ROBUST FILTERING FOR RANDOM SENSOR DELAY SYSTEMS [1]

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Abstract: In this paper, we consider the robust filtering problem for discrete time-varying systems with sensor delayed measurement subject to norm-bounded parameter uncertainties. The sensor delayed measurement is assumed to be a linear function of a stochastic variable that satisfies Bernoulli random binary distribution. An upper bound for the actual covariance of uncertain stochastic parameter system is derived, and is used for the estimation variance constraints. Such an upper bound is then minimized over the filter parameters for all stochastic sensor delays and admissible deterministic uncertainties. It is shown that the desired filter can be obtained in terms of solutions to two discrete Riccati difference equations, which are of a form suitable for recursive computation in online applications. An illustrative example is presented to show the applicability of the proposed method.

Keywords: Kalman filtering; robust filtering; random sensor delay

1. INTRODUCTION

Kalman filtering has proven to be very popular in a number of research areas such as signal processing and communication (Anderson and Moore, 1979). Since Kalman filtering algorithm is very sensitive to model structure drifts (Anderson and Moore, 1979), how to guarantee the robust performance of the Kalman filter in the presence of system parameter uncertainties has become an important issue, and has gained considerable attention from many researchers. A large volume of literature has been published on the general topic of robust and/or H_∞ filtering problems for systems with various parameter uncertainties, see e.g.(Fridman and Shaked, 2001; Hung and Yang, 2003; Petersen and Savkin, 1999; Shaked et al., 2001; Theodor and Shaked, 1996; Wang and Balakrishnan, 2002; Xie et al., 1994; Zhu et al., 2002) and references therein.

On the other hand, it is implicitly assumed in Kalman filtering approach that the sensor data, which may be corrupted by noise, always contain information about the *current* state of the plant. However, it is not always the case in engineering systems, biological systems, and economical systems, where the system measurements (outputs) may be *delayed*, and this could cause performance degradation or even instability of the traditional Kalman filters (Mahmoud, 2000; Malek-Zavarei and Jamshidi, 1987). Therefore, the filtering problem with delayed measurements has been a research subject recently, and many results have been published, most of which assume that the time-delay in the measurement is al-

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ways *deterministic*. Unfortunately, the time delay may occur in a *random* way in a large class of practical applications. For example, in real-time distributed decision-making and multiplexed data communication networks, the measurement device or the sensor is often randomly delayed, or the measurements are interrupted so that the measurements available to the predictor may not be up-to-date (Ray, 1994; Wang et al., 2004; Yaz and Ray, 1996). Hence, there is a need to develop new filtering methods for signal processing problems in a delayed environment of general network-based systems.

Up to now, there have been several papers discussing the filter design issue with randomly varying delayed measurements. In (Yaz and Ray, 1996), a linear unbiased state estimation problem has been dealt with for discrete-time systems with random sensor delay over both finite-horizon and infinite-horizon, where the full and reduced-order filters have been designed to achieve a certain estimation error covariance. The results of (Yaz and Ray, 1996) have been extended in (Wang et al., 2004) to the case where the parameter uncertainties (modeling error) have been taken into account. However, in (Wang et al., 2004), only the *stationary* (infinite-horizon) robust filtering problem has been studied. It is well know that finite-horizon filters could provide a better transient performance for the filtering process systems where the noise inputs are nonstationary. It is, therefore, our aim in this paper to further study the finite-horizon counterpart of (Wang et al., 2004). That is, we intend to tackle the finite-horizon filtering problem for uncertain discrete *time-varying* systems subject to both randomly varying sensor delay and parameter uncertainties. Different from (Wang et al., 2004), in this paper, the nominal system is allowed to be time-varying, and an optimization approach is used that is based on the solutions to two discrete Riccati difference equations.

In this paper, we are concerned with the robust filtering problem for discrete time-varying systems with sensor delayed measurement subject to norm-bounded parameter uncertainties. The sensor delayed measurement is assumed to be a linear function of a stochastic variable that satisfies Bernoulli random binary distribution. An upper bound for the actual covariance of uncertain stochastic parameter system is derived, and is used for the estimation variance constraints. Such an upper bound is then minimized over the filter parameters for all stochastic sensor delays and admissible deterministic uncertainties, which renders the filter design problem a *sub-optimal* one. A Riccati difference equation approach is developed to design the expected filter parameters. Such an approach is suitable for recursive computation in online applications. We illustrate the applicability of the proposed method by means of a simulation example.

