LMI-BASED SYNTHESIS OF ANTI-WINDUP FILTERS WITH POLE CONSTRAINTS [1]

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Abstract: This paper presents a design method for a linear anti-windup filter with filter pole constraints. The anti-windup design problem can be cast as an LMI optimization problem. The anti-windup filter is optimized for $\mathcal{L}_2$ performance while its poles are constrained in a predefined region. The constrained poles ensure the fast recovery of saturation free linear responses and practical implementation of the anti-windup filter in digital form. This design method guarantees global stability for asymptotically stable plants, and local stability for plants, which are not asymptotically stable. The effectiveness of the design method is demonstrated by a simulation example.

Keywords: Saturation control, Stability analysis, Control system synthesis, Constrained poles, Convex optimization.

1. INTRODUCTION

It is well known that a linear feedback closed-loop system may suffer performance degradation or even instability in the presence of input saturation. Therefore, it is necessary to consider saturation in control design. In linear control design, one can design a controller by penalizing the control effort so that the saturation limit is never reached for any given reference command. However, this usually leads to a conservative design. For example, higher controller gains may be possible for small reference commands without causing saturation.

An alternative approach to dealing with saturation is to first design a feedback controller without considering the presence of saturation. Then, an additional anti-windup filter is designed such that 1) the linear closed-loop system is unmodified in the absence of saturation, and 2) the closed-loop system is stable with small performance degradation during saturation.

This paper starts with a linear anti-windup scheme (Teel and Kapoor, 1997; Weston and Postlethwaite, 1998). This scheme satisfies the first goal of the anti-windup design. As for the second goal, the anti-windup filter is designed to achieve certain $\mathcal{L}_2$ performance as suggested by (Turner *et al.*, 2003; Turner and Postlethwaite, 2004). Furthermore, it will be shown that the poles of the anti-windup filter determine the speed of the unconstrained response recovery. Therefore, the anti-windup filter is also designed such that the poles are in a desired predefined region.

The remainder of this paper is organized as follows. Section 2 provides the structure of the anti-windup filter and the problem statement of the anti-windup synthesis. The solution to the anti-windup problem is presented in Section 3. The anti-windup problem with pole constraints is

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solved in Section 4. Section 5 gives an example to illustrate the design method. Finally, conclusions are given in Section 6.

2. PROBLEM STATEMENT

Consider a linear time-invariant plant $P(s)$ in state-space form

$$\begin{aligned} \dot{x}_p &= A_p\, x_p + B_p\, u_s + B_d\, d \\ y &= C_p\, x_p + D_p\, u_s + D_d\, d \end{aligned} \tag{1}$$

where $x_p \in \mathbf{R}^{n_p}$ is the state, $y \in \mathbf{R}^{n_y}$ is the output, $u_s \in \mathbf{R}^m$ is the control input, and $d \in \mathbf{R}^{n_d}$ is the disturbance input. Let a controller $C(s)$ be represented by

$$\begin{aligned} \dot{x}_c &= A_c\, x_c + B_c\, y_\ell + B_r\, r \\ u_\ell &= C_c\, x_c + D_c\, y_\ell + D_r\, r \end{aligned} \tag{2}$$

where $x_c \in \mathbf{R}^{n_c}$ is the controller state, $y_\ell \in \mathbf{R}^{n_y}$ is the controller input, $u_\ell \in \mathbf{R}^m$ is the controller output, and $r \in \mathbf{R}^{n_y}$ is the reference input.

Without considering the presence of saturation, let $u_s = u_\ell$ and $y_\ell = y$, then the unconstrained linear closed-loop system can be obtained as (let $\eta = [x_p^T\; x_c^T]^T$)

$$\begin{aligned} \dot{\eta} &= A_{cl}\, \eta + B_{rcl}\, r + B_{dcl}\, d \\ y_\ell &= C_{ycl}\eta + D_{yrcl}\, r + D_{ydcl}\, d \\ u_\ell &= C_{ucl}\eta + D_{urcl}\, r + D_{udcl}\, d \end{aligned} \tag{3}$$

where the details of the matrices in (3) are given in Appendix A. It is assumed that $C(s)$ has been designed to guarantee the well-posedness, internal stability, and desirable performance of the unsaturated linear closed-loop system.

