# ALMOST SURE STABILITY OF CONTINUOUS-TIME MARKOV JUMP LINEAR SYSTEMS: A RANDOMIZED APPROACH

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Abstract: In this paper, we study the almost sure stability of continuous-time jump linear systems with a finite-state Markov form process. A sufficient condition for almost sure stability is derived that refers to the statistics of the transition matrix over $m$ switches. It is shown that, if the system is exponentially almost sure stable, there exists a finite $m$ such that the criterion is satisfied. In order to evaluate the expected value appearing in the condition, an efficient Monte Carlo algorithm is worked out. 

Keywords: Markov Jump Linear Systems, Stochastic jump processes, Hybrid systems, Stability, Monte Carlo method

## 1. INTRODUCTION

Markov Jump Linear Systems (MJLS) are a class of stochastic hybrid systems in which the switches between different linear systems are governed by a finite-state Markov chain. They are well suited to represent dynamic systems subject to random switches between alternative configurations. As such, they are used to model technological and economic systems, including failure-prone plants and communication lines subject to random delays, (Chizeck *et al.*, 1986; Mariton, 1990; Krtolica *et al.*, 1991; Fang *et al.*, 1991; Gomez-Puig and Montalvo, 1997).

The stability theory of MJLS is rather complex in that there exist several stability notions that differ in conservativeness as well as ease of testability. The most important stability notions are Mean-Square (MS) stability, moment stability, and Almost Sure (AS) stability. MS-stability has to do with the asymptotic convergence to zero of the second moment of the state norm. There exist necessary and sufficient conditions involving either the solution of coupled Lyapunov equations or the location in the complex plane of the eigenvalues of suitable augmented matrices, (Feng *et al.*, 1992; Fang *et al.*, 1994; Fang and Loparo, 2002*b*).

Moment stability, also called $\delta$-moment stability, requires the convergence to zero of the moment of order $\delta$ (MS-stability is just a particular case for $\delta = 2$). Although there exist some practical sufficient conditions, a simple necessary and sufficient condition for testing $\delta$-moment stability (except for $\delta = 2$) is not available.

Finally, AS-stability holds if the sample path of the state converges to zero with probability one. Checking AS-stability involves the determination of the sign of the top Lyapunov exponent, which is usually a rather difficult task (Arnold *et al.*, 1986; Fang and Loparo, 2002*a*). In practice, one may exploit the fact that $\delta$-moment stability implies AS-stability and that, for $\delta$ tending to zero, AS and $\delta$-moment stability become equivalent, (Feng *et al.*, 1992; Fang *et al.*, 1994). However, in view of the lack of simple necessary and sufficient conditions for $\delta$-moment stability, the problem of assessing AS-stability in the least conservative way is still open. In practical applications the most useful notion would be AS-stability because it guarantees the convergence of almost all realizations of the sample path. Conversely, $\delta$-moment stability may be too conservative. Sufficient conditions for AS-stability that do not rely on $\delta$-moment stability are reported in (Costa and Fragoso, 1995), (Fang, 1997).

With reference to discrete-time MJLS, the authors of the present paper have recently derived a family of criteria for testing AS-stability whose conservativeness can be made arbitrarily small at the expense of computational complexity, (Bolzern *et al.*, 2004). A sufficient condition for AS-stability is applied to a lifted representation over $m$ steps of the original MJLS. It has been proven that if the system is AS-stable, a finite $m$ exists such that the criterion is fulfilled. As the lifting horizon $m$ grows, it becomes necessary to resort to a Monte Carlo strategy in order to calculate the expected value appearing in the stability criterion.

The extension of these recent results to the continuous-time case is all but straightforward. In fact, a lifted representation over a given time-interval would yield an infinite-state Markov chain. For this reason, a different approach is pursued in the present paper. A condition for testing the contractivity of the MJLS after $m$ switches is applied. Such a contractivity depends on the expected value of the logarithm of the norm of the transition matrix, an expectation that can be calculated only by means of a Monte Carlo strategy. The Monte Carlo algorithm requires only the stochastic simulation of $m$ switches of the continuous-time Markov chain and is stopped when a prescribed confidence on the fulfilment of the criterion has been reached. It is shown that, if the MJLS is AS-stable, there exists a finite value of $m$ such that the contractivity condition is satisfied. In other words, the sufficient condition asymptotically approaches necessity.

