FROM CONSERVATION LAWS TO PORT-HAMILTONIAN REPRESENTATIONS OF DISTRIBUTED-PARAMETER SYSTEMS

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Abstract: In this paper it is shown how the port-Hamiltonian formulation of distributed-parameter systems is closely related to the general thermodynamic framework of systems of conservation laws and closure equations. The situation turns out to be similar to the lumped-parameter case where the Dirac structure captures the basic interconnection laws, and the closure equations correspond to the constitutive relations of the energy-storing elements.

Keywords: Interconnected systems, modeling, energy storage, geometric theory.

1. INTRODUCTION

The treatment of infinite-dimensional Hamiltonian systems in the literature is mostly confined to systems with boundary conditions such that the energy exchange through the boundary is *zero*. On the other hand, in many applications the interaction with the environment (e.g. actuation or measurement) takes place through the boundary of the system. In (van der Schaft and Maschke, 2002; Maschke and van der Schaft, 2000), we have developed a framework to represent classes of physical distributed-parameter systems with boundary energy flow as infinite-dimensional *port-Hamiltonian systems*. Key in this is the notion of a *Dirac structure*. Dirac structures were originally introduced in (Courant, 1990; Dorfman, 1993) as a geometric structure generalizing both *symplectic* and *Poisson* structures. Later on (van der Schaft and Maschke, 1995; Dalsmo and van der Schaft, 1999; Maschke and van der Schaft, 1997; Bloch and Crouch, 1999) it was realized that in the finite-dimensional case Dirac structures can be employed to formalize Hamiltonian systems with *algebraic constraints*. In order to allow the inclusion of boundary variables in distributed-parameter systems the concept of (an infinite-dimensional) Dirac structure provides again the right type of generalization with respect to the existing framework (Olver, 1993) using Poisson structures.The aim of this paper is to show how this port-Hamiltonian formulation of distributed-parameter systems can be based on the thermodynamic framework for describing distributed-parameter systems as systems of conservation laws, see e.g. (Godlewsky and Raviart, 1996; Serre, 1999).

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2. CONSERVATION LAWS, INTERDOMAIN COUPLING AND BOUNDARY ENERGY FLOWS: MOTIVATIONAL EXAMPLES

In this section we shall introduce the main concepts by means of three classical examples of distributed-parameter systems.

Example 2.1. (Inviscid Burger's equation). The *viscous Burger's equation* is a scalar parabolic equation which represents the simplest model for a fluid flow (Serre, 1999). It is defined on a one-dimensional spatial domain (interval) $Z = [a, b] \subset \mathbb{R}$, with the state variable $\alpha(t, z) \in \mathbb{R}, z \in Z,\ t \in I$, where I is an interval of $\mathbb{R}$, satisfying the partial differential equation

$$\frac{\partial \alpha}{\partial t} + \alpha \frac{\partial \alpha}{\partial z} - \nu \frac{\partial^2 \alpha}{\partial z^2} = 0 \tag{1}$$

The *inviscid* ($\nu = 0$) Burger's equations may be alternatively expressed as

$$\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial z}\beta = 0 \tag{2}$$

where the state variable $\alpha(t, z)$ is called the *conserved quantity* and the function $\beta := \frac{\alpha^2}{2}$ the *flux* variable. Eq. (2) is called a *conservation law*, since by integration one obtains the *balance equation*

$$\frac{d}{dt}\int_a^b \alpha dz = \beta(a) - \beta(b) \tag{3}$$

Furthermore, according to the framework of Irreversible Thermodynamics (Prigogine, 1962), one may express the flux β as a function of the *generating force* which is the *variational derivative* of some functional $H(\alpha)$ of the state variable. This variational derivative plays the same role as the gradient of a function in the finite-dimensional case. The variational derivative $\frac{\delta H}{\delta \alpha}$ of a functional $H(\alpha)$ is uniquely defined by the requirement

$$H(\alpha + \epsilon\eta) = H(\alpha) + \epsilon \int_a^b \frac{\delta H}{\delta \alpha}\ \eta\ dz + O(\epsilon^2) \tag{4}$$

for any $\epsilon \in \mathbb{R}$ and any smooth function $\eta(z, t)$ such that $\alpha + \epsilon\eta$ satisfies the same boundary conditions as α (Olver, 1993). For the inviscid Burger's equation one has $\beta = \frac{\delta H}{\delta \alpha}$, where

$$H(\alpha) = \int_a^b \frac{\alpha^3}{6} dz \tag{5}$$

Hence the inviscid Burger's equation may be also expressed as

$$\frac{\partial \alpha}{\partial t} = -\frac{\partial}{\partial z}\frac{\delta H}{\delta \alpha} \tag{6}$$

