HYBRID CONTROL WITH SLIDING SECTOR

Yaodong Pan * **Satoshi Suzuki** **
Katsuhisa Furuta ***

* *The 21st Century COE Project Office, Tokyo Denki University, Hatoyama, Hiki-gun, Saitama 350-0394, Japan, Email: pan@ham.coe.dendai.ac.jp*
** *Kanda Branch of the 21st Century COE Project Office, Tokyo Denki University, 2-2 Kanda-Nishiki-cho, Chiyoda-ku, Tokyo 101-8457, Japan, Email: ssuzuki@ham.coe.dendai.ac.jp*
*** *Dept. of Computers and Systems, Tokyo Denki University, Hatoyama, Hiki-gun, Saitama 350-0394, Japan, Email: furuta@k.dendai.ac.jp*

Abstract: This paper proposes a variable structure controller (VS) with sliding sector for a hybrid system. At first, the extremum seeking control algorithm is used to find a Lyapunov function for the hybrid system. In general, a sliding sector is defined as a subset of the state space inside which some norm of state decreases. In the paper, then a sliding sector is designed for each subsystem of the hybrid system so that each state in the state space is inside at least one sliding sector with its corresponding subsystem, where the Lyapunov function found by the extremum seeking control is decreasing. Finally a variable structure control law is designed to switch the hybrid system among subsystems to ensure the decrease of the Lyapunov function in the state space. The resulted VS control system is quadratically stable as the Lyapunov function decreases in the state space. Simulation results are given to show the efficiency of the proposed VS controller.

Keywords: Variable Structure Control, Sliding Sector, Hybrid System, Extremum Seeking Control

1. INTRODUCTION

Hybrid dynamical systems (HDS) have been considered as a combination of continuous and discrete behavior with both continuous and discrete state variables(Tavernini, 1987) (Savkin *et al.*, 1996). In general, in a HDS, the discrete state determines the dynamical rule of the continuous state. Transition to another discrete state implies a change of this rule. Therefore a controller of the hybrid system may be designed to stabilize the hybrid system by switching the discrete state.

The Variable Structure (VS) Control system has been mainly considered in the form of sliding mode (Utkin, 1992). As an alternate design algorithm of the VS Controller, a sliding sector has been proposed to replace the sliding mode in the design of VS control system for a chattering free controller and for the implementation in discrete-time control systems (Furuta, 1990)(Furuta and Pan, 2000). It has been shown for any system including continuous- and discontinuous-time systems and stable and unstable systems that there exists a subset in the state space, inside which some norm of the system decreases without any control input. The subset is called a sliding sector, which can be designed by using the Riccati equation. The corresponding variable structure control law with the sliding sector is thus designed such that the system moves from the outside to the inside of the sliding sector and the control input is zero inside the sliding sector. The variable structure control system with the sliding sector has been shown to be quadratically stable and chattering-free(Furuta and Pan, 2000).

A VS controller for hybrid systems with sliding sector has been propsoed in (Pan *et al.*, 2004) where a sliding sector is designed for each subsystem of the hybrid system with respective to a predetermined norm of the state such that (1) inside every sliding sector, the same norm decreases without any control input for it corresponding subsystem and (2) for any state in the state space, at least one subsystem exists so that the state is inside the corresponding sliding sector of the subsystem. The variable structure control law is designed in (Pan *et al.*, 2004) to switch the hybrid system among subsystems to ensure the decrease of the norm in the state space. That is, according to the current state, find a subsystem such that the corresponding sliding sector covers the current state and the norm of the system state decreases with the maximum rate. It is clear that the resulted hybrid variable structure control system with sliding sector is quadratically stable as the norm decreases in the state space. The most important step in the design of such controller is to find a predetermined norm of the state, i.e. a Lyapunov function. It has not been solved in (Pan *et al.*, 2004) how to find the Lyapunov function although the conditions for the existence of the Lyapunov function were given.

Extremum seeking control approaches have been proposed to find a setpoint and/or track a varying setpoint where the output or a cost function of the system reaches the extremum (Tsien, 1954)(Blackman, 1962)(Astrom and Wittenmark, 1995)(Krstic, 2000). Using a sliding mode to realize the extremum seeking may improve the control performance (S.K.Korovin and Utkin, 1974)(Utkin, 1992) as the control performance of a sliding mode control system is determined by a pre-designed sliding mode and is invariant to parameter uncertainty and external disturbance when the sliding mode happens. The extremum seeking control with sliding mode ensures the convergence to the extremum in a pre-determined speed no matter how the system parameters change(Drakunov and Ümit Özgüner, 1992)(Pan and Ümit Özgüner, 2003).