Notation. The notation used here is fairly standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). A^{ii} denotes diagonal block sub-matrix of matrix A with respect to the ith row and ith column. x^i represents the ith element of vector x. $\mathrm{Cov}(x)$ means the covariance of x. The superscript "T" denotes the transpose. $\mathbb{E}\{x\}$ stands for the expectation of x. $\mathrm{Prob}\{\cdot\}$ means the occurrence probability of the event "$\cdot$". Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider a class of uncertain linear discrete time-varying systems

$$\breve{x}_{k+1} = (\breve{A}_k + \Delta\breve{A}_k)\breve{x}_k + \breve{B}_k w_k, \tag{1}$$

where $\breve{x}_k \in \mathbb{R}^n$ is a state vector, $w_k \in \mathbb{R}^n$ is a zero mean Gaussian white noise sequence with covariance $Q_k > 0$. The delayed sensor measurement is described by

$$\breve{y}_k = \breve{C}_k\breve{x}_k + \breve{v}_k, \tag{2}$$

$$y_k = (1-\gamma_k)\breve{y}_k + \gamma_k\breve{y}_{k-1}, \tag{3}$$

where $\breve{y}_k \in \mathbb{R}^p$ is an actual output vector, $y_k \in \mathbb{R}^p$ is a measured output vector, and $\breve{v}_k \in \mathbb{R}^p$ a zero mean Gaussian white noise sequence with covariance $\breve{R}_k > 0$ which is uncorrelated with w_k. The initial state $\breve{x}_0$ has the mean $\bar{x}_0$ and covariance P_0, and is uncorrelated with either $w(k)$ or $\breve{v}(k)$. $\breve{A}_k$,$\breve{B}_k$ and $\breve{C}_k$ are known real time-varying matrices with appropriate dimensions. $\Delta\breve{A}_k$ is a real-valued uncertain matrix satisfying

$$\Delta\breve{A}_k = \breve{H}_k F_k \breve{E}_k, \quad F_k F_k^T \leq I. \tag{4}$$

Here, $\breve{H}_k$ and $\breve{E}_k$ are known time-varying matrices of appropriate dimensions, and F_k represents the time-varying uncertainties. The parameter uncertainty in $\Delta\breve{A}_k$ is said to be admissible if (4) holds.

The stochastic variable $\gamma_k \in \mathbb{R}$ is a Bernoulli distributed white sequence taking values on 0 and 1 with

$$\mathrm{Prob}\{\gamma_k = 1\} = \mathbb{E}\{\gamma_k\} := \beta_k, \tag{5}$$

where $\beta_k \in \mathbb{R}$ is a known time-varying positive scalar, and $\gamma_k \in \mathbb{R}$ is assumed to be independent of w_k, $\breve{v}_k$, and $\breve{x}_0$. Therefore, we have

$$\text{Prob}\{\gamma_k = 0\} = 1 - \beta_k, \tag{6}$$
$$\sigma_\gamma^2 := \mathbb{E}\{(\gamma_k - \beta_k)^2\} = (1 - \beta_k)\beta_k. \tag{7}$$

Remark 1. The system measurement mode (3) was introduced in (Ray, 1994; Wang et al., 2004; Yaz and Ray, 1996), which can be used to represent the system output subject to randomly varying state delay. It can be easily seen that, at kth sampling time, the actual system output takes the value $\breve{y}_{k-1}$ with probability β_k, and the value $\breve{y}_k$ with probability $1 - \beta_k$.

By defining

$$x_k := \begin{bmatrix} \breve{x}_k \\ \breve{x}_{k-1} \end{bmatrix}, A_k := \begin{bmatrix} \breve{A}_k & 0 \\ I_n & 0 \end{bmatrix},$$
$$H_k := \begin{bmatrix} \breve{H}_k \\ 0 \end{bmatrix}, E_k := \begin{bmatrix} \breve{E}_k & 0 \end{bmatrix}, \Delta A_k := H_k F_k E_k,$$
$$C_k(\gamma_k) = \begin{bmatrix} (1-\gamma_k)\breve{C}_k & \gamma_k \breve{C}_{k-1} \end{bmatrix}, \quad B_k = \begin{bmatrix} \breve{B}_k \\ 0 \end{bmatrix},$$
$$D_k(\gamma_k) = \begin{bmatrix} (1-\gamma_k) I_p & \gamma_k I_p \end{bmatrix}, \quad v_k = \begin{bmatrix} \breve{v}_k \\ \breve{v}_{k-1} \end{bmatrix},$$