This defines an infinite-dimensional Hamiltonian system in the sense of (Olver, 1993) with respect to the skew-symmetric operator $\frac{\partial}{\partial z}$ that is defined on the functions with support contained in the interior of the interval Z.

From this formulation one derives that the Hamiltonian $H(\alpha)$ is *another* conserved quantity. Indeed, by integration by parts

$$\frac{d}{dt}H = \int_a^b \frac{\delta H}{\delta \alpha} \cdot -\frac{\partial}{\partial z}\frac{\delta H}{\delta \alpha} dz = \frac{1}{2}\left(\beta^2(a) - \beta^2(b)\right) \tag{7}$$

For later use we note that the right-hand side is a *quadratic function of the flux variables* evaluated at the boundary of the spatial domain Z.

The second example consists of a system of *two* conservations laws, corresponding to the case of two physical domains in interaction.

Example 2.2. (The p-system). (Serre, 1999), (Godlewsky and Raviart, 1996) The p-system is a model for e.g. a one-dimensional gas dynamics. Again, the spatial variable z belongs to an interval $Z \subset \mathbb{R}$, while the dependent variables are the specific volume $v(t, z) \in \mathbb{R}^+$, the velocity $u(t, z)$ and the pressure functional $p(v)$ (which for instance in the case of an ideal gas with constant entropy is given by $p(v) = Av^{-\gamma}$ where $\gamma \geq 1$). The *p-system* is then defined by the following system of partial differential equations

$$\begin{aligned} \frac{\partial v}{\partial t} - \frac{\partial u}{\partial z} &= 0 \\ \frac{\partial u}{\partial t} + \frac{\partial\, p(v)}{\partial z} &= 0 \end{aligned} \tag{8}$$

representing respectively conservation of mass and of momentum. By defining the state vector as $\alpha(t, z) = (v, u)^T$, and the vector-valued flux $\beta(t, z) = (-u, p(v))^T$ the p-system is rewritten as

$$\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial z}\beta = 0 \tag{9}$$

Again, according to the framework of Irreversible Thermodynamics, the flux vector may be written as function of the variational derivatives of some functional. Indeed, consider the energy functional $H(\alpha) = \int_a^b \mathcal{H}(v, u) dz$ where the energy density $\mathcal{H}(v, u)$ is given as the sum of the internal energy and the kinetic energy densities

$$\mathcal{H}(v, u) = \mathcal{U}(v) + \frac{u^2}{2}, \tag{10}$$

with $-\mathcal{U}(v)$ a primitive function of the pressure. (Note that for simplicity the mass density has been set equal to 1, and hence no difference is made between the velocity and the momentum.) The flux vector β may be expressed in terms of the variational derivatives of H as

$$\beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta v} \\ \frac{\delta H}{\delta u} \end{pmatrix} \tag{11}$$

The anti-diagonal matrix represents the canonical coupling between two physical domains: the kinetic and the potential (internal) domain. Thus the variational derivative of the total energy with respect to the state variable of one domain generates the flux variable for the other domain. Combining eqns. (9) and (11), the p-system may thus be written as the Hamiltonian system

$$\begin{pmatrix} \frac{\partial \alpha_1}{\partial t} \\ \frac{\partial \alpha_2}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix} \tag{12}$$

Using again integration by parts, one may derive the following *power balance equation*:

$$\frac{d}{dt}H = \beta_1(a)\,\beta_2(a) - \beta_1(b)\,\beta_2(b) \quad (13)$$

Notice again that the right-hand side of this power-balance equation is a quadratic function of the fluxes at the boundary of the spatial domain.