In this paper, the extremum seeking control algorithm will be used to find the Lyapunov function, based on which a hybrid control system with sliding sector is designed. A calculation example will be provided and the simulation results with the hybrid system will show the effectiveness of the proposed hybrid variable structure control method.

The organization of the paper is as follows. Section 2 describes the control objective of a hybrid system considered in this paper; Section 3 presents the extremum seeking control algorithm to find the Lyapunov function; Section 4 defines and designs the sliding sector for the hybrid system; Section 5 shows the stability of the proposed VS controller with the sliding sector for the hybrid system; Section 6 gives the simulation result.

2. HYBRID SYSTEM

Consider a hybrid system described by the state space equation as follows

$$\dot{x}(t) = A_\sigma x(t) \tag{1}$$

where $x(t) \in R^n$ is the continuous state variable, σ denotes a discrete state variable, i.e. "switching signal" taking values as $\sigma = 1, 2, \cdots, N$. and a finite set of matrices $\cap A := \{A_\sigma : \sigma = 1, 2, \cdots, N\}$ is given.

It is assumed that the system matrices $\{A_\sigma : \sigma = 1, 2, \cdots, N\}$ may be stable or unstable but there exist positive constants α_i

$$0 < \alpha_i < 1 \quad (i = 1, 2, \cdots, N)$$
$$\sum_{i=1}^{N} \alpha_i = 1$$

such that the linear combination of matrices A_i $(i = 1, 2, \cdots, N)$

$$A = \sum_{i=1}^{N} \alpha_i A_i \tag{2}$$

is stable, i.e.

$$Re\lambda_i(A) < 0, \quad (i = 1, 2, \cdots, n) \tag{3}$$

where $Re\lambda_i(A)$ $(i = 1, 2, \cdots, n)$ is the real part of the i-th eigenvalue of A.

The control objective is thus to let the system (1) be asymptotically stable by switching the discrete signal σ, i.e., to find a Lyapunov function

$$L(t) = x^T(t)Px(t) \tag{4}$$

for some positive definite symmetric matrix $P \in R^{n\times n}$ such that

$$\dot{L}(t) \le 0, \forall x \in R^n \quad (x(t)=0 \quad \text{if} \quad \dot{L}(t)=0) \tag{5}$$

with a variable structure control rule

$$\sigma = F(x). \tag{6}$$

3. LYAPUNOV FUNCTION FOUND BY EXTREMUM SEEKING CONTROL

Define a cost function J as

$$J = \max_{1\le i\le n} Re\lambda_i(A) \tag{7}$$

where A is the matrix defined in (2). As the cost function J is a function on $\alpha_1, \alpha_2, \cdots, \alpha_N$ which are assumed to be time functional during adjusting, thus it may be denoted as

$$J = J(\alpha_1(t), \alpha_2(t), \cdots, \alpha_N(t)).$$

It is assumed that there exists a setpoint at $\alpha_1^*, \alpha_2^*, \cdots, \alpha_N^*$ such that the cost function J reaches its minimum value as $J^* = J^*(\alpha_1^*, \alpha_2^*, \cdots, \alpha_N^*)$ and the minimum value J^* is negative, i.e.

$$J(\alpha_1, \alpha_2, \cdots, \alpha_N) \ge J^*(\alpha_1^*, \alpha_2^*, \cdots, \alpha_N^*) \tag{8}$$
$$\forall \alpha_i \in (0,1), \ (i=1,2,\cdots,N)$$
$$J^*(\alpha_1^*, \alpha_2^*, \cdots, \alpha_N^*) = \max_{1\le i\le n} Re\lambda_i(A^*) < 0 \tag{9}$$

where the matrix A^* is determined by

$$A^* = \sum_{i=1}^{N} \alpha_i^* A_i. \tag{10}$$

Then an extremum seeking controller to minimize the cost function J by adjusting the parameters $\alpha_1(t), \alpha_2(t), \cdots, \alpha_N(t)$ is given by

$$\begin{aligned} \dot{\alpha}_1(t) &= -ksgn\sin(h\pi s(t)) \\ \dot{\alpha}_2(t) &= -ksgn\sin(2h\pi s(t)) \\ \vdots &= \quad \vdots \\ \dot{\alpha}_N(t) &= -ksgn\sin(2^{N-1}h\pi s(t)) \end{aligned} \tag{11}$$

where k and h are constant, $sgn()$ is the sign function, and $s(t)$ is a switching function defined as

$$s(t) = J(\alpha_1(t), \alpha_2(t), \cdots, \alpha_N(t)) - g(t), \tag{12}$$

the decerasing reference signal $g(t)$ is given by

$$\dot{g}(t) = -\epsilon, \tag{13}$$

ϵ is a positive constant.

