {G_j^{(3)}}^T A_j - W_j C_j & \boldsymbol{\chi}_{ij}^{(3)T} & {G_{ij}^{(2)}}^T - P_{ij}^{(2)T} & {G_j^{(3)}}^T - P_{ij}^{(3)} \\ & * & -P_{ij}^{(1)} + \tilde{U}_{ijkl}^{(1)} & -P_{ij}^{(2)} + \tilde{U}_{ijkl}^{(2)} & 0 & -\varepsilon A_i^T G_{ij}^{(1)} & -\varepsilon \boldsymbol{\chi}_{ij}^{(2)} \\ & * & * & -P_{ij}^{(3)} + \tilde{U}_{ijkl}^{(3)} + E^T E & 0 & 0 & -\varepsilon(A_j^T G_j^{(3)} - C_j^T W_j^T) \\ & * & * & * & -\gamma^2 I & -\varepsilon B_i^T G_{ij}^{(1)} & -\varepsilon \boldsymbol{\chi}_{ij}^{(3)} \\ & * & * & * & & -\varepsilon({G_{ij}^{(1)}}^T + G_{ij}^{(1)}) & -\varepsilon G_{ij}^{(2)} \\ & * & * & * & & * & -\varepsilon({G_j^{(3)}}^T + G_j^{(3)}) \end{bmatrix} < 0 \quad (16)$$

where $\boldsymbol{\chi}_{ij}^{(1)} = {G_{ij}^{(2)}}^T A_i + {G_j^{(3)}}^T (A_i - A_j) - W_j(C_i - C_j)$, $\boldsymbol{\chi}_{ij}^{(2)} = A_i^T G_{ij}^{(2)} + (A_i - A_j)^T G_j^{(3)} - (C_i - C_j)^T W_j^T$ and $\boldsymbol{\chi}_{ij}^{(3)} = B_i^T G_{ij}^{(2)} + B_i^T G_j^{(3)} - D_i^T W_j^T$.

In this situation, the estimator gains can be given by:

$$L_j = G_j^{(3)-T} W_j, \; j \in \mathcal{I}$$

The above result applies partition-dependent Lyapunov functions and will be less conservative than the method stated in Remark 4. However, there is still structural constraint on G_{ij}.

In the following, we will show how to remove the structural constraint completely by an iterative LMI approach.

In fact, we can easily see the following conditions are equivalent to the inequality (12) of the theorem 3.

$$\begin{bmatrix} -Q_{kl} & \tilde{A}_{ij} & \tilde{B}_{ij} \\ * & \boldsymbol{\vartheta}_{ijkl} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \; \forall i, j, k, l \in \Psi \quad (17)$$

$$P_{kl} Q_{kl} = I, \; k, l \in \breve{\Omega} \quad (18)$$

Note that (18) can be weakened to the following well-known semi-definite programming relaxation:

$$\begin{bmatrix} -P_{kl} & I \\ I & -Q_{kl} \end{bmatrix} \leq 0, \; k, l \in \breve{\Omega} \quad (19)$$

Observe that the condition $P_{kl} Q_{kl} = I$ is equivalent to $trace(P_{kl} Q_{kl}) = 2n$, thus we can solve the equality constraint by solving the following optimization problem

$$min \sum_{k,l \in \breve{\Omega}} trace(P_{kl} Q_{kl}), \; subject \; to \; (19) \quad (20)$$

The above problem is not convex since the cost function in (20) is bilinear. This bilinear problem has been investigated by many researchers for static output feedback control of continuous-time systems. In fact, some efficient computational algorithms, such as the cone complementarity linearization methods(Ghaoui *et al.*, 2001) and sequential linear programming matrix method (SLPMM)(Leibfritz, 2001), have been known. In this paper, we borrow the main idea of SLPMM because SLPMM always generates a strictly decreasing sequence of the objective function value which is bounded below by some integer, and thus it is convergent.

Now we extend the SLPMM to solve the state estimation problem and have the following steps:

Algorithm 1. SLPMM FOR ESTIMATOR DESGIN

Step 1 Obtain an initial set (P_{kl}^0, Q_{kl}^0) by solving (19) and (17) for $\forall (i, j, k, l) \in \Psi$.