Remark 2.3. In fact, every non-linear wave equation

$$\frac{\partial^2 g}{\partial t^2} - \frac{\partial}{\partial z}\left(\sigma\left(\frac{\partial g}{\partial z}\right)\right) = 0$$

may be expressed as a p-system using the change of variables $u = \frac{\partial g}{\partial t}$, $v = \frac{\partial g}{\partial z}$ and $p(v) = -\sigma(v)$.

The last example is the *vibrating string.* It is again a system of two conservation laws representing the canonical interdomain coupling between the kinetic energy and the elastic potential energy. However in this example the *classical* choice of the state variables leads to express the total energy as a function of some of the *spatial derivatives* of the state variables. We shall analyze how the dynamic equations and the power balance are expressed in this case and we shall subsequently draw some conclusions on the choice of the state variables.

Example 2.4. (Vibrating string). Consider an elastic string subject to traction forces at its ends, with spatial variable $z \in Z = [a, b] \subset \mathbb{R}$. Denote by $u(t, z)$ the displacement of the string and the velocity by $v(t, z) = \frac{\partial u}{\partial t}$. Using the vector of state variables $x(t, z) = (u, v)^T$, the dynamics of the vibrating string is described by the system of partial differential equations

$$\frac{\partial x}{\partial t} = \begin{pmatrix} v \\ \frac{1}{\mu}\frac{\partial}{\partial z}\left(T\frac{\partial u}{\partial z}\right) \end{pmatrix} \quad (14)$$

where the first equation is simply the definition of the velocity and the second one is Newton's second law. Here T denotes the elasticity modulus, and μ the mass density. The time-variation of the state may be expressed as a function of the variational derivative of the total energy as in the preceeding examples. Indeed, the total energy is $H(x) = U(u) + K(v)$, where the elastic potential energy U is a function of the *strain* $\frac{\partial u}{\partial z}(t, z)$

$$U(u) = \int_a^b \frac{1}{2}\,T\,\left(\frac{\partial u}{\partial z}\right)^2\,dz \quad (15)$$

and the kinetic energy K depends on the velocity $v(t, z) = \frac{\partial u}{\partial t}$ as

$$K(v) = \int_a^b \frac{1}{2}\,\mu v(t, z)^2\,dz \quad (16)$$

Thus the total system (14) may be expressed as

$$\frac{\partial x}{\partial t} = \begin{pmatrix} 0 & \frac{1}{\mu} \\ -\frac{1}{\mu} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} \quad (17)$$

where $\frac{\delta H}{\delta u} = \frac{\delta U}{\delta u} = -\frac{\partial}{\partial z}\left(T\,\frac{\partial u}{\partial z}\right)$ is the elastic force and $\frac{\delta H}{\delta v} = \frac{\delta K}{\delta v} = \mu v$ is the momentum.

In this formulation, the system is *not* anymore expressed as a system of conservation laws since the time-derivative of the state variables is a function of the variational derivatives of the energy *directly*, and *not* the spatial derivative of a function of the variational derivatives as before. Instead of being a simplification, this reveals a drawback for the case of non-zero energy flow through the boundary of the spatial domain. Indeed, in this case the *variational derivative has to be completed by a boundary term* since the Hamiltonian functional depends on the *spatial derivatives of the state.* For example, in the computation of the variational derivative of the elastic energy U one obtains by integration by parts that $U(u + \epsilon\eta) - U(u)$ equals

$$-\epsilon\int_a^b \frac{\partial}{\partial z}\left(T\,\frac{\partial u}{\partial z}\right)\eta\,dz + \epsilon\left[\eta\,\left(T\,\frac{\partial u}{\partial z}\right)\right]_a^b + O(\epsilon^2) \quad (18)$$

and the second term in this expression yields an extra boundary term.

