

FURTHER RESULTS ON OUTPUT REGULATION BY PURE ERROR FEEDBACK ¹

C.I. Byrnes * A. Isidori ^{*,**,*} L. Marconi ^{***}

** Department of Electrical and Systems Engineering,
Washington University, St. Louis, MO 63130.*

*** Dipartimento di Informatica e Sistemistica, Università
di Roma "La Sapienza", Italy.*

**** Center for Research on Complex Automated Systems
(CASYS), DEIS - University of Bologna, Italy.*

Abstract: This work addresses the design of internal model-based regulators for nonlinear systems by error feedback in the general “non-equilibrium” framework proposed in (Byrnes and Isidori, 2003). The material presented here is specifically meant to complement the work in (Byrnes and Isidori, 2004) where a procedure for designing *nonlinear* internal model-based regulators was presented for systems having relative degree 1. *Copyright©2005 IFAC.*

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1. INTRODUCTION

The topic of output regulation for nonlinear systems has been the subject of relevant research efforts in the recent past. Besides others, it is worth mentioning the work (Serrani et al., 2001), in which adaptive internal models have been introduced to cope with uncertainties in the exosystem structure, the works (Chen and Huang, 2004) and (Delli Priscoli, 2004), in which procedures to design nonlinear internal models are presented, and the work (Pavlov et al., 2004), in which incremental stability concepts are employed to infer regulation properties.

In this very active and dynamic scenario a relevant contribution has been recently given in (Byrnes and Isidori, 2003) where the foundations for a non-

equilibrium theory of nonlinear output regulation have been laid. One of the first contributions stemming from this approach has been the design method proposed in (Byrnes and Isidori, 2004). Specifically, in this work, it has been shown how the theory of high-gain observers can be effectively used to design *nonlinear* internal models in a general “non-equilibrium” framework in which the zero dynamics of regulated plant and the dynamics of the exosystem do not possess an equilibrium point but rather a possible complex, though bounded, attractor.

In this paper we wish to complement the results presented in (Byrnes and Isidori, 2004), which were restricted to the simplified case of systems with unitary relative degree, to the more general case in which the relative degree of the regulated plant is bigger than one and pure error feedback is sought. This will be achieved through a number of subsequent steps which will be followed to design the regulator. First (see section 3) it will be shown how the problem of designing a

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regulator by partial state feedback (namely by assuming knowledge of the regulation error and of a number of its time derivatives) can be reduced to a problem of output regulation for a suitably defined system having relative degree 1 with respect to a dummy regulation error given by a linear combination of the error and its time derivatives. Then (see section 4) we run the procedure proposed in (Byrnes and Isidori, 2004) to design a partial state feedback regulator for the relative degree 1 system thus defined. Finally (see section 5) an “high-gain” observer will be designed in order to replace the higher time derivatives of the error with suitable estimates. As stressed above, all the analysis will be derived in the general non-equilibrium framework proposed in (Byrnes and Isidori, 2003) to which the reader is referred for more details about the definitions and tools used throughout this paper.

2. PROBLEM STATEMENT AND STANDING ASSUMPTIONS

In this paper we consider nonlinear systems modeled by equations of the form

$$\begin{aligned} \dot{z} &= f_0(w, z) + f_1(w, z, e_1)e_1 \\ \dot{e}_1 &= e_2 \\ &\vdots \\ \dot{e}_{r-1} &= e_r \\ \dot{e}_r &= q(w, z, e_1, \dots, e_r) + u \\ e &= e_1 \\ y &= \text{col}(e_1, \dots, e_r), \end{aligned} \quad (1)$$

with state $(z, e_1, \dots, e_r) \in \mathbb{R}^n \times \mathbb{R}^r$, control input $u \in \mathbb{R}$, regulated output $e \in \mathbb{R}$, measured output $y \in \mathbb{R}^r$, in which the exogenous inputs $w \in \mathbb{R}^s$ are generated by an exosystem modeled by equations of the form