Fig. 1 shows an anti-windup scheme (Teel and Kapoor, 1997; Weston and Postlethwaite, 1998). The function $sat(\cdot)$ represents a decentralized saturation function

$$sat(u) := [sat_1(u_1), \ldots, sat_m(u_m)]^T \tag{4}$$

where $sat_i(u_i) = u_i / \max\{1, \frac{|u_i|}{u_{i,\max}}\}$, and $u_{i,\max}$ is the saturation limit for the ith control input. The structure of the anti-windup filter $F(s)$ is expressed as (5).

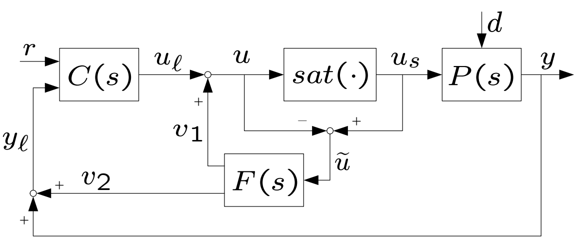

Fig. 1. Anti-windup scheme

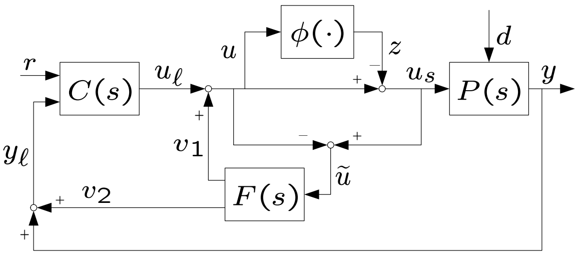

Fig. 2. Extraction of the deadzone function.

$$\begin{aligned} \dot{x}_a &= A_f\, x_a + B_p\, \widetilde{u} \\ v_1 &= K\, x_a \\ v_2 &= -(C_p + D_p K)x_a - D_p\, \widetilde{u} \end{aligned} \tag{5}$$

where $A_f = A_p + B_p K$ and $\widetilde{u} = u_s - u$. In the absence of saturation, i.e. $\widetilde{u} \equiv 0$, $v_1 \equiv 0$ and $v_2 \equiv 0$, if the initial condition $x_a(0) = 0$. Then, the closed-loop system in Fig. 1 behaves as the unconstrained linear closed-loop system (3).

In the presence of saturation, the nonlinear closed-loop system in Fig. 1 can be analyzed by replacing $sat(\cdot)$ with $I - \phi(\cdot)$ as shown in Fig. 2, where $\phi(u) := u - sat(u)$ is a deadzone function. The linear part of the closed-loop system from the inputs z, r, and d to the signals u_ℓ, y_ℓ, and u can be found as

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} A_{cl} & 0 \\ 0 & A_f \end{bmatrix} \xi + \begin{bmatrix} 0 & B_{rcl} & B_{dcl} \\ -B_p & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ r \\ d \end{bmatrix} \\ y_\ell &= \begin{bmatrix} C_{ycl} & 0 \end{bmatrix} \xi + D_{yrcl}\, r + D_{ydcl}\, d \\ u_\ell &= \begin{bmatrix} C_{ucl} & 0 \end{bmatrix} \xi + D_{urcl}\, r + D_{udcl}\, d \\ u &= \begin{bmatrix} C_{ucl} & K \end{bmatrix} \xi + D_{urcl}\, r + D_{udcl}\, d \end{aligned} \tag{6}$$

with the coordinate $\xi = [(x_p - x_a)^T\; x_c^T\; x_a^T]^T$. It can be observed from (6) that u_ℓ and y_ℓ are not affected by z and behave as the signals u_ℓ and y_ℓ in the unconstrained linear closed-loop system (3) as long as the initial conditions $x_p(0)$ and $x_c(0)$ are the same as the ones in (3) and $x_a(0) = 0$. Notice that the signal u can be expressed as

$$u = u_\ell + K(sI - A_f)^{-1}(-B_p)z \tag{7}$$

Notice also that $\widetilde{u} = -(u - u_s) = -z$. It follows that the internal stability analysis of Fig. 2 is equivalent to that of Fig. 3, since the unconstrained linear closed-loop system is assumed to