Alternatively we shall now formulate the vibrating string as a system of two conservation laws. Take as alternative vector of state variables $\alpha(t, z) = (\epsilon, p)^T$, where ϵ denotes the *strain* $\alpha_1 = \epsilon = \frac{\partial u}{\partial z}$ and p the *momentum* $\alpha_2 = p = \mu v$. Recall that in these variables the total energy is written as

$$H_0 = \int_a^b \frac{1}{2}\left(T\,\alpha_1^2 + \frac{1}{\mu}\alpha_2^2\right)dz \quad (19)$$

and directly depends on the state variables and *not* on their spatial derivatives. Furthermore, one defines the flux variables to be the *stress* $\beta_1 = \frac{\delta H_0}{\delta \alpha_1} = T\;\alpha_1$ and the *velocity* $\beta_2 = \frac{\delta H_0}{\delta \alpha_1} = \frac{\alpha_2}{\mu}$. In matrix notation, the flux vector β is thus expressed as a function of the variational derivatives $\frac{\delta H_0}{\delta \alpha}$ by

$$\beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\delta H_0}{\delta \alpha} \quad (20)$$

Hence the vibrating string may be alternatively expressed by the system of two conservation laws

$$\frac{\partial \alpha}{\partial t} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \frac{\delta H_0}{\delta \alpha} \quad (21)$$

satisfying the power balance equation (13).

3. SYSTEMS OF TWO CONSERVATION LAWS IN CANONICAL INTERACTION

Let us now consider the *general class* of distributed-parameter systems consisting of two conservation laws with the canonical coupling as in the above examples of the p-system and the vibrating string. First, for 1-dimensional spatial domains, we introduce the concept of *interconnection structure* and *port variables* which are fundamental to the definition of port-Hamiltonian systems. In the second part we give the definition of systems of two conservation laws defined on *n−dimensional* spatial domains.

3.1 1-D spatial domain

Consider as before for the p-system and the vibrating string a system of two conservation laws arising from the modelling of two physical domains in canonical interaction:

$$\frac{\partial \alpha}{\partial t} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \frac{\delta H_0}{\delta \alpha} \tag{22}$$

where $\alpha = (\alpha_1(t,z), \alpha_2(t,z))^T$. Let us now define an *interconnection structure* for this system in the sense of network port-based modelling (Karnopp *et al.*, 1990) (Maschke *et al.*, 1992) (van der Schaft and Maschke, 1995). Define the vector of *flow variables* f to be the time-variation of the state and the vector of *effort variables* e to be the vector of variational derivatives, that is

$$f = \frac{\partial \alpha}{\partial t}, \quad e = \frac{\delta H_0}{\delta \alpha} \tag{23}$$

The flow and effort vectors are *power-conjugated* since their product is the time-variation of the total energy:

$$\begin{aligned} \frac{d}{dt} H_0 &= \int_a^b \left(\frac{\delta H_0}{\delta \alpha_1} \frac{\partial \alpha_1}{\partial t} + \frac{\delta H_0}{\delta \alpha_2} \frac{\partial \alpha_2}{\partial t} \right) dz \\ &= \int_a^b (e_1 \; f_1 + e_2 \; f_2) \, dz \end{aligned} \tag{24}$$

Considering the right-hand side of the power balance equation (13) it is clear that the energy exchange of the system with its environment is determined by the flux variables at the boundary of the domain. Therefore let us define two boundary variables by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} e_2 \\ e_1 \end{pmatrix} = \begin{pmatrix} \frac{\delta H_0}{\delta \alpha_2} \\ \frac{\delta H_0}{\delta \alpha_1} \end{pmatrix} = \begin{pmatrix} v \\ \sigma \end{pmatrix} \tag{25}$$

These boundary variables are power-conjugated since their product $\beta_1 \beta_2 = e_b f_b = \sigma v$ equals the right-hand side of the power balance equation (13). Considering the four power-conjugated variables $f_1, f_2, f_\partial, e_1, e_2, e_\partial$, the power balance equation (13) implies

$$\int_a^b (e_1 f_1 + e_2 f_2) \, dz + e_\partial(b) f_\partial(b) - e_\partial(a) f_\partial(a) = 0 \tag{26}$$

This bilinear product between the power conjugated variables is analogous to the product between the circuit variables expressing the *power continuity* relation in circuits and network models (Karnopp *et al.*, 1990) (Breedveld, 1984). Such products are also central in the definition of implicit Hamiltonian systems (Courant, 1990) (Dorfman, 1993) and port-Hamiltonian systems in finite dimensions (van der Schaft and Maschke, 1995) (Maschke and van der Schaft, 1997), and the same will hold for infinite-dimensional port-Hamiltonian systems (Maschke and van der Schaft, 2000) (van der Schaft and Maschke, 2002).