$$\dot{w} = s(w). \quad (2)$$

The functions $f_0(\cdot), f_1(\cdot), q(\cdot), s(\cdot)$ in (1) and (2) are assumed to be at least continuously differentiable. The initial conditions of (1) range on a set $Z \times E$, in which Z is a fixed compact subset of \mathbb{R}^n and $E = \{(e_1, \dots, e_r) : |e_i| \leq c\}$, with c a fixed number. The initial conditions of the exosystem (2) range on the compact set W of \mathbb{R}^s . In this framework the problem of *output regulation by partial state feedback* is to design a regulator of the form

$$\begin{aligned} \dot{\zeta} &= \varphi(\zeta, y) \\ u &= \gamma(\zeta, y) \end{aligned}$$

such that for all initial conditions $w(0) \in W$ and $(z(0), e_1(0), \dots, e_r(0)) \in Z \times E$ the trajectories of the closed-loop system are bounded and $\lim_{t \rightarrow \infty} e(t) = 0$.

In the more challenging case in which the information available to the regulator is not y (namely the error and its time derivatives) but rather only the error e , the problem in question is called an *output regulation problem by error feedback*.

Augmenting (1) with (2) yields a system which, viewing u as input and e as output, has relative degree r . The associated “augmented” zero dynamics, which is forced by the control

$$u = -q(w, z, 0, \dots, 0), \quad (3)$$

is given by

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f_0(w, z) \end{aligned} \quad (4)$$

For sake of compactness, throughout the paper, we will rewrite the latter as

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}), \quad (5)$$

where $\mathbf{z} = \text{col}(w, z)$. Accordingly, we set $\mathbf{Z} = W \times Z$ and we denote by $\mathbf{q}_0(\mathbf{z})$ the function $-q(w, z, 0, \dots, 0)$ in (3).

In what follows, we retain all the basic assumptions that were introduced in (Byrnes and Isidori, 2003) and (Byrnes and Isidori, 2004) to express certain properties of the augmented zero dynamics (4). We use some concepts and notations introduced in (Byrnes and Isidori, 2003) to which the reader is referred for details. The assumptions in question are the following ones:

Assumption (i): the set W is a differential submanifold with boundary of \mathbb{R}^s , invariant for (2). ◁

Assumption (ii): there exists a compact subset \mathcal{Z} of $W \times \mathbb{R}^n$ which contains the positive orbit of the set \mathbf{Z} under the flow of (5), and $\omega(\mathbf{Z})$ is a differential submanifold (with boundary) of $W \times \mathbb{R}^n$. Moreover there exists a number $d_1 > 0$ such that

$$\mathbf{z} \in W \times \mathbb{R}^n, \quad |\mathbf{z}|_{\omega(\mathbf{Z})} \leq d_1 \quad \Rightarrow \quad \mathbf{z} \in \mathbf{Z}. \quad \triangleleft$$

Remark. Since the positive orbit of the set \mathbf{Z} under the flow of (5) is bounded by hypothesis, the set $\omega(\mathbf{Z})$ is a nonempty, compact and invariant subset of $W \times \mathbb{R}^n$ which uniformly attracts all trajectories of (5) with initial conditions in \mathbf{Z} . It can also be shown (as in (Byrnes and Isidori, 2003)) that for every $w \in W$ there is $z \in \mathbb{R}^n$ such that $(w, z) \in \omega(\mathbf{Z})$. ◁

In what follows, for convenience, the set $\omega(\mathbf{Z})$ will be simply denoted as \mathcal{A} . The last condition in assumption (ii) implies that \mathcal{A} is stable in the

sense of Lyapunov. The next hypothesis is that the set \mathcal{A} is locally exponentially attractive.