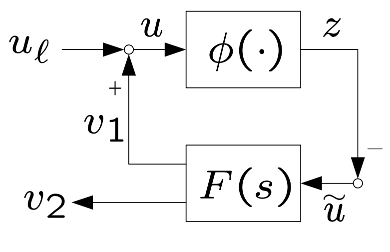

Fig. 3. Stability analysis

be stable. The output of the nonlinear system in Fig. 3 is v_2. Notice that $v_2 = y_\ell - y$, which represents the difference between the unconstrained linear output y_ℓ and the saturated output y. Therefore, Fig. 3 shows how the unconstrained control effort u_ℓ induces the difference v_2. It is desirable to have v_2 small in some sense, that is, to have saturated output y close to the unsaturated linear output y_ℓ. As suggested in (Turner *et al.*, 2003; Turner and Postlethwaite, 2004), the performance of the anti-windup filter can be evaluated by the $\mathcal{L}_2$ gain of the operator from u_ℓ to v_2. The anti-windup synthesis problem studied in this paper is to design the filter $F(s)$ (or the gain K) so that the nonlinear system in Fig. 3 is globally (or locally) stable with a finite $\mathcal{L}_2$ gain of the operator from u_ℓ to v_2.

3. ANTI-WINDUP SYNTHESIS

The anti-windup synthesis problem is solved by utilizing the circle criterion. First, consider a decentralized nonlinear function $\psi(u)$. If the nonlinearity $\psi(\cdot)$ belongs to a sector $[0, \Lambda]$, where $\Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_m\}$ is a diagonal matrix, and $\lambda_i > 0$, then for any $W = \operatorname{diag}\{w_1, \ldots, w_m\} \succ 0$ and $u \in \mathbf{R}^m$,

$$z^T W(\Lambda u - z) \geq 0 \tag{8}$$

with $z = \psi(u)$. Notice that the deadzone function $\phi(\cdot)$ belongs to the sector $[0, I]$.

Given a sector $[0, \Lambda]$, where $0 < \lambda_i \leq 1$, the following theorem provides a tool to find the gain K such that the interconnection of $\psi(\cdot)$ and $F(s)$ as shown in Fig. 3 with $\phi(\cdot)$ replaced with $\psi(\cdot)$ is globally stable with a finite $\mathcal{L}_2$ gain for all $\psi(\cdot)$'s belonging to the sector $[0, \Lambda]$. Notice that, when $\Lambda = I$, the theorem is the continuous time version of the result in (Turner *et al.*, 2003).

Theorem 1. Given a sector $[0, \Lambda]$, where $0 < \lambda_i \leq 1$, there exists a gain K such that the interconnection of $\psi(\cdot)$ and $F(s)$ is globally stable with a finite $\mathcal{L}_2$ gain for all $\psi(\cdot)$'s belonging to the sector $[0, \Lambda]$, if there exist $Q = Q^T \in \mathbf{R}^{n_p \times n_p} \succ 0$, $\Sigma = \operatorname{diag}\{\sigma_1, \ldots, \sigma_m\} \succ 0$, $M \in \mathbf{R}^{m \times n_p}$, and a scalar $\gamma > 0$ such that

$$LMI\,(Q, \Sigma, M, \gamma) \prec 0 \tag{9}$$

(the details of $LMI\,(Q, \Sigma, M, \gamma)$ are given on the top of the next page), then $K = MQ^{-1}$.