The paper is organized as follows. In Section 2, some basic definitions are recalled. The new AS-stability condition is worked out in Section 3, while Section 4 deals with the randomized algorithm. An illustrative example is discussed in Section 5. The paper ends with some concluding remarks (Section 6).

## 2. CONTINUOUS-TIME MJLS - PRELIMINARIES

Consider the continuous-time Markov jump linear system (MJLS)

$$\dot{x}(t) = A(\sigma(t))x(t) \qquad (1)$$

where $x(t) \in I\!R^n$, and the form process $\sigma(t)$ is a finite-state, time homogeneous, Markov stochastic process taking values in a finite set $S = \{1, 2, \ldots, M\}$, with stationary transition probabilities $Pr\{\sigma(t+h) = j|\sigma(t) = i\} = q_{ij}h+o(h), i \neq j$, where $h > 0$, and $q_{ij}$ is the transition rate from mode $i$ at time $t$ to mode $j$ at time $t + h$. Letting

$$q_{ii} = -\sum_{j=1, j\neq i}^{M} q_{ij}$$

and defining $Q = [q_{ij}]$, the matrix $Q$ is called the infinitesimal generator of the Markov process. Let $\tau_k, k = 0, 1, \ldots$ be the successive sojourn times between jumps. Then, assuming that after the $k$-th jump the system stays in mode $i$, $\tau_k$ is exponentially distributed, with parameter $\lambda = -q_{ii}$. Given an initial probability distribution

$$\pi_0 = [\, \pi_{01}\ \pi_{02}\ \ldots\ \pi_{0M}\, ]'$$

$\pi_{0i} := Pr\{\sigma(0) = i\}$, the probability distribution $\pi(t)$ obeys the differential equation $\dot{\pi}(t) = Q'\pi(t)$. Under the assumption that the Markov process is irreducible (see e.g. (Bremaud, 1998)), there exists a unique invariant distribution $\bar{\pi} = [\, \bar{\pi}_1\ \bar{\pi}_2\ \ldots\ \bar{\pi}_M\, ]'$ to whom $\pi(t)$ converges for any $\pi(0)$. Such a distribution provides the steady-state probability distribution for the (ergodic) form process $\sigma(t)$. Hereafter, it is assumed that the Markov process is irreducible.

In the following, $\Phi(t, \tau)$ will denote the state transition matrix over the interval $[\tau, t]$. Note that

$\Phi(\tau, t)$ depends on the time history of the form process $\sigma(t)$, and, as such, is a random matrix.

*Definition 2.1.* The MJLS (1) is said to be (exponentially) almost surely stable (AS-stable) if there exists $\rho > 0$ such that, for any $x(0) \in \mathbb{R}^n$ and any initial distribution $\pi(0)$, it results that

$$Pr\{\limsup_{t\to\infty} \frac{1}{t} ln\|x(t)\| \leq -\rho\} = 1$$

□

It is immediate to see that Definition 2.1 implies that $Pr\{\lim_{t\to\infty} \|x(t)\| = 0\} = 1$, i.e. almost all realizations of the stochastic process $x(t)$ converge to zero. Moreover, it is well known (Fang, 1994) that the system is exponentially AS stable if and only if $\bar{\lambda} = E[\lambda] < 0$, where $\lambda$ is the *top Lyapunov exponent*, i.e.

$$\lambda = \limsup_{t\to\infty} \frac{1}{t} ln\|\Phi(t, 0)\|$$

It is worth pointing out that the above limit converges almost surely to $\bar{\lambda}$, irrespective of the initial probability distribution $\pi(0)$.