Assumption (iii): There exist $M \geq 1$, $a > 0$ and $d_2 \leq d_1$ such that

$$\mathbf{z}_0 \in W \times \mathbb{R}^n, \quad |\mathbf{z}_0|_{\mathcal{A}} \leq d_2 \quad \Rightarrow \quad |\mathbf{z}(t, \mathbf{z}_0)|_{\mathcal{A}} \leq M e^{-at} |\mathbf{z}_0|_{\mathcal{A}}$$

in which $\mathbf{z}(t, \mathbf{z}_0)$ denotes the solution of (5) passing through \mathbf{z}_0 at time $t = 0$. \triangleleft

The next assumption, usually referred to as *immersion assumption*, involves a property of the system (5) with output $\mathbf{q}_0(\mathbf{z})$. In particular it is assumed that the output of the system in question can be viewed as the output of a nonlinear system which is uniformly observable in the sense of (Gauthier and Kupka, 2001). More precisely we assume what follows.

Assumption (iv): There exists a positive integer q , a differentiable map $\tau : \mathcal{Z} \rightarrow \mathbb{R}^q$, a locally Lipschitz function $f : \mathbb{R}^q \rightarrow \mathbb{R}$ and two maps $\Phi : \mathbb{R}^q \rightarrow \mathbb{R}^q$, $\Gamma : \mathbb{R}^q \rightarrow \mathbb{R}$ such that for all $\mathbf{z} \in \mathcal{A}$

$$\frac{\partial \tau}{\partial \mathbf{z}} \mathbf{f}_0(\mathbf{z}) = \Phi(\tau(\mathbf{z})) \quad \mathbf{q}_0(\mathbf{z}) = \Gamma(\tau(\mathbf{z}))$$

with

$$\Phi(s) = \begin{pmatrix} s_2 \\ \vdots \\ s_q \\ f(s_1, s_2, \dots, s_q) \end{pmatrix} \quad \Gamma(s) = s_1. \triangleleft \quad (6)$$

Under these assumptions the theory presented in (Byrnes and Isidori, 2003), complemented with the result in (Byrnes and Isidori, 2004), has shown how the design of the regulator can be achieved in the simplified case of systems with relative degree 1 (for which the problem in question by *partial state* and by *error* feedback coincide). This paper aims to complement the works (Byrnes and Isidori, 2003) and (Byrnes and Isidori, 2004), by showing how the design of the error feedback regulator can be obtained for systems with higher relative degree. This is the goal of the next sections.

3. REDUCING TO THE CASE OF RELATIVE DEGREE 1

In this section we show how the problem of output regulation by *partial state feedback* for a system with relative degree $r > 1$ can be reformulated into a similar problem for a system having relative degree 1 characterized by a “dummy” regulation error given by a linear combination of the first $r - 1$ time derivatives of the error. This would

allow us to recover the design procedure proposed in (Byrnes and Isidori, 2004), relying upon the case of relative degree 1, for the design of a partial state feedback regulator. To this end, suppose that $r > 1$ and consider the change of variables

$$e_r \mapsto \tilde{e} := e_r + \sum_{i=1}^{r-1} g^{r-i} a_{i-1} e_i \quad (7)$$

where g is a *positive* design parameter and a_i , $i = 0, \dots, r - 2$, are such that all roots of the polynomial $\lambda^{r-1} + a_{r-2} \lambda^{r-2} + \dots + a_1 \lambda + a_0 = 0$ have negative real part. This changes system (1) into a system of the form

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(w, \tilde{z}, g) + \tilde{f}_1(w, \tilde{z}, \tilde{e}) \tilde{e} \\ \dot{\tilde{e}} &= \tilde{q}(w, \tilde{z}, \tilde{e}, g) + u \end{aligned} \quad (8)$$

in which $\tilde{z} = \text{col}(z, e_1, \dots, e_{r-1})$

$$\tilde{f}_0(w, \tilde{z}, g) = \begin{pmatrix} f_0(w, z) + f_1(w, z, e_1) e_1 \\ e_2 \\ \vdots \\ e_{r-1} \\ - \sum_{i=1}^{r-1} g^{r-i} a_{i-1} e_i \end{pmatrix}$$

$$\tilde{f}_1(w, \tilde{z}, \tilde{e}) = (0 \ 0 \ \dots \ 0 \ 1)^T$$

and

$$\tilde{q}(w, \tilde{z}, \tilde{e}, g) = q(w, z, e_1, \dots, e_r) - \sum_{i=1}^{r-1} g^{r-i} a_{i-1} \tilde{e}_{i+1}$$

with $\tilde{e}_i = e_i$, $i = 2, \dots, r - 1$, $\tilde{e}_r = \tilde{e} - \sum_{i=1}^{r-1} g^{r-i} a_{i-1} e_i$.