Proof: Let $V = x_a^T P x_a$ be a Lyapunov function, where $P = P^T \in \mathbf{R}^{n_p \times n_p} \succ 0$. Let W be any diagonal matrix $\operatorname{diag}\{w_1, \ldots, w_m\} \succ 0$. If the matrix R in (10) is negative definite, it follows that $\dot{V} + |v_2|^2 - \gamma |u_\ell|^2 < 0$ since $z^T W[\Lambda u - z] \geq 0$.

$$\begin{aligned}
&\dot{V} + 2z^T W[\Lambda u - z] + |v_2|^2 - \gamma |u_\ell|^2 \\
&= \begin{bmatrix} x_a \\ z \\ u_\ell \end{bmatrix}^T \underbrace{\begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{12}^T & R_{22} & W\Lambda \\ 0 & \Lambda W & -\gamma I \end{bmatrix}}_{R} \begin{bmatrix} x_a \\ z \\ u_\ell \end{bmatrix}
\end{aligned} \tag{10}$$

$$\begin{aligned}
R_{11} =& (A_p + B_p K)^T P + P(A_p + B_p K) \\
&+ (C_p + D_p K)^T (C_p + D_p K) \\
R_{12} =& -PB_p + K^T \Lambda W - (C_p + D_p K)^T D_p \\
R_{22} =& -2W + D_p^T D_p
\end{aligned}$$

Then, it can be concluded that $x_a = 0$ is a globally asymptotically stable equilibrium when $u_\ell \equiv 0$, and $\|v_2\|_2 < \sqrt{\gamma}\,\|u_\ell\|_2$, when $u_\ell \in \mathcal{L}_2$ and $x_a(0) = 0$. Finally, it can be shown that $R \prec 0$ if and only if $LMI\,(Q, \Sigma, M, \gamma) \prec 0$ with $Q = P^{-1}$, $\Sigma = W^{-1}$, and $M = KQ$ (see Appendix B). □

Finally, the anti-windup synthesis problem is solved in two cases:

1) When the open-loop plant $P(s)$ is asymptotically stable, select $\Lambda = I$, and solve the LMI feasible problem (9) with $Q \succ 0$, $\Sigma \succ 0$ and $\gamma > 0$ to obtain K. If the LMI feasible problem is solvable, the anti-windup problem is solved globally, since the deadzone function $\phi(\cdot)$ belongs to the sector $[0, I]$. In fact, the LMI feasible problem is always solvable in this case. A simple solution is to choose $K = M = 0 \in \mathbf{R}^{m \times n_p}$. This corresponds to an internal model control scheme (IMC) (Zheng *et al.*, 1994). Notice that, although the synthesis of the IMC scheme is simple, it can lead to poor performance, if some poles of $P(s)$ are close to the imaginary axis.
2) When $P(s)$ is not asymptotically stable, select $0 \prec \Lambda \prec I$, i.e. $0 < \lambda_i < 1\ \forall i$, so that the LMI feasible problem is solvable. Fig. 4 shows the intersection of the deadzone function $\phi(\cdot)$ and the sector $[0, \Lambda]$ for the ith input. Notice that $\phi_i(u_i)$ belongs to the sector $[0, \lambda_i]$ only when $|u_i| \leq \frac{u_{i,\max}}{1-\lambda_i}$. Therefore, the anti-windup problem is only solved locally in this case. Finally, it is desirable to select λ_i as close to one as possible to enlarge the constrained range of u_i.

4. POLE CONSTRAINTS OF THE ANTI-WINDUP FILTER

The anti-windup filter found in the previous section guarantees that the nonlinear system in Fig. 3 is globallly (or locally) stable with $\|v_2\|_2 < \sqrt{\gamma}\,\|u_\ell\|_2$ for some $\gamma > 0$. Furthermore, the performance of the anti-windup filter can be optimized by minimizing γ. However, the method presented

$$LMI\,(Q,\Sigma,M,\gamma)=\begin{bmatrix} (A_pQ+B_pM)+(A_pQ+B_pM)^T & -B_p\Sigma+M^T\Lambda & 0 & QC_p^T+M^TD_p^T \\ -\Sigma B_p^T+\Lambda M & -2\Sigma & \Lambda & -\Sigma D_p^T \\ 0 & \Lambda & -\gamma I & 0 \\ C_pQ+D_pM & -D_p\Sigma & 0 & -I \end{bmatrix}$$