The analysis of almost-sure stability of system (1) can be addressed by using the concept of matrix measure (see e.g. (Desoer and Vidyasagar, 1975)). The matrix measure $\mu(A)$ of a square matrix $A$ is defined as

$$\mu(A) = \lim_{T\to 0} \frac{\|I + AT\| - 1}{T} \tag{2}$$

where $I$ is the identity matrix. A simple interpretation is that the measure $\mu(A)$ is the derivative of the norm of $\exp(At)$ at $t = 0$. It is easy to see that, if $\mu(A) < 0$, then $A$ is "instantaneously norm-contractive", and, consequently, Hurwitz stable. If the matrix norm used in (2) is the usual spectral norm, $\mu(A)$ coincides with $\lambda_{max}(A + A')/2$, where $\lambda_{max}$ denotes the maximum eigenvalue (see e.g. (Desoer and Vidyasagar, 1975)).

In (Fang and Loparo, 2002*b*), the following sufficient condition for AS-stability can be found.

*Proposition 1.* If $\sum_{i=1}^{M} \bar{\pi}_i \mu(A_i) < 0$, then the MJLS (1) is AS-stable. □

Such a condition is tantamount to requiring that $E_{\bar{\pi}}[\mu(A(\sigma))] < 0$, namely that the system is "instantaneously norm-contractive" on the average. As is well known, the condition of Proposition 1 is not in general necessary, with the exception of one-dimensional systems.

## 3. MAIN RESULT

In this section, we will work out a new sufficient condition for AS-stability that is less restrictive than Proposition 1. The key idea is to consider the evolution of the state $x(t)$ over an interval of time corresponding to $m$ transitions and to impose that the system is average norm-contractive over such an interval. To this aim, define $T_m = \sum_{k=0}^{m-1} \tau_k$, $T_0 = 0$ as the (random) time to the $m$-th transition, and recall that $\Phi(T_m, 0)$ is the (random) state transition matrix over the interval $[0, T_m]$.

Given the continuous-time Markov process $\sigma(t)$, the associated discrete-time Markov chain $\sigma_k = \sigma(T_k)$ is called embedded Markov chain (Bremaud, 1998). This discrete-time Markov chain describes the time evolution of the form process at the commutation instants, and it is characterized by the transition probabilities

$$p_{ij} = Pr\{\sigma_{k+1} = j | \sigma_k = i\} = \begin{cases} -\dfrac{q_{ij}}{q_{ii}} \;, & i \neq j \\ 0 \;, & i = j \end{cases}$$

The invariant probability distribution of the embedded Markov chain is denoted by $\bar{\vartheta}$, and its entries are

$$\bar{\vartheta}_i = \frac{\bar{\pi}_i q_{ii}}{\sum_{j=1}^{M} \bar{\pi}_j q_{jj}}$$

Now, we are in a position to prove the new sufficient condition for AS-stability.

*Proposition 2.* System (1) is AS-stable if, for some integer $m > 0$,

$$E_{\bar{\vartheta}}\left[ln\|\Phi(T_m, 0)\|\right] < 0 \tag{3}$$

*Proof.* In the proof, exponential AS stability will be established by showing that the top Lyapunov exponent $\bar{\lambda}$ is negative. To this end, we first recall a technical result presented in (Fang, 1994). Precisely, in view of ergodicity,

$$\bar{\lambda} = \frac{1}{\bar{\tau}} \lim_{k\to\infty} \frac{1}{k} E_{\bar{\vartheta}}[ln\|\Phi(T_k, 0)\|] \tag{4}$$

where

$$\bar{\tau} = E[\tau] > 0, \quad \tau = \lim_{k\to\infty} \frac{T_k}{k} \quad a.s.$$