It follows that the *interconnection structure* underlying the system (22) (analogous to Kirchhoff's laws for circuits) is defined by (25) together with

$$f = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} e \tag{27}$$

This is the one-dimensional Stokes-Dirac structure of (van der Schaft and Maschke, 2002).

3.2 N-dimensional spatial domain

Let the spatial domain $Z \subset \mathbb{R}^n$ be an n-dimensional smooth manifold with smooth $(n-1)$-dimensional boundary ∂Z. Denote by $\Omega^k(Z)$ the vector space of (differential) k-forms on Z (respectively by $\Omega^k(\partial Z)$ the vector space of k-forms on ∂Z). Denote furthermore $\Omega = \bigoplus_{k \geq 0} \Omega^k(Z)$ the algebra of differential forms over Z and recall that it is endowed with an exterior product $\wedge$ and an exterior derivation d.

Definition 3.1. A *system of conservation laws* is defined by a set of *conserved quantities* $\alpha_i \in \Omega^{k_i}(Z)$, $i \in \{1, \ldots, N\}$ where $N \in \mathbb{N}, k_i \in \mathbb{N}$, defining the state space $\mathcal{X} = \bigotimes_{i=1,..,N} \Omega^{k_i}(Z)$, and satisfying a set of *conservation laws*

$$\frac{\partial \alpha_i}{\partial t} + \mathrm{d}\beta_i = g_i \tag{28}$$

where $\beta_i \in \Omega^{k_i - 1}(Z)$ denote the set of *fluxes* and $g_i \in \Omega^{k_i}(Z)$ denote the set of *distributed interaction forms.* In general, the fluxes β_i are defined by so-called *closure equations*

$$\beta_i = J(\alpha_i, z) \;, \; i = 1, .., N \tag{29}$$

leading to a closed form for the dynamics of the conserved quantities α_i. The integral form of the conservation laws yields the *balance equations*

$$\frac{d}{dt} \int_S \alpha_i + \int_{\partial S} \beta_i = \int_S g_i \tag{30}$$

for any surface $S \subset Z$ of dimension equal to the degree of α_i.

Remark 3.2. A common case is that $Z = \mathbb{R}^3$ and that the conserved quantities are 3-forms, that is, the balance equation is evaluated on volumes of the 3-dimensional space. In this case Eqn. (28) takes in vector calculus notation the familiar form

$$\frac{\partial \alpha_i}{\partial t}(z,t) + \mathrm{div}_z \beta_i = g_i \ , \ i = 1,..,n \tag{31}$$

However, conservation laws may correspond to differential forms of any degree. Maxwell's equations are an example where the conserved quantities are differential forms of degree 2.

In the sequel, as in the case of the 1-dimensional spatial domain, we shall consider a particular class of systems of conservation laws where the closure equations are such that fluxes are (linear) functions of the variational derivatives of the Hamiltonian functional. First recall the general definition of the *variational derivative* of a functional $H(\alpha)$ with respect to the differential form $\alpha \in \Omega^p(Z)$ (generalizing the definition given before).

Definition 3.3. Consider a density function $\mathcal{H} : \Omega^p(Z) \times Z \to \Omega^n(Z)$ where $p \in \{1,..,n\}$, and denote by $H := \int_Z \mathcal{H} \in \mathbb{R}$ the associated functional. Then the uniquely defined differential form $\frac{\delta H}{\delta \alpha} \in \Omega^{n-p}(Z)$ which satisfies for all $\Delta\alpha \in \Omega^p(Z)$ and $\varepsilon \in \mathbb{R}$

$$H(\alpha+\varepsilon\Delta\alpha) = \int_Z \mathcal{H}(\alpha) + \varepsilon \int_Z \left[\frac{\delta H}{\delta \alpha} \wedge \Delta\alpha\right] + \mathrm{O}\left(\varepsilon^2\right)$$

is called the *variational derivative* of H with respect to $\alpha \in \Omega^p(Z)$.