Let the initial conditions of (8) range on a set of the form $Z \times Z_e \times \tilde{E}$, in which $Z_e = \{(e_1, \dots, e_{r-1}) : |e_i| \leq c\}$ and $\tilde{E} = \{\tilde{e} : |\tilde{e}| \leq \tilde{c}\}$ with

$$\tilde{c} \geq (1 + g^{r-1} a_0 + g^{r-2} a_1 + \dots + g a_{r-2}) c$$

(note the dependence on the choice of the a_i 's and of g). Let system (8) be augmented with (2) and consider a regulation problem with fictitious *regulated* output \tilde{e} and *measured* output $\tilde{y} = \tilde{e}$. The system, viewed as a system with input u and output \tilde{e} , has relative degree 1 and its zero dynamics, forced by the control

$$\tilde{u} = -\tilde{q}(w, \tilde{z}, 0, g), \quad (9)$$

is given by

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{\tilde{z}} &= \tilde{f}_0(w, \tilde{z}, g). \end{aligned} \quad (10)$$

In accordance with the notation used in the previous section, in the following we will find conve-

nient to rewrite system (10) in the more compact form

$$\dot{\tilde{\mathbf{z}}} = \tilde{\mathbf{f}}_0(\tilde{\mathbf{z}}) \quad (11)$$

having set $\tilde{\mathbf{z}} = \text{col}(w, \tilde{z})$ and to write $\tilde{\mathbf{q}}_0(\tilde{\mathbf{z}})$ instead of $-\tilde{q}(w, \tilde{z}, 0, g)$ in (9).

Simple arguments can now be used to show that a controller solving the problem of output regulation for the system (8) easily yields a controller solving the problem for the original system (1) by partial state feedback. As a matter of fact, suppose that a controller of the form

$$\begin{aligned} \dot{\zeta} &= \varphi(\zeta, \tilde{e}) \\ u &= \gamma(\zeta, \tilde{e}) \end{aligned} \quad (12)$$

has been found which solves the problem at issue for system (8). Then, it is immediate to realize that the controller

$$\begin{aligned} \dot{\zeta} &= \varphi(\zeta, e_r + \sum_{i=1}^{r-1} g^{r-i} a_{i-1} e_i) \\ u &= \gamma(\zeta, e_r + \sum_{i=1}^{r-1} g^{r-i} a_{i-1} e_i) \end{aligned} \quad (13)$$

solves the problem of output regulation for the original plant (1) by *partial state feedback*. To this end note, first of all, that (13) is an admissible controller for (1), because it is driven only by the components e_1, \dots, e_r of the measured output y of (1). Trivially, the composition of (1) with (13) differs from the composition of (8) with (12) only by a linear change of coordinates, and for any initial state of (1) in $Z \times E$, the corresponding initial state of (8) is in $Z \times Z_e \times \tilde{E}$. Thus all trajectories of (1), controlled by (13), with initial conditions in $Z \times E$ are bounded. The trajectories in question are such that $\lim_{t \rightarrow \infty} \tilde{e}(t) = 0$. But since

$$\begin{aligned} \dot{e}_1 &= e_2 \\ &\vdots \\ \dot{e}_{r-1} &= -\sum_{i=1}^{r-1} g^{r-i} a_{i-1} e_i + \tilde{e} \end{aligned}$$

and the a_i 's are coefficients of a Hurwitz polynomial, it is readily concluded that also $\lim_{t \rightarrow \infty} e_1(t) = 0$. Therefore (13) solves the problem of output regulation for the system (1) by *partial output feedback*.