Now, let $k = Nm + h$, $0 \leq h \leq m - 1$ and observe that

$$\frac{1}{k} E_{\bar{\vartheta}}[ln\|\Phi(T_k, 0)\|] = \frac{1}{Nm + h} E_{\bar{\vartheta}}[ln\|\Phi(T_k, 0)\|]$$
$$\leq \frac{1}{Nm + h} E_{\bar{\vartheta}}\left[ln\|\Phi(T_h, 0)\|\right]$$
$$+ \frac{1}{Nm + h} E_{\bar{\vartheta}}\left[\sum_{j=1}^{N} ln\|\Phi(T_{jm+h}, T_{(j-1)m+h})\|\right]$$

$$= \frac{1}{Nm+h} E_{\bar{\vartheta}}[ln\|\Phi(T_h, 0)] + \frac{N\bar{\alpha}}{Nm+h}$$

where

$$\bar{\alpha} = E_{\bar{\vartheta}}[ln\|\Phi(T_m, 0)\|] < 0 \qquad (5)$$

Then, taking the limit as $N \to \infty$, it is immediate to see that

$$\bar{\lambda} \leq \frac{\bar{\alpha}}{m\bar{\tau}} < 0 \qquad (6)$$

□

By increasing $m$, a sequence of sufficient conditions is obtained. The following result shows that, if the system is AS-stable, it suffices to check only a finite (yet unknown) number of such conditions.

*Proposition 3.* System (1) is exponentially AS-stable if and only if there exists a finite value of $m$ such that condition (3) holds.

*Proof.* The sufficiency part coincides with Proposition 2. As for necessity, assume that system (1) is AS-stable. Then $\bar{\lambda} < 0$, and, in view of (4), there exists a finite $\bar{k}$ such that, for any $m \geq \bar{k}$, condition (3) holds.

## 4. RANDOMIZED ALGORITHM FOR ASSESSMENT OF ALMOST SURE STABILITY

When applying condition (3), one is faced with the problem of calculating the expected value. An exact formula is not available, so that a Monte Carlo strategy is proposed.

First of all, the user has to select a confidence level $\delta$ (e.g. $\delta = 0.01$). The quantity $1 - \delta$ is the degree of belief that has to be reached before a decision is taken that condition (3) is fulfilled or not. Then, for a given $m \geq 1$, the decision algorithm is as follows.

- **Step 0:** let $j = 1$, $\bar{y}_0 = \gamma_0^2 = 0$, and let $N_1$ be a large integer;
- **Step 1:** simulate $m$ steps of the joint process $\{\sigma_k, \tau_k\}$, using the invariant distribution $\bar{\vartheta}$ as the initial distribution for $\sigma_0$; in this way, the sequence $\{\sigma_0, \tau_0\} \ldots \{\sigma_{m-1}, \tau_{m-1}\}$ is generated;
- **Step 2:** compute $y_j = ln\|\Phi(T_m, 0)\|$, where $\Phi(T_m, 0)$ is the state transition matrix;
- **Step 3:** compute

$$\bar{y}_j = \frac{1}{j}\left((j-1)\,\bar{y}_{j-1} + y_j\right)$$

$$\gamma_j^2 = \frac{1}{j}\left((j-1)\,\gamma_{j-1}^2 + y_j^2\right)$$

$$s_j = \sqrt{\gamma_j^2 - \bar{y}_j^2}$$

- **Step 4:** if $j \geq N_1$ then compute

$$z_j = -\frac{\bar{y}_j}{s_j}\sqrt{j}$$

  Letting $F(z)$ be the standard normal probability distribution function,
  – if $F(z_j) > 1 - \delta$, then decide that condition (3) is fulfilled; end
  – if $F(z_j) < \delta$, then decide that condition (3) is not fulfilled; end
- **Step 5:** let $j = j + 1$ and go to Step 1.

*Remark 4.* The simulation of the joint process in Step 1 can be performed as follows. At the $k$-th iteration, let $i = \sigma_k$ denote the current logical state. First compute the sojourn time $\tau_k$ by drawing from the exponential distribution with parameter $|q_{ii}|$. Then, the destination state $\sigma_{k+1}$ is drawn from the embedded discrete distribution $p_{ij}$. This procedure is legitimate because the sojourn time and the destination state can be proven to be statistically independent.