Step 2 Given P_{kl}^t and Q_{kl}^t, where t is a counter, solve the following optimization problem for some $P_{kl} > 0, Q_{kl} > 0$:

$$\begin{aligned} & min \sum_{k,l \in \breve{\Omega}} trace(P_{kl} Q_{kl}^t + P_{kl}^t Q_{kl}), \\ & subject \; to \; (19) \; and \; (17) \; for \; \forall (i, j, k, l) \in \Psi \end{aligned} \quad (21)$$

Step 3 If $\sum_{k,l \in \breve{\Omega}} trace(P_{kl} Q_{kl}^t + P_{kl}^t Q_{kl} - 2P_{kl}^t Q_{kl}^t) \leq \epsilon$, where ϵ is a pre-defined sufficiently small positive scalar, substitute P_{kl} into (12). If (12) is feasible, we obtain a proper estimator gain. Stop. Otherwise let $\epsilon = \epsilon / \kappa$, where $\kappa > 1$ is a given scalar. If $\epsilon < \epsilon_0$, where ϵ_0 is a given sufficiently small positive valve value, stop. We fail to find an estimator gain.

Step 4 Compute $\alpha \in [0 \; 1]$ by solving

$$min \sum_{k,l \in \breve{\Omega}} trace((1 - \alpha) P_{kl} + \alpha P_{kl}^t)((1 - \alpha) Q_{kl} + \alpha Q_{kl}^t))$$

Set $P_{kl}^{t+1} = (1 - \alpha) P_{kl} + \alpha P_{kl}^t$, $Q_{kl}^{t+1} = (1 - \alpha) Q_{kl} + \alpha Q_{kl}^t$. $t = t + 1$. Go to Step 2.

4. BILINEAR MATRIX INEQUALITY APPROACH

In existing controller, observer and estimator design methods for discrete-time PWL systems, the assumption that $P_{ij} > 0$, for $i, j \in \Omega$ prevails. But based on the stability theory in Lemma 1 for discrete-time PWL systems, it is not a necessary condition. A sufficient condition for a proper piecewise quadratic Lyapunov function $V(\xi)$ only requires that its components $V_k(\xi)$ be positive in each partition. In this section we shall do away with the assumption that $P_{ij} > 0$.

Theorem 10. Consider the system defined by (1). For a given scalar $\gamma > 0$, the given observer gains $L_j,\ j \in \mathcal{I}$ solve the H_∞ estimation problem if there exists a set of solution $(P_{ij} = P_{ij}^T, V_{ij} \succeq 0, U_{ijkl} \succeq 0)$, to (12) and

$$P_{ij} - \tilde{F}_{ij}^T V_{ij} \tilde{F}_{ij} > 0 \tag{22}$$

for $\forall i, j, k, l \in \Psi$.

Since P_{ij} is not required to be positive definite, we cannot apply the Schur complement to (12). Observe that the terms $\tilde{A}_{ij}^T P_{kl} \tilde{A}_{ij}$, $\tilde{A}_{ij}^T P_{kl} \tilde{B}_{ij}$ and $\tilde{B}_{ij}^T P_{kl} \tilde{B}_{ij}$ in (12) are not even bilinear as $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ involve the estimator gains to be determined. To overcome this obstacle, we introduce the following technical lemma.

Lemma 11. Inequality (12) of Theorem 10, is equivalent to the following inequality for some $(P_{ij} = P_{ij}^T, \Upsilon_{ijkl}, \Psi_{ij}, U_{ijkl} \succeq 0)$ for $\forall i, j, k, l \in \Psi$:

$$\begin{bmatrix} \varrho_{ijkl} + \hat{A}_{ij}^T \Upsilon_{ijkl} + \Upsilon_{ijkl}^T \hat{A}_{ij} & -\Upsilon_{ijkl}^T + \hat{A}_{ij}^T \Psi_{kl} \\ * & P_{kl} - \Psi_{kl} - \Psi_{kl}^T \end{bmatrix} < 0 \tag{23}$$

where $\varrho_{ijkl} = \begin{bmatrix} -P_{ij} + \tilde{F}_{ij}^T U_{ijkl} \tilde{F}_{ij} + \tilde{E}_{ij}^T \tilde{E}_{ij} & 0 \\ * & -\gamma^2 I \end{bmatrix}$, $\hat{A}_{ij} = \begin{bmatrix} \tilde{A}_{ij} & \tilde{B}_{ij} \end{bmatrix}$.

Thus the following result follows.