In the light of these considerations, what is left to show in order to prove the desired claim (namely the fact that there is no loss of generality in addressing the problem for systems having relative degree 1 as far as a partial state feedback solution is sought) is that the zero dynamics (10) and the

associated map (9) inherit, from (4) and (3), the appropriate properties which make the solution of the problem of output regulation possible. Specifically, we will prove that if (4) and (3) satisfy assumptions (i) through (iv) above, then (10) and (9) satisfy an *identical* set of assumptions, *provided that* the parameter g is chosen sufficiently large. This is formalized in the next Lemma.

Lemma 1. Suppose that assumptions (i) through (iv) hold for (4) and (3). Then there exists $g^* > 0$ such that for all fixed $g \geq g^*$ the following hold:

(ii)' there exists a compact subset \tilde{Z} of $W \times \mathbb{R}^n \times \mathbb{R}^{r-1}$ which contains the positive orbit of the set $W \times Z \times Z_e$ under the flow of (11). Moreover, there exists a number $\tilde{d}_0 > 0$ such that

$$\tilde{\mathbf{z}} \in W \times \mathbb{R}^n \times \mathbb{R}^{r-1}, |\tilde{\mathbf{z}}|_{\tilde{\mathcal{A}}} \leq \tilde{d}_0 \Rightarrow \tilde{\mathbf{z}} \in Z \times Z_e$$

in which $\tilde{\mathcal{A}} = \omega(\mathbf{Z} \times Z_e)$.

(iii)' there exist $\tilde{M} \geq 1, \tilde{\lambda} > 0$ such that

$$\tilde{\mathbf{z}}_0 \in W \times \mathbb{R}^n \times \mathbb{R}^{r-1}, |\tilde{\mathbf{z}}_0|_{\tilde{\mathcal{A}}} \leq \tilde{d}_0 \Rightarrow |\tilde{\mathbf{z}}(t)|_{\tilde{\mathcal{A}}} \leq \tilde{M} e^{-\tilde{\lambda}t} |\tilde{\mathbf{z}}_0|_{\tilde{\mathcal{A}}}$$

in which $\tilde{\mathbf{z}}(t)$ denotes the solution of (11) passing through $\tilde{\mathbf{z}}_0$ at time $t = 0$.

(iv)' There exist a C^1 map

$$\begin{aligned} \tilde{\tau} &: \tilde{Z} \rightarrow \mathbb{R}^q \\ \tilde{\mathbf{z}} &\mapsto \tilde{\tau}(\tilde{\mathbf{z}}) \end{aligned}$$

such that

$$\frac{\partial \tilde{\tau}(\tilde{\mathbf{z}})}{\partial \tilde{\mathbf{z}}} \tilde{\mathbf{f}}_0(\tilde{\mathbf{z}}) = \Phi(\tilde{\tau}(\tilde{\mathbf{z}})) \quad \tilde{\mathbf{q}}_0(\tilde{\mathbf{z}}) = \Gamma(\tilde{\tau}(\tilde{\mathbf{z}})) \quad (14)$$

for all $\tilde{\mathbf{z}} \in \tilde{\mathcal{A}}$.

4. THE REGULATOR FOR THE CASE OF RELATIVE DEGREE 1

The problem of designing a regulator of the form (12) for systems of the form (8) enjoying the properties (i), (ii)', (iii)' and (iv)' specified in Lemma 1, has been discussed in (Byrnes and Isidori, 2004). In this section we briefly present the structure of the regulator, referring the interested reader to (Byrnes and Isidori, 2004) for more details.