*Remark 5.* The decision strategy underlying the algorithm relies on the Bayesian paradigm. More precisely, a decision is taken when the posterior probability that (3) holds becomes either greater than $1 - \delta$ or less than $\delta$. To this purpose, we exploit the fact that, for large samples, the posterior distribution of the expected value $E_{\bar{\pi}}\left[ln\|\Phi(T_m, 0)\|\right]$ becomes normal with expectation equal to the sample mean $\bar{y}_j$ and variance $s_j^2/j$. To be sure that the sample size is large enough for the normal approximation to hold, we introduced a minimum sample size $N_1$, that has to be reached before any decision is taken.

In the algorithm, it is assumed that the prior distribution of the samples $y_j$ is normal with zero mean and infinite variance so that the posterior depends only on the extracted samples. In absence of a-priori information, it is reasonable to assume a prior with large (or infinite) variance, so that the posterior depends solely on the data $y_j$. In practice, taking $N_1$ equal to some hundreds may suffice.

*Remark 6.* It should be noted that the decision algorithm terminates with probability one, provided that $E_{\bar{\pi}}\left[ln\|\Phi(T_m, 0)\|\right] \neq 0$. Indeed, when such expectation is nonzero, the absolute value of the random variable $z_j$ tends to infinity with probability one.

*Remark 7.* When the algorithm terminates with the decision that the condition (3) is not fulfilled, this by no way means that the system is not AS-stable. Rather, it is worth running the algorithm again with a larger value of $m$. In doing this, one may exploit the data generated with the previous value of $m$, extending the simulations of Step 1 by adding one more transition of the Markov chain.

*Remark 8.* The randomized algorithm hinges on the estimation of $\bar{\alpha}$, defined in (5). In view of (6), testing the negativity of $\bar{\alpha}$ is equivalent to testing the negativity of an upper bound of the top Lyapunov exponent. One may want to compare the proposed algorithm with the direct estimation of the top Lyapunov exponent from a single realization of the stochastic process, by exploiting the asymptotic convergence property (4). The main difficulty is to provide a confidence level of the estimate. Conversely, the generation of independent samples $y_j$ carried out within the randomized algorithm opens the way to the assessment of confidence levels by invoking the central limit theorem. In conclusion, it appears that many short independent realizations are more effective than a single long realization.

## 5. EXAMPLE

As an example, consider the MJLS (1) with $M = 2$, and

$$A(1) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A(2) = \begin{bmatrix} -15 & 1 \\ 0 & -2 \end{bmatrix},$$
$$Q = \begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix}$$

The unique invariant distribution of the continuous-time form process is $\bar{\pi} = \begin{bmatrix} 5/6 \ 1/6 \end{bmatrix}'$, and the invariant distribution of the embedded Markov chain turns out to be $\bar{\vartheta} = \begin{bmatrix} 1/2 \ 1/2 \end{bmatrix}'$. First of all, observe that this system is not MS-stable. In fact, the matrix $A(1) + 0.5q_{11}I$ is not stable, so that the necessary condition for MS-stability given in (Fang, 1994) is violated. Nevertheless, it could be AS-stable.

Since $\mu(A(1)) = 2$, and $\mu(A(2)) = -1.9808$, the sufficient condition for AS-stability given in Proposition 1 is not fulfilled (it results that $\bar{\pi}_1\mu(A(1)) + \bar{\pi}_2\mu(A(2)) = 1.3365 > 0$). Hence, we apply the randomized algorithm of Section 4 for values of $m$ ranging from 1 to 10. For each $m$, the algorithm was run 100 times, letting the confidence level be $\delta = 0.01$, and the minimum sample size $N_1 = 100$. The algorithm was stopped whenever the number $j$ of iterations exceeded 10000. Recalling the rationale of the computational scheme, this happens when the prescribed confidence cannot be reached, so that the algorithm keeps on extracting new samples. These cases have been classified as ND ("not-decided"). When the algorithm decides that the AS stability condition (3) is fulfilled, the outcome is classified as ASD ("almost sure decision"). Conversely, if it is decided that the AS stability condition (3) is not fulfilled, the outcome is labelled as NASD (not ASD).