Theorem 1. The above extremum seeking controller ensures that the system converges to a sliding mode

$$s(t) = \gamma\pi$$

for some number $\gamma = 0, \pm1, \pm2, \cdots$ in a finite time and then the cost function $J(t)$ asymptotically converges to the minimum point J^*.

Proof: The proof of this theorem is omitted because of the page limitation. See (Pan and Ümit Özgüner, 2004) for the proof. ∎

Consider an autonomous linear time-invariant continuous-time single input system described by the following state equation.

$$\dot{x}(t) = A^* x(t) \tag{14}$$

where $x(t) \in R^n$ is the state variable and A^* is the system matrix determined by the extremum seeking control algorithm presented in Theorem 1.

It is clear that the system in (14) is stable as all eigenvalues of A^* are in the left half plane of the state space, i.e.

$$Re\lambda_i(A^*) < 0, \quad (i=1,2,\cdots,n)$$

thus there exists a positive definite symmetric matrix P and a positive semi-definite symmetric matrix $R = C^T C$ such that

$$\dot{L}(t) = x^T(A^{*T}P + PA^*)x \le -x^T R x, \quad \forall x \in R^n \tag{15}$$

where $P \in R^{n\times n}$, $R \in R^{n\times n}$, $C \in R^{l\times n}$, $l \ge 1$, (C, A^*) is an observable pair, and $L(t)$ is a Lyapunov function candidate defined as

$$L(t) = ||x||_P^2 = x^T P x > 0, \forall x \in R^n, x \ne 0 \tag{16}$$

which will be used to design a sliding sector in the next section.

4. SLIDING SECTOR

For each subsystem

$$\dot{x}(t) = A_\sigma x(t), \quad \sigma = 1,2,\cdots,N, \tag{17}$$

the inequality

$$\dot{L}(t) = x^T(A_\sigma^T P + PA_\sigma)x \le -x^T R x, \quad \forall x \in R^n$$

may not hold especially when the subsystem is unstable. It is possible to decompose the state space for each subsystem into two parts such that one part satisfies the condition $\dot{L}(t) = x^T(A_\sigma^T P + PA_\sigma)x > -x^T Rx$ for some element $x \in R^n$, and the other part satisfies the condition $\dot{L}(t) = x^T(A_\sigma^T P + PA_\sigma)x \leq -x^T Rx$ for some other element $x \in R^n$. The latter elements form a special subset in which the Lyapunov function candidate $L(t)$ decreases.

We define a P-norm, denoted by $||x||_P$ as the square root of the Lyapunov function candidate $L(t)$ in (16), i.e.

$$||x||_P = \sqrt{L(t)} = \sqrt{(x^T P x)}, \quad x \in R^n. \quad (18)$$

Then the P-norm $||x||_P$ decreases inside this special subset for each subsystem as

$$\frac{d}{dt}||x||_P^2 = x^T(A_\sigma^T P + PA_\sigma)x \leq -x^T Rx., \quad \sigma = 1, 2, \cdots, N.$$

Accordingly, we call this special subset a PR_σ-sliding sector because the matrices P and R together with the system parameter A_σ ($\sigma = 1, 2, \cdots, N$) determine the property of this subset. The definition of the PR_σ-Sliding Sector found in (Furuta and Pan, 2000) is given as follows.

Definition 2. The *PR_σ-Sliding Sector* is defined on the state space R^n for each subsystem in Equation (17) as

$$\mathcal{S}_\sigma = \{x| \; |x^T(A_\sigma^T P + PA_\sigma)x \leq -x^T Rx, x \in R^n \;\}, (19)$$

inside which the P-norm of the system (17) decreases and satisfies

$$\dot{L}(t) = \frac{d}{dt}(x^T(t)Px(t)) = x^T(A_\sigma^T P + PA_\sigma)x \leq -x^T(t)Rx(t), \quad \forall x(t) \in \mathcal{S}_\sigma.$$

where P and R are the matrices as described above and $\sigma = 1, 2, \cdots, N$.

It has been shown in (Furuta and Pan, 1995) that such sliding sector defined above exists for any system including stable and unstable systems.

5. VARIABLE STRUCTURE CONTROLLER

Based on the sliding sectors defined in the last section, a variable structure controller is designed in the following theorem.