we combine the uncertain system (1) and the sensor delayed measurement (2)-(3) as follows:

$$x_{k+1} = (A_k + \Delta A_k)x_k + B_k w_k, \tag{8}$$
$$y_k = C_k(\gamma_k)x_k + D_k(\gamma_k)v_k, \tag{9}$$

where v_k is a zero mean Gaussian white noise sequence with covariance

$$R_k := \begin{bmatrix} \breve{R}_k & 0 \\ 0 & \breve{R}_{k-1} \end{bmatrix}, \tag{10}$$

and is independent of w_k, γ_k, and $\breve{x}_0$. Since $C_k(\gamma_k)$ and $D_k(\gamma_k)$ involve the stochastic variable γ_k, the system (8)-(9) is in fact a stochastic parameter system.

Denoting

$$\bar{C}_k = \mathbb{E}[C_k(\gamma_k)] = \begin{bmatrix} (1-\beta_k)\breve{C}_k & \beta_k \breve{C}_{k-1} \end{bmatrix}, \tag{11}$$

and

$$\bar{D}_k = \mathbb{E}[D_k(\gamma_k)] = \begin{bmatrix} (1-\beta_k) I_p & \beta_k I_p \end{bmatrix}, \tag{12}$$

we can rewrite (9) as

$$y_k = \bar{C}_k x_k + \bar{D}_k v_k + \tilde{C}_k(\gamma_k)x_k + \tilde{D}_k(\gamma_k)v_k \tag{13}$$

where

$$\tilde{C}_k(\gamma_k) := C_k(\gamma_k) - \bar{C}_k = \begin{bmatrix} (\beta_k - \gamma_k)\breve{C}_k & (\gamma_k - \beta_k)\breve{C}_{k-1} \end{bmatrix}$$
$$= (\gamma_k - \beta_k)\begin{bmatrix} -\breve{C}_k & \breve{C}_{k-1} \end{bmatrix} = (\gamma_k - \beta_k)C_{ek}, \tag{14}$$
$$\tilde{D}_k(\gamma_k) := D_k(\gamma_k) - \bar{D}_k = \begin{bmatrix} (\beta_k - \gamma_k) I_p & (\gamma_k - \beta_k) I_p \end{bmatrix}$$
$$= (\gamma_k - \beta_k)\begin{bmatrix} -I_p & I_p \end{bmatrix} = (\gamma_k - \beta_k)D_{ek}, \tag{15}$$
$$C_{ek} := \begin{bmatrix} -\breve{C}_k & \breve{C}_{k-1} \end{bmatrix}, \quad D_{ek} := \begin{bmatrix} -I_p & I_p \end{bmatrix}. \tag{16}$$

It can be shown that $\tilde{C}_k(\gamma_k) \in \mathbb{R}^{p\times 2n}$ and $\tilde{D}_k(\gamma_k) \in \mathbb{R}^{p\times 2p}$ are zero mean stochastic matrix sequences. In this paper, a full-order filter is of the following structure:

$$\hat{x}_{k+1} = \hat{A}_k \hat{x}_k + \hat{K}_k(y_k - \bar{C}_k \hat{x}_k), \tag{17}$$

where $\hat{x}_k \in \mathbb{R}^{2n}$ is the state estimate of the stochastic parameter system (8)-(13), and $\hat{A}_k$ and $\hat{K}_k$ are filter parameters to be determined.

Remark 2. It can be noticed that the system under consideration is both stochastic and uncertain, whereas the designed filter depends on neither the stochastic parameter nor the parameter uncertainty, which facilitates its implementation.

The objective of this paper is twofold. First, we intend to design a finite-horizon filter (17) such that there exists a sequence of positive-definite matrices $\Theta_k (0 < k \leq N)$ satisfying

$$\mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] \leq \Theta_k, \quad \forall k. \tag{18}$$

That is, a finite upper bound for the estimation error variance is guaranteed. Second, we shall minimize such a bound Θ_k in the sense of the matrix norm, and then obtain an optimized filter. This problem will be referred to as a finite-horizon robust filtering problem.