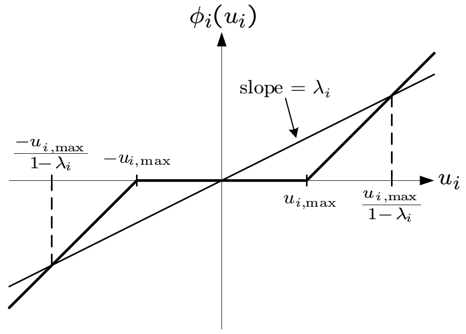

Fig. 4. Constrained range of u_i

in the previous section provides no direct control over the anti-windup filter poles, i.e. the eigenvalues of A_f. The poles of the resulting anti-windup filter may not be in a "good" region. Notice in Fig. 3 that the poles of the anti-windup filter determine the transient response of v_2 decaying to zero, when $\widetilde{u}=0$. In other words, the poles of the anti-windup filter determine the transient recovery of the unconstrained linear output response y_ℓ, when the actuator is no longer saturated. Therefore, it is desirable to solve the anti-windup synthesis problem with the poles constrained in a certain region. The region is chosen to guarantee that the transient response is satisfactory and the poles of the anti-windup filter are not too fast to be practically implemented in digital control systems.

A region $\mathcal{D}(\alpha,\rho,\theta)$ as shown Fig. 5 is considered in this paper. A complex eigenvalue $a+j\,b$ is in the region $\mathcal{D}(\alpha,\rho,\theta)$ if $a<-\alpha<0$, $|a+j\,b|<\rho$, and $(\tan\theta)\,a<-|b|$. It has been shown that all eigenvalues of A_f are in the region $\mathcal{D}$ if and only if there exists an $X=X^T\in\mathbf{R}^{n_p\times n_p}\succ 0$ such that (11)∼(13) hold (Chilali and Gahinet, 1996).

$$A_fX+XA_f^T+2\alpha X\prec 0 \tag{11}$$

$$\begin{bmatrix} -\rho X & A_fX \\ XA_f^T & -\rho X \end{bmatrix}\prec 0 \tag{12}$$

$$\begin{bmatrix} \sin\theta(A_fX+XA_f^T) & \cos\theta(A_fX-XA_f^T) \\ \cos\theta(A_fX-XA_f^T)^T & \sin\theta(A_fX+XA_f^T) \end{bmatrix}\prec 0 \tag{13}$$

In order for (11)∼(13) and (9) to be tractable in the LMI context, let $X=Q$ and $KX=M$, then (11)∼(13) can be written as (14)∼(16).

$$(A_pQ+B_pM)+(A_pQ+B_pM)^T+2\alpha Q\prec 0 \tag{14}$$

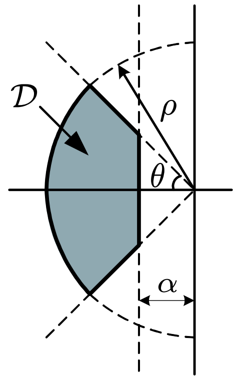

Fig. 5. Desired pole region

$$\begin{bmatrix} -\rho Q & A_pQ+B_pM \\ (A_pQ+B_pM)^T & -\rho Q \end{bmatrix}\prec 0 \tag{15}$$

$$\begin{bmatrix} \begin{array}{c}\sin\theta((A_pQ+B_pM)\\ +(A_pQ+B_pM)^T)\end{array} & \begin{array}{c}\cos\theta((A_pQ+B_pM)\\ -(A_pQ+B_pM)^T)\end{array} \\ \begin{array}{c}\cos\theta((A_pQ+B_pM)\\ -(A_pQ+B_pM)^T)^T\end{array} & \begin{array}{c}\sin\theta((A_pQ+B_pM)\\ +(A_pQ+B_pM)^T)\end{array} \end{bmatrix}\prec 0 \tag{16}$$

It follows that a sufficient condition for the existence of the gain K to solve the problem defined in *Theorem 1* with the pole constraint $\mathcal{D}(\alpha,\rho,\theta)$ is to find $Q=Q^T\in\mathbf{R}^{n_p\times n_p}\succ 0$, $\Sigma=\mathrm{diag}\{\sigma_1,\ldots,\sigma_m\}\succ 0$, $M\in\mathbf{R}^{m\times n_p}$, and a scalar $\gamma>0$ such that (9) and (14)∼(16) hold.