Theorem 3. The variable structure control to stabilize the hybrid system given in Equation (subsys01) is determined by

$$\sigma = k, \quad 1 \leq k \leq N \quad (20)$$

if

$$x(t) \in \mathcal{S}_k$$

and

$$x^T(A_k^T P + PA_k)x \leq x^T(A_\sigma^T P + PA_\sigma)x, \quad \forall \sigma = 1, 2, \cdots, N.$$

Proof: Consider the Lyapunov function defined in (16), i.e.

$$L(t) = ||x||_P^2 = x^T P x > 0, \quad \forall x \in R^n, x \neq 0.$$

Its derivative for the autonomous system in (14) is

$$\dot{L}(t) = x^T(A^{*T} P + PA^*)x \leq -x^T Rx, \quad \forall x \in R^n$$

As we have

$$x^T(A^{*T} P + PA^*)x + x^T Rx = \sum_{\sigma=1}^{N} \lambda_\sigma(x^T(A_\sigma^T P + PA_\sigma)x + x^T Rx) \leq 0,$$

there exists a $\sigma = k$ such that

$$x^T(A_k^T P + PA_k)x \leq -x^T Rx.$$

Therefore we choose the discrete state σ to be equal to k, then the derivative function of the Lyapunov function in (16) for the hybrid system in (1) satisfies the following inequality:

$$\dot{L}(t) = x^T(A_k^T P + PA_k)x \leq -x^T Rx, \quad \forall x \in R^n$$

which means the proposed control law in (20) quadratically stabilizes the hybrid system in (1). ■

6. SIMULATION

Consider the third order hybrid system used in (Pan *et al.*, 2004) with system matrices as

$$A_1 = \begin{bmatrix} 3.3167 & 8.3333 & 5.8333 \\ -2.6533 & -6.0667 & -2.6667 \\ 1.0150 & 2.3000 & 0.9500 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1.6000 & 47.0000 & 121.0000 \\ -0.6500 & -23.2000 & -60.5000 \\ 0.1800 & 7.2000 & 18.9000 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} -8.6667 & -48.3333 & -100.0000 \\ 4.3333 & 24.1667 & 50.0000 \\ -1.5500 & -8.5000 & -17.5000 \end{bmatrix}$$

Eigenvalues of each subsystem are as follows.

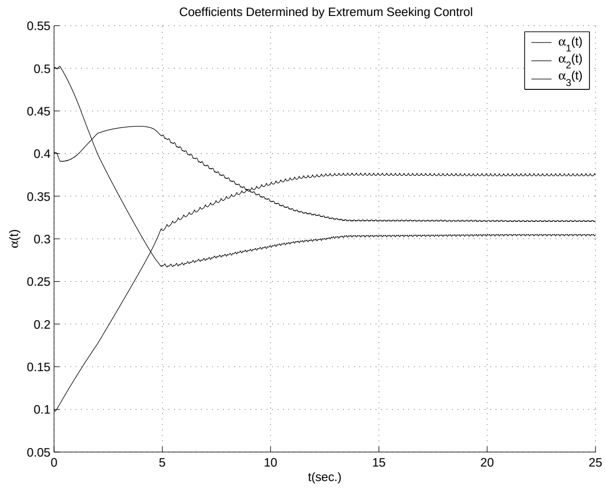

Fig. 1. Evolution of $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$ with Extremum Seeking Control

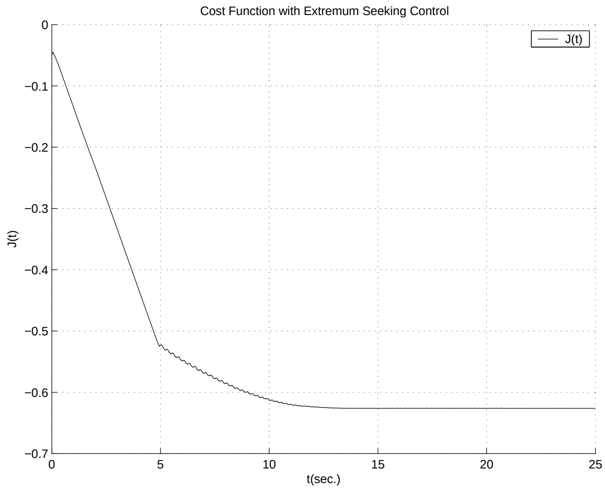

Fig. 2. Evolution of Cost Function with Extremum Seeking Control

$$\lambda(A_1) = -2.0050, 0.0050, 0.2000$$
$$\lambda(A_2) = -3.0297, 0.0297, 0.3000$$
$$\lambda(A_3) = -2.5000, 0, 0.5000.$$