3. COVARIANCE AND ITS UPPER BOUND

It is noted that in the last section, the system parameters in (13) contain stochastic terms due to sensor delayed measurement. Therefore, we need to derive the estimation error covariance and then obtain a corresponding upper bound. For this purpose, we define a new state vector by

$$\tilde{x}_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}, \tag{19}$$

and then an augmented state-space model combining the system (8) and the filter (17) can be expressed by

$$\tilde{x}_{k+1} = (\tilde{A}_k + \tilde{H}_k F_k \tilde{E}_k)\tilde{x}_k + \tilde{A}_{ek}\tilde{x}_k$$
$$+ \tilde{B}_{1k} w_k + \tilde{B}_{2k} v_k + \tilde{B}_{ek} v_k, \tag{20}$$

where

$$\tilde{A}_k = \begin{bmatrix} A_k & 0 \\ \hat{K}_k \bar{C}_k & \hat{A}_k - \hat{K}_k \bar{C}_k \end{bmatrix}, \quad \tilde{H}_k = \begin{bmatrix} H_k \\ 0 \end{bmatrix},$$
$$\tilde{E}_k = \begin{bmatrix} E_k & 0 \end{bmatrix}, \quad \tilde{A}_{ek} = \begin{bmatrix} 0 & 0 \\ \hat{K}_k \tilde{C}_k(\gamma_k) & 0 \end{bmatrix},$$
$$\tilde{B}_{1k} = \begin{bmatrix} B_k \\ 0 \end{bmatrix}, \tilde{B}_{2k} = \begin{bmatrix} 0 \\ \hat{K}_k \bar{D}_k \end{bmatrix}, \tilde{B}_{ek} = \begin{bmatrix} 0 \\ \hat{K}_k \tilde{D}_k(\gamma_k) \end{bmatrix}.$$

Note that $\tilde{A}_k, \tilde{H}_k, \tilde{E}_k, \tilde{B}_{1k}$ and $\tilde{B}_{2k}$ are deterministic parameters, and $\tilde{A}_{ek}$ and $\tilde{B}_{ek}$ are stochastic parameters having zero mean values. Hence, the augmented system (20) is still a stochastic parameter system. Denote the state covariance matrix of the augmented system (20) by

$$\tilde{\Sigma}_k := \mathbb{E}[\tilde{x}_k\tilde{x}_k^T] = \mathbb{E}\left\{\begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}\begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}^T\right\}. \quad (21)$$

Since $\tilde{A}_{ek}$ and $\tilde{B}_{ek}$ are zero mean stochastic matrix sequences in (20), we have the following Lyapunov equation that governs the evolution of the covariance matrix $\tilde{\Sigma}_k$ from (20):

$$\begin{aligned}\tilde{\Sigma}_{k+1} = &(\tilde{A}_k + \tilde{H}_k F_k \tilde{E}_k)\tilde{\Sigma}_k(\tilde{A}_k + \tilde{H}_k F_k \tilde{E}_k)^T + \Psi_k \\ &+\tilde{B}_{1k}Q_k\tilde{B}_{1k}^T + \tilde{B}_{2k}R_k\tilde{B}_{2k}^T + \Phi_k, \quad (22)\end{aligned}$$

where

$$\begin{aligned}\Psi_k :&= \mathbb{E}[\tilde{A}_{ek}\tilde{\Sigma}_k\tilde{A}_{ek}^T] \\ &= \delta_k \begin{bmatrix} 0 & 0 \\ \hat{K}_k C_{ek} & 0 \end{bmatrix} \tilde{\Sigma}_k \begin{bmatrix} 0 & 0 \\ \hat{K}_k C_{ek} & 0 \end{bmatrix}^T, \quad (23) \\ \Phi_k :&= \mathbb{E}[\tilde{B}_{ek}R_k\tilde{B}_{ek}^T] \\ &= \delta_k \begin{bmatrix} 0 \\ \hat{K}_k D_{ek} \end{bmatrix} R_k \begin{bmatrix} 0 \\ \hat{K}_k D_{ek} \end{bmatrix}^T, \quad (24)\end{aligned}$$

with

$$\delta_k = (1-\beta_k)\beta_k \quad (25)$$

It is noted that the deterministic uncertainty F_k appears in (22). Therefore, it is impossible to have the exact value of the covariance matrix $\tilde{\Sigma}_k$. An alternative way is to find a set of upper bounds for $\tilde{\Sigma}_k$, and then obtain the minimum with respect to the filter parameters $\hat{A}_k$ and $\hat{K}_k$.