The anti-windup synthesis problem with the pole constraint $\mathcal{D}(\alpha,\rho,\theta)$ can be solved for the two cases ($P(s)$ is asymptotically stable and $P(s)$ is not asymptotically stable) as explained at the end of Section 3.

5. EXAMPLE

This example is adopted from (Grimm *et al.*, 2002). Consider a mass-spring-damper system

$$\begin{aligned} \dot{x}&=\begin{bmatrix} 0 & 1 \\ -k/m & -f/m \end{bmatrix}x+\begin{bmatrix} 0 \\ 1/m \end{bmatrix}u \\ y&=\begin{bmatrix} 1 & 0 \end{bmatrix}x \end{aligned} \tag{17}$$

where $x=[x_1\,x_2]$ represents the position and velocity of the mass, $m=0.1kg$ is the mass, $k=1kg/s^2$ is the spring constant, $f=0.005kg/s$ is the damping coefficient, and u represents the input force exerted on the mass.

Let r be the reference signal for the output y. Consider the following controller

$$u = C_1(s)\,(C_2(s)r - y) \qquad (18)$$

where the feedback controller $C_1(s)$ and feedforward controller $C_2(s)$ are

$$C_1(s) = 200\frac{(s+5)^2}{s(s+80)}, \quad C_2(s) = \frac{5}{s+5} \quad (19)$$

The reference signal r was chosen to switch between ± 0.9 meters every 10 seconds and go back to zero after 30 seconds. The solid line in Fig. 6 shows the unconstrained linear closed-loop response y. The response y tracks the reference r well with zero steady-state error. A saturation limit $\pm 1\ kg{\cdot}m/s^2$ was applied to the input force u, and the constrained response without anti-windup compensation is shown by the dotted line in Fig. 6. Clearly, the stability of the closed-loop system is lost.

Three anti-windup filters were designed in this example utilizing the methods: 1) IMC scheme, 2) anti-windup synthesis without the pole constraints (Section 3), and 3) anti-windup synthesis with the pole constraints (Section 4). The results are as following.

1) IMC scheme: $K = [0\ 0]$. The time response is shown by the dash-dotted line in Fig. 6. Although the constrained closed-loop system is stable, the time response has very large oscillations, and decays to the unconstrained linear response slowly. This is due to the fact that the poles of the IMC anti-windup filter are the same as that of the open-loop plant, which are $-0.025 \pm j\,3.162$.
2) Without pole constraints: $\Lambda = I$ was chosen, since the open-loop plant is asymptotically stable. The anti-windup filter (or the gain K) was obtained by minimizing γ subject to (9) with $Q \succ 0$, $\Sigma \succ 0$ and $\gamma > 0$. The minimization problem was solved by using YALMIP (Löfberg, 2004) with SeDuMi solver(Sturm, 2001). The minimal finite gain $\sqrt{\gamma}$ was obtained to be 63.25. The time response of this case is shown by the dashed line in Fig. 6. The response approaches the unconstrained linear response very slowly without oscillations. The poles of the anti-windup filter are -1772.69 and -0.031. One pole is too fast compared to the feedback controller dynamics, and the other causes the slow response.
3) With pole constraints: $\Lambda = I$, $\alpha = 0.02$ and $\rho = 9.5$ were chosen. Notice that the pole constraint (16) was not used, since appropriate poles were already resulted from using only (14) and (15) in this example. The anti-windup filter (or the gain K) was obtained by minimizing γ subject to (9), (14) and (15) with $Q \succ 0$, $\Sigma \succ 0$ and $\gamma > 0$. The minimal finite gain $\sqrt{\gamma}$ was obtained to be 63.25. The poles of the anti-windup filter in this case are -2.394 and -4.248. The time response is shown by the bold line in Fig. 6. The response is the best among the three cases. It approaches the unconstrained linear response fast and smooth. This confirms the effectiveness of the proposed approach.