Let $f_c : \mathbb{R}^q \rightarrow \mathbb{R}$ be a locally Lipschitz function with compact support which agrees on $\tilde{\tau}(\tilde{\mathcal{A}})$ with the function $f(\cdot)$ introduced in assumption (iv), and consider, for (12), the candidate controller

$$\begin{aligned} \dot{\xi} &= \varphi(\xi) + \psi(\xi)v \\ u &= \gamma(\xi) + v \quad v = -k\tilde{e} \end{aligned} \quad (15)$$

with

$$\varphi(\xi) = \begin{pmatrix} \xi_2 \\ \vdots \\ \xi_q \\ f_c(\xi_1, \xi_2, \dots, \xi_q) \end{pmatrix}, \quad \psi(\xi) = \begin{pmatrix} \kappa c_{q-1} \\ \kappa^2 c_{q-2} \\ \dots \\ \kappa^d c_0 \end{pmatrix}$$

and $\gamma(\xi) = \Gamma(\xi)$, and where the c_i 's such that the polynomial $\lambda^q + c_{q-1}\lambda^{q-1} + \dots + c_0 = 0$ is Hurwitz. Then the following result has been proven in (Byrnes and Isidori, 2004).

Proposition 1. Consider system (8), (2) with initial conditions in the compact set $W \times Z \times Z_e \times \tilde{E}$ and where g is fixed so that properties (ii)'-(iv)' of Lemma (1) hold. Consider the regulator (15) with $\varphi(\cdot)$, $\psi(\cdot)$ and $\gamma(\cdot)$ as specified before and initial conditions in a compact set Ξ . There exists a $\kappa^* > 0$ and, for all $\kappa \geq \kappa^*$, there exists a $k^* > 0$ such that, for all $k \geq k^*$, in the closed loop system

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(w, \tilde{z}, g) + \tilde{f}_1(w, \tilde{z}, \tilde{e})\tilde{e} \\ \dot{\tilde{\xi}} &= \varphi(\tilde{\xi}) - \psi(\tilde{\xi})k\tilde{e} \\ \dot{\tilde{e}} &= \tilde{q}(w, \tilde{z}, \tilde{e}, g) + \gamma(\tilde{\xi}) - k\tilde{e} \end{aligned} \quad (16)$$

augmented with (2), the set $W \times Z \times Z_e \times \Xi \times \tilde{E}$ is attracted by the invariant set $\text{graph}(\tilde{\tau}|_{\tilde{\mathcal{A}}}) \times \{0\}$, which is also locally exponentially stable.

5. FROM PARTIAL STATE TO PURE ERROR FEEDBACK

In this section we show how the knowledge of the time derivatives of the error implicitly assumed in the regulator (15) through the term \tilde{e} , can be replaced by suitable estimates.

Following (Khalil and Esfandiari, 1993) the idea is to implement the controller (15) by substituting the term $v = -k\tilde{e}$ with the saturated estimate

$$v = -k\sigma_L(\hat{e}_r + \sum_{i=1}^{r-1} g^{r-i} a_{i-1} \hat{e}_i) \quad (17)$$

in which the estimates \hat{e}_i , $i = 1, \dots, r$, are provided by the so-called ‘‘dirty-derivatives’’ observer

$$\begin{aligned} \dot{\hat{e}}_1 &= \hat{e}_2 + c_0\lambda(e_1 - \hat{e}_1) \\ \dot{\hat{e}}_2 &= \hat{e}_3 + c_1\lambda^2(e_1 - \hat{e}_1) \\ &\vdots \\ \dot{\hat{e}}_r &= c_{r-1}\lambda^r(e_1 - \hat{e}_1) \end{aligned} \quad (18)$$

in which λ is a *positive* design parameter and c_i , $i = 0, \dots, r-1$, are such that all roots of the polynomial $s^r + c_{r-1}s^{r-1} + \dots + c_1s + c_0 = 0$ have negative real part, and $\sigma_L(\cdot)$ is the piecewise linear saturation function defined as

$$\sigma_L(s) = \begin{cases} s & \text{if } |s| \leq L \\ L \text{sgn}(s) & \text{otherwise.} \end{cases}$$