In the following, we will provide an upper bound for $\tilde{\Sigma}_k$. Before giving the upper bound, we present two lemmas.

Lemma 1. (Xie et al., 1994) Given matrices A, H, E and F with compatible dimensions such that $FF^T \le I$. Let X be a symmetric positive definite matrix and $\alpha > 0$ be an arbitrary positive constant such that $\alpha^{-1}I - EXE^T > 0$, then the following inequality holds:

$$\begin{aligned}&(A + HFE)X(A + HFE)^T \\ &\le A(X^{-1} - \alpha E^T E)^{-1}A^T + \alpha^{-1}HH^T. \quad (26)\end{aligned}$$

Lemma 2. (Theodor and Shaked, 1996) For $0 \le k \le N$, suppose $X = X^T > 0$, and $s_k(X) = s_k^T(X) \in \mathbb{R}^{n\times n}$, $h_k(X) = h_k^T(X) \in \mathbb{R}^{n\times n}$. If there exists $Y = Y^T > X$ such that

$$s_k(Y) \ge s_k(X), \quad (27)$$

and

$$h_k(Y) \ge s_k(Y), \quad (28)$$

then the solutions M_k and N_k to the following difference equations

$$M_{k+1} = s_k(M_k), N_{k+1} = h_k(N_k), M_0 = N_0 > 0, \quad (29)$$

satisfy $M_k \le N_k$.

The following corollary can be obtained immediately from Lemma 1 and (22), which provides a matrix recursive inequality for computing the actual covariance.

Corollary 1. If there exists an α_k such that $\alpha_k^{-1}I - \tilde{E}_k\tilde{\Sigma}_k\tilde{E}_k^T > 0$, then the following inequality

$$\begin{aligned}\tilde{\Sigma}_{k+1} \le &\tilde{A}_k(\tilde{\Sigma}_k^{-1} - \alpha_k\tilde{E}_k^T\tilde{E}_k)^{-1}\tilde{A}_k^T + \alpha_k^{-1}\tilde{H}_k\tilde{H}_k^T \\ &+\tilde{B}_{1k}Q_k\tilde{B}_{1k}^T + \tilde{B}_{2k}R_k\tilde{B}_{2k}^T + \Psi_k + \Phi_k \quad (30)\end{aligned}$$

holds from (22).

Corollary 1 has "eliminated" the uncertainty F_k in matrix equation (22). In the following, in order to design the quadratic filter associated with a positive definite matrix satisfying a Riccati-like inequality (Xie et al., 1994), we proceed to propose the notion of "identity quadratic filter" for the uncertain system (20) that is associated with a sequence of positive definite matrices satisfying a Riccati-like equation (understood as "identity" here) for all $\hat{A}_k$ and $\hat{K}_k$.

Definition 1. The filter (17) is said to be an identity quadratic filter associated with a sequence of matrices $\Sigma_k = \Sigma_k^T \ge 0$ $(0 \le k \le N)$ if, for some positive scalars α_k $(0 \le k \le N)$), the sequence Σ_k satisfies

$$\begin{aligned}\Sigma_{k+1} = &\tilde{A}_k(\Sigma_k^{-1} - \alpha_k\tilde{E}_k^T\tilde{E}_k)^{-1}\tilde{A}_k^T + \alpha_k^{-1}\tilde{H}_k\tilde{H}_k^T \\ &+\tilde{B}_{1k}Q_k\tilde{B}_{1k}^T + \tilde{B}_{2k}R_k\tilde{B}_{2k}^T + \Psi_k + \Phi_k, \quad (31)\end{aligned}$$

and

$$\alpha_k^{-1}I - \tilde{E}_k\Sigma_k\tilde{E}_k^T > 0. \quad (32)$$

Remark 3. In this paper, our primary objective is to find an upper bound for state estimation error variance and then minimize such an upper bound. It will be shown in the sequel that, if we could design an identity quadratic filter of the form (17), i.e., there exist positive definite solutions Σ_k to (31) and (32), then Σ_k is an expected upper bound, and a solution to the optimization problem can be found. Hence, it is important to investigate the existence as well as the solving algorithm for the solution to the recursive matrix equation (31).