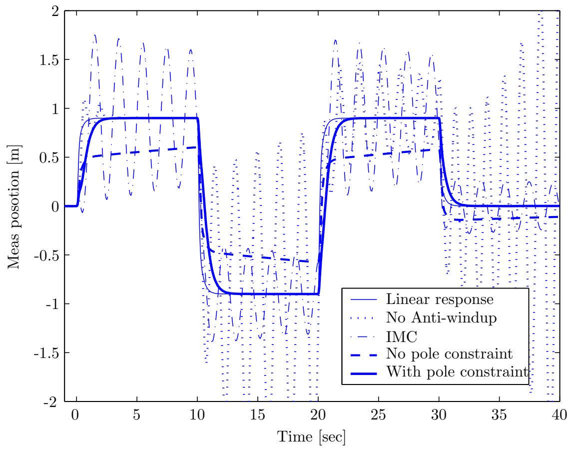

Fig. 6. The time response of the output y

6. CONCLUSIONS

This paper presented a design method for an anti-windup filter with filter pole constraints. The anti-windup filter is designed to ensure the stability and small performance degradation of the closed-loop system in the presence of saturation, while the filter poles are constrained in a predefined region. The anti-windup design problem can be converted to an LMI optimization problem. This design method can be applied to stable or unstable open-loop plants, while only local stability can be concluded for the plants which are not asymptotically stable. Finally, an example was provided to demonstrate the effectiveness of the proposed method.

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APPENDIX A

The details of the matrices in (3) are as follows.

$$A_{cl} = \begin{bmatrix} A_p + B_pD_cE_1C_p & B_pE_2C_c \\ B_cE_1C_p & A_c + B_cE_1D_pC_c \end{bmatrix} \tag{20}$$

$$B_{rcl} = \begin{bmatrix} B_pE_2D_r \\ B_cE_1D_pD_r + B_r \end{bmatrix} \tag{21}$$

$$B_{dcl} = \begin{bmatrix} B_pD_cE_1D_d + B_d \\ B_cE_1D_d \end{bmatrix} \tag{22}$$

$$C_{ycl} = \begin{bmatrix} E_1C_p & E_1D_pC_c \end{bmatrix} \tag{23}$$

$$C_{ucl} = \begin{bmatrix} D_cE_1C_p & E_2C_c \end{bmatrix} \tag{24}$$

$$D_{yrcl} = E_1D_pD_r \tag{25}$$

$$D_{ydcl} = E_1D_d \tag{26}$$

$$D_{urcl} = E_2D_r \tag{27}$$

$$D_{udcl} = D_cE_1D_d \tag{28}$$

where $E_1 = (I - D_pD_c)^{-1}$ and $E_2 = (I - D_cD_p)^{-1}$ exist due to the assumption that the unconstrained linear closed-loop system is well-posed.

APPENDIX B

Let

$$T = \begin{bmatrix} Q & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & I \end{bmatrix} \tag{29}$$

then $R \prec 0$ if and only if $T^TRT \prec 0$, since T is a full rank matrix. With $Q = P^{-1}$, $\Sigma = W^{-1}$, and $M = KQ$, T^TRT can be found as

$$T^TRT = \begin{bmatrix} Y_{11} & Y_{12} & 0 \\ Y_{12}^T & Y_{22} & \Lambda \\ 0 & \Lambda & -\gamma I \end{bmatrix} \tag{30}$$

$$\begin{aligned} Y_{11} &= (A_pQ + B_pM) + (A_pQ + B_pM)^T \\ &\quad + (C_pQ + D_pM)^T(C_pQ + D_pM) \\ Y_{12} &= -B_p\Sigma + M^T\Lambda - (C_pQ + D_pM)^TD_p\Sigma \\ Y_{22} &= -2\Sigma + \Sigma D_p^TD_p\Sigma \end{aligned}$$

Finally, it can be known that

$$T^TRT \prec 0 \Longleftrightarrow \quad LMI(Q, \Sigma, M, \gamma) \prec 0 \tag{31}$$