The results are summarized in Table 1. For each $m$ the table reports the percentage of ASD, NASD and ND outcomes. By averaging all the $\bar{y}_j$ generated within the 100 runs, a "grand-average" $\bar{\bar{y}}$ was computed. In turn, this was used to evaluate $\bar{\bar{y}}/(m\bar{\tau})$, which estimates the upper bound (6) for the top Lyapunov exponent $\bar{\lambda}$. It can be seen (last column of Table 1) that for the selected values of $m$ such a bound is still far from converging. Indeed, by running simulations with much larger values of $m$, it was found that the top Lyapunov exponent is $\bar{\lambda} \simeq -0.8$. It is remarkable that a decision on AS stability can be obtained even if the top Lyapunov exponent is not estimated with great accuracy.

In view of the results of Table 1, it appears that the system is actually AS stable. Figure 1 shows a histogram of the values of $\bar{y}$ calculated at the end of each experiment for $m = 6$. Note that these values are all negative. The number of realizations over which $\bar{y}$ is computed in each run is not fixed, as the algorithm terminates when the prescribed confidence level is reached. This explains the nongaussian shape of the histogram in Figure 1. The lack of values in the neighborhood of zero is due to the fact that the algorithm is reluctant to take a decision when $\bar{y}$ is close to zero, and keeps on extracting new random realizations until a clear cut situation is reached.

Table 1. Experimental results of 100 runs of the Monte Carlo algorithm for different values of $m$

| $m$ | % ASD | % NASD | % ND | bound on $\bar{\lambda}$ |
|---|---|---|---|---|
| 1 | 0.00 | 100.00 | 0.00 | 1.3235 |
| 2 | 0.00 | 100.00 | 0.00 | 0.2784 |
| 3 | 0.00 | 100.00 | 0.00 | 0.2445 |
| 4 | 6.00 | 78.00 | 16.00 | 0.0330 |
| 5 | 14.00 | 13.00 | 73.00 | 0.0008 |
| 6 | 100.00 | 0.00 | 0.00 | -0.0958 |
| 7 | 100.00 | 0.00 | 0.00 | -0.1122 |
| 8 | 100.00 | 0.00 | 0.00 | -0.1843 |
| 9 | 100.00 | 0.00 | 0.00 | -0.1976 |
| 10 | 100.00 | 0.00 | 0.00 | -0.2274 |

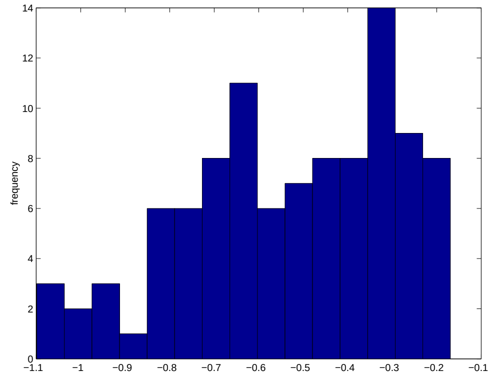


Fig. 1. Histogram of $\bar{y}$ in the Monte Carlo algorithm for $m = 6$

## 6. CONCLUDING REMARKS

A sufficient condition for almost sure stability of continuous-time MJLS has been studied. Roughly speaking, the condition relies on the average contractivity of the system over $m$ transitions. It is shown that, if the system is AS-stable, there exists a finite $m$ such that the condition is fulfilled. Therefore, in some sense, the sufficient condition approaches necessity as $m$ grows. The computation is carried out by means of a Monte Carlo algorithm guaranteeing a prescribed confidence level. Only the evolution of the logical state is required in the Monte Carlo simulation. As a by-product, the algorithm provides also an upper bound of the top Lyapunov exponent.

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