Based on Definition 1 and Lemma 2, we have the following conclusion, which shows that the solution Σ_k to (31)-(32) indeed provides an upper bound for the error covariance matrix $\tilde{\Sigma}_k$ in (22).

Theorem 1. Given $\tilde{\Sigma}_k$ and Σ_k satisfying (22) and (31)-(32), respectively. If $\Sigma_0 = \tilde{\Sigma}_0$, then we have

$$\tilde{\Sigma}_k \le \Sigma_k. \tag{33}$$

Furthermore, in the light of Definition 1 and Theorem 1, we have the following corollary readily.

Corollary 2. The following inequality holds:

$$\begin{aligned}&\mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] = \begin{bmatrix} I & -I \end{bmatrix} \tilde{\Sigma}_k \begin{bmatrix} I & -I \end{bmatrix}^T \\ &\le \begin{bmatrix} I & -I \end{bmatrix} \Sigma_k \begin{bmatrix} I & -I \end{bmatrix}^T, \quad \forall k.\end{aligned} \tag{34}$$

From Theorem 1 and Corollary 2, it is clear that, if (31) has symmetric positive definite solutions Σ_k such that $\alpha_k^{-1} I - \tilde{E}_k \Sigma_k \tilde{E}_k^T > 0$, then the upper bound for the state estimation error variance can be obtained as Σ_k. Such solutions are, of course, not unique in general. In next section we will try to solve (31) while selecting the filter parameters $\hat{A}_k$ and $\hat{K}_k$ so that the upper bound obtained is minimized.

4. FINITE-HORIZON SUB-OPTIMAL FILTER DESIGN

In this section, we will design the filter based on the upper bound for the state estimation error variance. Firstly, we will provide sufficient conditions for the existence of the identity quadratic filter (17) which satisfies the constraints for the upper bound of actual state estimation error variance. Secondly, we will design the filter that optimizes the upper bound of actual state estimation error variance.

An identity quadratic filter is found in the following theorem.

Theorem 2. Let $\alpha_k > 0$ be a sequence of positive scalars. If the following two discrete-time Riccati-like difference equations

$$\begin{aligned}\Theta_{k+1} = &-A_k(\Theta_k^{-1} - \alpha_k E_k^T E_k)^{-1} \bar{C}_k^T R_{1,k}^{-1} \bar{C}_k (\Theta_k^{-1} - \\ &\alpha_k E_k^T E_k)^{-1} A_k^T + A_k(\Theta_k^{-1} - \alpha_k E_k^T E_k)^{-1} A_k^T \\ &+ \alpha_k^{-1} H_{1,k} H_{1,k}^T + B_k Q_k B_k^T, \quad \Theta_0 = S_1,\end{aligned} \tag{35}$$

and

$$\begin{aligned}P_{k+1} = &A_k(P_k^{-1} - \alpha_k E_k^T E_k)^{-1} A_k^T + \alpha_k^{-1} H_{1,k} H_{1,k}^T \\ &+ B_k Q_k B_k^T, \quad P_0 = S_0 \ge S_1,\end{aligned} \tag{36}$$

have positive-definite solutions Θ_k and P_k such that

$$\alpha_k^{-1} I - E_k P_k E_k^T > 0, \tag{37}$$

then there exists an identity quadratic filter (17) with parameters

$$\hat{A}_k = A_k + (A_k - \hat{K}_k \bar{C}_k)\Theta_k E_k^T(\alpha_k^{-1} I - E_k \Theta_k E_k^T)^{-1} E_k, \tag{38}$$

and

$$\hat{K}_k = A_k(\Theta_k^{-1} - \alpha_k E_k^T E_k)^{-1} \bar{C}_k^T R_{1,k}^{-1}, \tag{39}$$

where

$$\begin{aligned}R_{1,k} = &\bar{D}_k R_k \bar{D}_k^T + \delta_k D_{ek} R_k D_{ek}^T + \delta_k C_{ek} P_k C_{ek}^T \\ &+ \bar{C}_k(\Theta_k^{-1} - \alpha_k E_k^T E_k)^{-1} \bar{C}_k^T,\end{aligned} \tag{40}$$

such that the state estimation error variance satisfies boundedness condition

$$\mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] \le \Theta_k, \qquad \forall k. \tag{41}$$

In the following theorem, we will prove that the filter (17) with parameters (38) and (39) is an optimal filter.