Definition 3.4. Systems of two conservation laws with canonical interdomain coupling are systems of two conservation laws involving a pair of conserved quantities $\alpha_p \in \Omega^p(Z)$ and $\alpha_q \in \Omega^q(Z)$, differential forms on the n-dimensional spatial domain Z of degree p and q respectively, satisfying $p + q = n + 1$ ('complementarity of degrees'). The closure equations generated by a *Hamiltonian density function* $\mathcal{H} : \Omega^p(Z) \times \Omega^q(Z) \times Z \to \Omega^n(Z)$ resulting in the Hamiltonian $H := \int_Z \mathcal{H} \in \mathbb{R}$ are given by

$$\begin{pmatrix} \beta_p \\ \beta_q \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & (-1)^r \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{pmatrix} \tag{32}$$

where $r = pq+1$, $\varepsilon \in \{-1,+1\}$, depending on the sign convention of the considered physical domain.

In the same way as for systems defined on 1-dimensional spatial domains, one may define for $n-$ spatial domains pairs of power conjugated variables. Define the vector of *flow variables* to be the time-variation of the state, and the *effort variables* to be the variational derivatives

$$\begin{pmatrix} f_p \\ f_q \end{pmatrix} = \begin{pmatrix} \frac{\partial \alpha_p}{\partial t} \\ \frac{\partial \alpha_q}{\partial t} \end{pmatrix} \quad \begin{pmatrix} e_p \\ e_q \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{pmatrix} \tag{33}$$

Their product equals again the time-variation of the Hamiltonian

$$\frac{dH}{dt} = \int_Z (e_p \wedge f_p + e_q \wedge f_q) \tag{34}$$

Using the conservation laws (28) for $g_i = 0$, the closure relations (32) and the properties of the exterior derivative d and Stokes' theorem, one obtains

$$\begin{aligned} \frac{dH}{dt} &= \int_Z \varepsilon\beta_q \wedge (-d\beta_p) + (-1)^r \beta_p \wedge \varepsilon(-d\beta_q) \\ &= -\varepsilon \int_Z \beta_q \wedge d\beta_p + (-1)^q \beta_q \wedge d\beta_p \\ &= -\varepsilon \int_{\partial Z} \beta_q \wedge \beta_p \end{aligned} \tag{35}$$

Finally, as before we define the power-conjugated pair of *flow and effort variables on the boundary* as the *restriction* of the flux variables to the boundary ∂Z of the domain Z:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \beta_q|_{\partial Z} \\ \beta_p|_{\partial Z} \end{pmatrix} \tag{36}$$

On the total space of power-conjugated variables, that is, the differential forms (f_p, e_p) and (f_q, e_q) on the domain Z and the differential forms (f_∂, e_∂) defined on the boundary ∂Z, one defines an *interconnection structure* (underlying the system of two conservation laws with canonical interdomain coupling of Definition 3.4) by Eqn. (36) together with

$$\begin{pmatrix} f_q \\ f_p \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & (-1)^r \ d \\ d & 0 \end{pmatrix} \begin{pmatrix} e_q \\ e_p \end{pmatrix} \tag{37}$$

This interconnection is power-continuous since by (36) and (37)

$$\int_Z (e_p \wedge f_p + e_q \wedge f_q) + \varepsilon \int_{\partial Z} f_\partial \wedge e_\partial = 0 \tag{38}$$

The above power-continuous interconnection structure can be formalized as a geometric structure, called Stokes-Dirac structure (van der Schaft and Maschke, 2002), leading to the definition of distributed-parameter port-Hamiltonian systems.

4. CONCLUSIONS AND FINAL REMARKS

In this paper we have related the framework for compositional modelling of distributed-parameter systems as port-Hamiltonian systems, to the basic thermodynamic framework of conservation laws and closure equations. The situation turns out to be quite similar to the lumped-parameter case where the Dirac structure incorporates the basic topological interconnection laws (Kirchhoff's laws, Newton's third law) together with other power-conserving interconnection constraints (see e.g. (Maschke and van der Schaft, 1997) (Maschke and van der Schaft, 1997) (van der Schaft and Maschke, 1995)), and the closure equations correspond to the constitutive relations of the energy-storing elements.

A prominent property of Dirac structures is that they are closed under power-conserving interconnection. This enables to link port-Hamiltonian systems (lumped- or distributed-parameter) to each other into a new port-Hamiltonian system. This leads to consider control strategies where the controller system is also a port-Hamiltonian system. Initial results along these lines have shown to be very promising, see e.g. (Rodriguez *et al.*, 2001; van der Schaft and Maschke, 2001).

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