Now let $\hat{e} := \text{col}(\hat{e}_1, \hat{e}_2, \dots, \hat{e}_r)$, $\eta := \text{col}(e_1, e_2, \dots, e_r)$ and consider the change of variables

$$\hat{e} \mapsto x = K_\lambda^{-1}(\eta - \hat{e})$$

where $K_\lambda = \text{diag}(1, \lambda, \dots, \lambda^{r-1})$. Setting $\mathbf{p} = \text{col}(w, \tilde{z}, \xi, \tilde{e})$, simple calculations show that the overall closed loop system (2)–(16)–(18), in the new coordinates, reads as

$$\begin{aligned} \dot{\mathbf{p}} &= f(\mathbf{p}) + \ell(\mathbf{p}, x, \lambda) \\ \dot{x} &= q(\mathbf{p}, \lambda) + r(\mathbf{p}, x, \lambda) + \lambda Ax \end{aligned} \quad (19)$$

in which the system $\dot{\mathbf{p}} = f(\mathbf{p})$ coincides with (2)–(16), $\ell(\mathbf{p}, x, \lambda)$, $q(\mathbf{p}, \lambda)$ and $r(\mathbf{p}, x, \lambda)$ are defined as follows

$$\begin{aligned} \ell(\mathbf{p}, x, \lambda) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \psi(\xi)k(\tilde{e} - \sigma_L(\tilde{e} - H_g K_\lambda x)) \\ k(\tilde{e} - \sigma_L(\tilde{e} - H_g K_\lambda x)) \end{pmatrix} \\ q(\mathbf{p}, \lambda) &= B\lambda^{-r+1}(q(w, z, e_1, \dots, e_{r-1}, \\ &\quad \tilde{e} - \sum_{i=1}^{r-1} g^{r-i} a_{i-1} e_i)) + \gamma(\xi) - k\tilde{e} \\ r(\mathbf{p}, x, \lambda) &= Bk(\tilde{e} - \sigma_L(\tilde{e} - H_g K_\lambda x)), \end{aligned}$$

with H_g such that $\tilde{e} = H_g \eta$, A is a Hurwitz matrix and $B = \text{col}(0, \dots, 0, 1)$.

With this in mind, set $\mathbf{P} = W \times Z \times Z_e \times \Xi \times \tilde{E}$ and let X be a compact set of initial conditions for x . Pick g , κ and k so that, according to the results of Lemma 1 and Proposition 1, in the system $\dot{\mathbf{p}} = f(\mathbf{p})$, the set $(\tilde{\tau}|_{\tilde{\mathcal{A}}}) \times \{0\}$ is locally exponentially stable and attracts \mathbf{P} .

System $\dot{\mathbf{p}} = f(\mathbf{p})$ possesses a (locally Lipschitz) Lyapunov function $V(\mathbf{p})$ which is proper on the domain \mathcal{D} of attraction of $(\tilde{\tau}|_{\tilde{\mathcal{A}}}) \times \{0\}$ (see e.g. (Byrnes et al., 2003) for details). Now, pick a such that $\mathbf{P} \subset V^{-1}([0, a])$ and choose the ‘‘saturation level’’ L so that

$$L \geq \max_{\mathbf{p} \in V^{-1}([0, a+1])} |\tilde{e}|. \quad (20)$$

As a consequence of Assumption (iv) and of the definition of $q(\mathbf{p}, \lambda)$ it is immediately realized that there exists a number α such that for all $\mathbf{p} \in V^{-1}([0, a+1])$ and for all $\lambda > 1$, the following estimate holds

$$|q(\mathbf{p}, \lambda)| \leq \alpha |\mathbf{p}|_{\text{graph}(\tilde{\tau}|_{\tilde{\mathcal{A}}}) \times \{0\}}.$$

Moreover, as a consequence of the definitions of $\ell(\mathbf{p}, x, \lambda)$ and $r(\mathbf{p}, x, \lambda)$ and of the choice (20), it turns out that there exist positive numbers $\bar{\ell}$ and \bar{r} such that for all $\mathbf{p} \in V^{-1}([0, a+1])$, and $\lambda \geq 1$

$$|\ell(\mathbf{p}, x, \lambda)| \leq \bar{\ell}|x| \quad |r(\mathbf{p}, x, \lambda)| \leq \bar{r}|x|.$$