Theorem 3. If (35) and (36) have positive-definite solutions Θ_k and P_k such that$\alpha_k^{-1} I - E_k P_k E_k^T > 0$. Then the identity quadratic filter (17) with parameters (38) and (39) minimizes the bound Θ_k.

Remark 4. Theorem 2 and Theorem 3 provide the optimal filter design by optimizing the upper bound for state estimation error variance. One-step ahead variance bound is optimized by selecting the filter parameters $\hat{A}_k$ and $\hat{K}_k$ as in (38) and (39) under given scaling parameter α_k. The optimization is step by step by solving the Riccati-like difference equations (35) and (36).

5. A NUMERICAL EXAMPLE

Consider the following discrete time-varying uncertain system with random sensor delay measurement:

$$\begin{cases} \breve{x}_{k+1} = \Big(\begin{bmatrix} 0 & 0.1\sin(6k) \\ 0.2 & 0.3 \end{bmatrix} \\ + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} F_k \begin{bmatrix} 0.2 & 0.1 \end{bmatrix} \Big) \breve{x}_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} w_k, \\ \breve{y}_k = \big(\begin{bmatrix} 0.5 + 0.3\sin(6k) & 1 \end{bmatrix} \breve{x}_k + \breve{v}_k, \\ y_k = (1 - \gamma_k)\breve{y}_k + \gamma_k \breve{y}_{k-1} \\ \breve{x}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \end{cases}$$

where $F_k = \sin(0.6k)$ is a deterministic perturbation matrix satisfying $F_k F_k^T \le I$, and both w_k and $\breve{v}_k$ are zero mean Gaussian white noise sequences with unity covariances. The stochastic variable $\gamma_k \in \mathbb{R}$ is a Bernoulli distributed white sequence taking values on 0 and 1 with Prob$\{\gamma_k = 1\} = \mathbb{E}\{\gamma_k\} = 0.95$.

We choose $\alpha_k = 3$ in this example. Using Theorem 2 under the initial conditions of $S_0 = 2I_4$ and $S_1 = I_4$, the filter is obtained by solving (35) and (36). The plots of upper bounds Θ_k^{11} and Θ_k^{22} as well as the actual variances for the states x_k^1 and

x_k^2 are given in Fig. 1. It can be seen that the actual variances for the states stay below their upper bounds. Therefore the proposed design method provides an expected variance constraint. These plots confirm that the proposed design requirements are achieved.

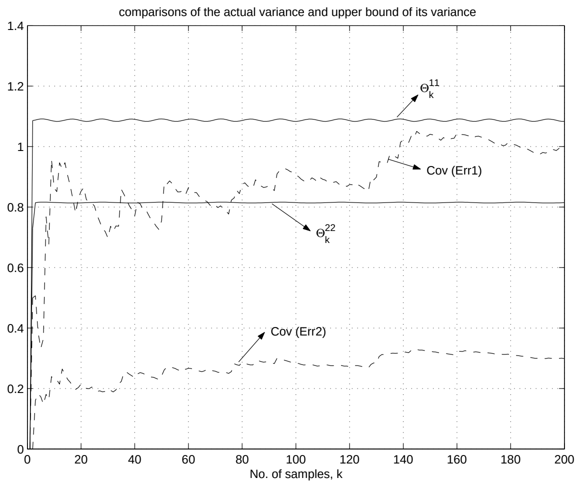

Fig. 1. The actual variances and their upper bounds, where Err1=$x_k^1 - \hat{x}_k^1$ and Err2=$x_k^2 - \hat{x}_k^2$.

6. CONCLUSIONS

A new robust filtering problem with sensor delayed measurement has been considered for discrete time-varying systems subject to norm-bounded parameter uncertainties. An algorithm has been provided for designing a finite-horizon filter which guarantees an optimized upper bound on the state estimation error variance, for all stochastic sensor delays and admissible deterministic uncertainties. Simulation results demonstrate the feasibility of our algorithm. One of our future research topics would be the design of reduced-order filters within the same framework.

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