Every subsystem is unstable as there exist unstable eigenvalues for each subsystem. Using the extremum seeking control algorithm presented in Section 3, as shown in Figures 1 and 2, the minimum value J^* is found to be equal to -0.6263 at the setpoint

$$\alpha_1^* = 0.3752$$
$$\alpha_2^* = 0.3041$$
$$\alpha_3^* = 0.3208$$

with the initial condition as

$$\alpha_1(0) = 0.1$$
$$\alpha_2(0) = 0.4$$
$$\alpha_3(0) = 0.5,$$

and parameters of the extremum seeking controller as

$$k = 0.05$$
$$h = 100.$$

The system matrix A^* at the setpoint is

$$A^* = \sum_{i=1}^{3} \alpha_i^* A_i = \begin{bmatrix} -1.0493 & 1.9140 & 6.9048 \\ 0.1969 & -1.5787 & -3.3586 \\ -0.0617 & 0.3257 & 0.4899 \end{bmatrix}$$

with eigenvalues being

$$-0.8361, -0.6756, -0.6263$$

which is better than the one used in (Pan *et al.*, 2004) where the parameters are chosen as

$$\alpha_1 = 0.5$$
$$\alpha_2 = 0.2$$
$$\alpha_3 = 0.3,$$

and eigenvalues are at

$$-0.9452, -0.6660, -0.4288$$

Choose the positive definite matrix R as

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

then the positive definite solution P of the Lyapunov function is determined by

$$P = \begin{bmatrix} 43.3071 & -23.6845 & -1.4972 \\ -23.6845 & 14.1840 & 0.8259 \\ -1.4972 & 0.8259 & 1.4226 \end{bmatrix}.$$

The simulation result with the proposed VS controller for the hybrid system are given in Figures 3 and 4 where the initial condition are taken as

$$x(0) = \begin{bmatrix} 1.0 \\ 2.0 \\ -1.0 \end{bmatrix}.$$

Figure 3 is the time response where the continuous state $x(t)$ converges to the origin and Figure 4 is the plot of the discrete state σ which is adjusted among 1, 2, and 3. From Figures 3 and 4, it is clear that the proposed VS controller with sliding sector stabilizes the hybrid system where each subsystem is an unstable one.

7. CONCLUSION

In this paper, a variable structure controller with sliding sector is designed for the hybrid system.

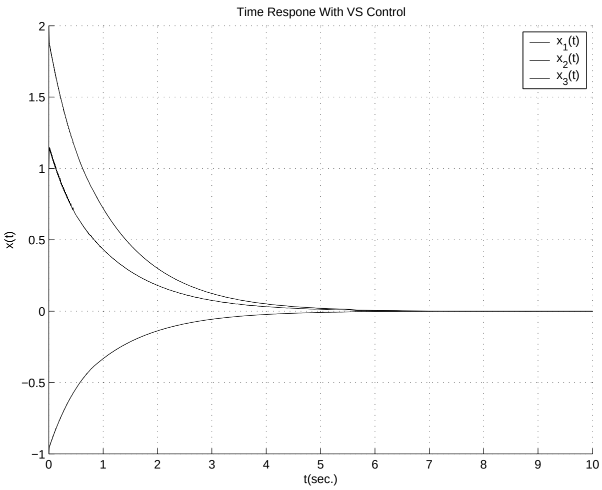

Fig. 3. Evolution of Continuous State Variables $x_1(t)$, $x_2(t)$ and $x_3(t)$

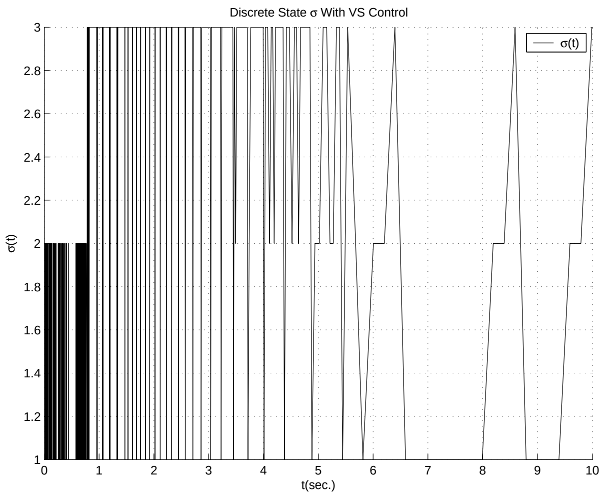

Fig. 4. Evolution of Discrete State Variables of $\sigma(t)$

The extremum seeking control is used to find a Lyapunov function for the VS controller design. The resulted VS control system is quadratically stable. The simulation result shows the efficiency of the proposed method.

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