According to this, it is immediately realized that system (19) fits in the framework of Appendix A. In particular system (19) can be identified with system (.1) with the set \mathcal{S} replaced here by $\text{graph}(\tilde{\tau}|_{\tilde{\mathcal{A}}}) \times \{0\}$. Hence, by Proposition 3 in Appendix A, it turns out that a large value of λ renders the trajectories of (19) originating from $\mathbf{P} \times X$ bounded and the consequent ω -limit set $\omega(\mathbf{P} \times X)$ coincides with the set $\text{graph}(\tilde{\tau}|_{\tilde{\mathcal{A}}}) \times \{0\} \times \{0\}$. This, in particular, implies that the problem of output regulation by error feedback is solved. This is formalized in the next final proposition.

Proposition 2. Consider system (2)–(8) with initial conditions in the compact set $W \times Z \times Z_e \times \tilde{E}$ and where g is fixed so that the properties indicated in Lemma 1 hold. Consider the regulator (15) with $\varphi(\cdot)$, $\psi(\cdot)$ and $\gamma(\cdot)$ as specified before, v chosen as in (17), (18), and initial conditions in a compact set Ξ . Pick $\kappa \geq \kappa^*$ and $k \geq k^*$, with κ^* and k^* such that the properties in Proposition 1 hold. Pick a compact set of initial conditions X and let L be defined as indicated above. Then there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$ the proposed regulator solves the output regulation problem.

6. CONCLUSIONS

In this paper we have addressed the design of an internal-model based regulator by error feedback for nonlinear systems. The specific aim of the paper was to present a number of auxiliary results which complement the work in (Byrnes and Isidori, 2003) and (Byrnes and Isidori, 2004) by allowing for regulated systems with arbitrary relative degree. All the analysis has been conducted in the “non-equilibrium” framework proposed in (Byrnes and Isidori, 20043).

APPENDIX A

System (2)–(16)–(18) are special cases of a system of the form

$$\begin{aligned} \dot{\mathbf{p}} &= f(\mathbf{p}) + \ell(\mathbf{p}, x, \lambda) \\ \dot{x} &= q(\mathbf{p}, \lambda) + r(\mathbf{p}, x, \lambda) + \lambda Ax. \end{aligned} \quad (.1)$$

with initial conditions in a compact set $\mathbf{P} \times X$. Moreover, the “subsystem”

$$\dot{\mathbf{p}} = f(\mathbf{p}) \quad (.2)$$

has the following property:

Property P: there is a compact invariant set \mathcal{S} , contained in the interior of \mathbf{P} , which is locally exponentially stable and uniformly attracts \mathbf{P} .

As a consequence, as shown e.g. in (Byrnes et al., 2003), there exists a (locally Lipschitz) Lyapunov

function $V(\mathbf{p})$ which is proper on the domain \mathcal{D} of attraction of \mathcal{S} . Let a be such that $\mathbf{P} \subset V^{-1}([0, a])$.

Proposition 3. Consider system (.1) and assume that the associated subsystem (.2) has the property P indicated above. Assume, in addition, that there exist positive numbers $\bar{\ell}$, \bar{r} and α such that

$$\begin{aligned} |\ell(\mathbf{p}, x, \lambda)| &\leq \bar{\ell}|x| & |r(\mathbf{p}, x, \lambda)| &\leq \bar{r}|x| \\ |q(\mathbf{p}, \lambda)| &\leq \alpha |\mathbf{p}|_{\mathcal{A}} \end{aligned}$$

for all $\mathbf{p} \in V^{-1}([0, a + 1])$, all x and all $\lambda \geq 1$. Then, there exists $\lambda^* > 0$ such that, for all $\lambda \geq \lambda^*$, the positive orbit of $\mathbf{P} \times X$ under the flow of (.1) is bounded and $\omega(\mathbf{P} \times X)$, the ω -limit set of the set $\mathbf{P} \times X$ under the flow of (.1), coincides with $\mathcal{S} \times \{0\}$. Furthermore $\omega(\mathbf{P} \times X)$ is locally exponentially stable.

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