Theorem 12. Consider the system defined by (1). Given a scalar $\gamma > 0$, there exists an estimator (5) that solves the H_∞ estimation problem if for $\forall i, j, k, l \in \Psi$, there exists a solution $(L_j, P_{ij} = P_{ij}^T, \Upsilon_{ijkl}, \Psi_{ij}, V_{ij} \succeq 0, U_{ijkl} \succeq 0)$ to the LMIs (22) and BMIs (23).

Remark 13. There are several existing (interative) algorithms to BMI problems, such as the branch and bound algorithm (Beran *et al.*, 1997), V-K iterative algorithm (Goh *et al.*, 1994), path-following algorithm (Hassibi *et al.*, 1999), and method-of-centers-like algorithm (Kanev *et al.*, 2004) for local region, branch and bound algorithm (Beran *et al.*, 1997) and trust region strategy (J. Thevenet, 2004) for global optimization. We can also apply the commercial software: PENBMI to solve this problem (Stingl, 2004). We omit the detail steps here.

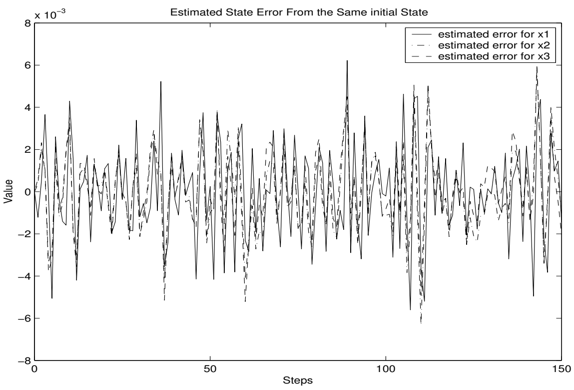

Fig. 1. Trajectories of Estimated State Error

5. EXAMPLES

Example 1. Consider the system (1) with the following parameters

$$A_1 = \begin{bmatrix} 0.9 & -0.1 \\ 0.1 & 0.9 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.9 & 0.1 \\ -0.1 & 0.9 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = \begin{bmatrix} 1 \end{bmatrix},\ F_1 = -F_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T,\ E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem 9 generates an optimal $\gamma = 17.45$ with $\varepsilon = 3.875$ while Remark 4 gives the optimal γ of 22.96. Note that Theorem 9 performs a one-dimensional search over the scaling parameter ε. It can be easily done by applying a numerical optimization algorithm, such as *fminsearch* in Matlab. Using SLPMM algorithm, we can get $\gamma = 16.13$. However, using the path-following algorithm for Theorem 12, we can further improve the result to $\gamma = 15.88$. A pair of possible estimator gains are $L_1 = \begin{bmatrix} 0.1093 \\ 0.0105 \end{bmatrix},\ L_2 = \begin{bmatrix} 0.2267 \\ 0.0141 \end{bmatrix}$.

Example 2. Consider the system (1) with the following parameters

$$A_1 = \begin{bmatrix} 0 & 0.89 & 0.5 \\ -0.12 & 0.89 & 0 \\ -0.1 & 0 & 0.9 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0.89 & 0.5 \\ 0.12 & 0.89 & 0 \\ -0.1 & 0 & 0.9 \end{bmatrix},\ C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, F_1 = -F_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 9 generates the optimal γ of 0.17694, which is better than the result $\gamma = 0.18750$ from Remark 4. Using SLPMM algorithm, we can get $\gamma = 0.17112$. However, Theorem 12 gives $\gamma = 0.16945$. The corresponding estimator gains are $L_1 = \begin{bmatrix} 1.01846 \\ 0.99763 \\ 1.00151 \end{bmatrix},\ L_2 = \begin{bmatrix} 0.98911 \\ 1.00550 \\ 1.00323 \end{bmatrix}$. One sample of estimation error e_t when the input noise is a white noise with unit power and initial conditions $x_0 = [10\ 10\ -10]^T$, $\hat{x}_0 = x_0$, is shown in Figure 1.

6. CONCLUSION

In this paper, we have considered the non-synchronized H_∞ state estimation problem for a class of discrete-time piecewise linear systems. A Luenberger-type of estimator has been proposed to achieve the H_∞ performance. The less conservative designs are achieved by applying the *S-procedure* and partition-dependent slack variables in optimization. Our results are given in terms of LMIs, which can be easily solved using convex optimization, or BMI, which can be solved by iterative algorithms.

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